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VARIABLE STRUCTURE CONTROL OF INFINITE DIMENSIONAL SYSTEMS WITH APPLICATIONS

Xuezhang Hou
Department of Mathematics
Towson University
Baltimore, MD 21252-0001, USA
xhou@towson.edu

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ABSTRACT: The variable structure control problem of distributed parameter system is investigated with frequent domain approach in this paper. Under the conditions more general than in Orlov and Utkin [11] and Yueming and Qijie [18], an equivalent control theorem is proposed and proved. Finally, we apply the equivalent control theorem to a heat system and a robot system so that both the heat and robot systems are exponentially stable under the variable structure control.

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1. INTRODUCTION

The variable structure system is a system whose structure is intentionally changed with a discontinuous control and it drives the phase trajectory to a hyperplane or manifold El-Ghezawi et al [6], Fu and Liao [7]. This method is well-known for its robustness to disturbance and parameter variations Verghese et al [15], Yeung et al [16], Yoo and Chung [17], Zak and Hui [19]. Conventionally, the variable structure control is based on the state-space approach in which a Lyapunov function need to be constructed so that the derivative of the Lyapunov function negative definite. As the method provides robustness characteristics, there exists a major problem, that is, the chattering phenomenon, usually encountered in the practical implementation Fu and Liao [7], Slotine and Weiping [13]. This phenomenon is highly undesirable because it may excite the high-frequency unmodelled dynamics.

In this paper, the variable structure control is dealt with an approach of operator semigroup in the frequency domain. Under the conditions more general than in Orlov and Utkin [11] and Yueming and Qijie [18], we proposed and proved an equivalent control theorem. Finally, we apply the equivalent control theorem to a heat system

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and a robot system so that both systems are exponentially stable under the variable structure control.

We start with considering the following infinite dimensional system:

$$\begin{cases} \dfrac{\partial y(y,t)}{\partial t} = Ay(x,t) + Bu(y,t) + f(y,t), \quad 0 \le t \le T < \infty\,, \\ y(x,0) = y_0(x)\,, \end{cases} \tag{1.1}$$

where $y \in V$, V is a reflexive Banach space composed of the whole functions from R^q to R^n, A is a linear operator acting in V and in V_1 respectively, B is a bounded linear operator acting on V_1 into V.

Let $S = S(y) = Cy,\ S(y) \in R^n$, C be a bounded linear operator acting on V into V_2, $V_2 \subset V_1$ be a Banach subspace. On the manifold $S = Cy = 0$, the control $u(y,t)$ of the system (1) is not continuous.

We see that the boundary layer was introduced and the effectiveness of equivalent control method (namely, equivalent control theorem) was proved in reference Orlov and Utkin [11]. In Yueming and Qijie [18], however the invertible condition of CB table place of the invertible conditions of A and A_0 in Orlov and Utkin [11], where the commutable conditions of A and P remains to be retained. In the present paper, with the assumptions that A is an infinitesimal generator of an analytic semigroup, we shall prove the equivalent control theorem by perturbation theory for analytic semigroups without the commutable conditions for A and P. Finally, we apply the equivalent control theorem to the variable structure system concerning heat process, and obtain that the solution of the system of heat process is exponentially stable under the appropriate conditions for the sliding mode.

2. EQUIVALENT CONTROL THEOREM

In this section we start with introducing the equivalent control method.

Considering the δ-neighborhood of sliding mode $S = Cy = 0$ (δ is an arbitrary given positive number), using the continuous control $\tilde{u}(y,t)$ to take place of $u(y,t)$ in system (1.1), we have

$$\begin{cases} \dot{y} = Ay + B\tilde{u}(y,t) + f(y,t)\,, \\ y(0) = y_0\,, \end{cases} \tag{2.1}$$

where $\dot{y} = \partial y/\partial t$, and the solution of (2.1) belongs to the boundary layer $\|S(y)\| \le \delta$.

Let $\dot{S}(y) = C\dot{y} = 0$. Applying C to the first equation of (2.1) leads to the equivalent control $u_{eq}(y,t)$:

$$u_{eq}(y,t) = -(CB)^{-1}C(Ay + f(y,t))\,, \tag{2.2}$$

with assumption that the $(CB)^{-1}$ exists. Substituting $u_{eq}(y,t)$ into (2.1) yields

$$\dot{y} = [I - B(CB)^{-1}C]Ay + [I - B(CB)^{-1}C]f(y,t), \tag{2.3}$$

which is called the equivalent control equation.

Denote $P = B(CB)^{-1}C$, $A_0 = (I-P)A$ and $f_0(y,t) = (I-P)f(y,t)$, then (2.3) is equivalent to the following evolution equation:

$$\dot{y} = A_0 y + f_0(y,t)\,. \tag{2.4}$$

We turn now to prove the following result.

Theorem 2.1. (Equivalent Control Theorem) *Let the following conditions be satisfied:*

(i) A is an infinitesimal generator of C_0-semigroup semigroup $T(t), t \geq 0$ on V;

(ii) $(CB)^{-1}$ exists, and PA is a closed operator;

(iii) $f(y,t)$ satisfies Lipschitz condition in y with the constant L;

(iv) the control $u(y,t)$ is bounded in any bounded region, and the solution of (2.1) is unique and bounded in the boundary layer $\|S(Y)\| \leq \delta$.

Then for each solution $\tilde{y}(t)$ of (2.4) satisfying $S(\tilde{y}_0) = 0$, $\tilde{y}_0 \in \dot{D}(A_0)$, $\|y_0 - \tilde{y}_0\| \leq \delta$, $y_0 \in D(A)$, we have

$$\lim_{t\to 0} \|y(t) - \tilde{y}(t)\| = 0, \quad \textit{uniformly on} \quad [0,T]\,. \tag{2.5}$$

Proof. Since P is bounded, it is clear that

$$\|PAy\| \leq \|P\|\|Ay\|, \quad y \in D(A). \tag{2.6}$$

Conditions (i) and (ii) together with the above inequality imply that $A_0 = A - PA$ is an infinitesimal generator of an analytic semigroup $\tilde{T}(t)$, $t \geq 0$ on V in virtue of Pazy [12], Theorem 3.2.1, and so there are constants M_1, ω_1, M_2, ω_2, satisfying

$$\|T(t)\| \leq M_1 e^{\omega_1 t}, \quad \|\tilde{T}(t)\| \leq M_2 e^{\omega_2 t}\,. \tag{2.7}$$

In the boundary layer, it is easy to see that

$$\tilde{u}(y,t) = -(CB)^{-1}C(Ay + f(y,t)) + (CB)^{-1}C\dot{y}\,. \tag{2.8}$$

Substituting this into (2.1), we find that

$$\dot{y} \;=\; (I-P)Ay + (I-P)f(y,t) + P\dot{y} = \; A_0 y + (I-P)f(y,t) + P\dot{y}, \tag{2.9}$$

and therefore, the solution $y(t)$ of (2.1) can be expressed as follows (see Pazy [12])

$$y(t) = \tilde{T}(t)y_0 + \int_0^t \tilde{T}(t-s)(L-P)f(\tilde{y}(s),s)ds + \int_0^t \tilde{T}(t-s)P\dot{y}(s)ds, \tag{2.10}$$

the solution $\tilde{y}(t)$ of (2.4) can be written as follows

$$\tilde{y}(t) = \tilde{T}(t)\tilde{y}_0 + \int_0^t \tilde{T}(t-s)(L-P)f(\tilde{y}(s),s)ds. \tag{2.11}$$

Using integration by parts, and estimating the back term of the right side of (2.5) in view of Pazy [12], Theorem 1.2.4, we obtain

$$\int_0^t \tilde{T}(t-s)P\dot{y}(s)ds = Py(t) - \tilde{T}Py_0 + \int_0^t A_0\tilde{T}(t-s)Py(s)ds$$
$$= Py(t) - \tilde{T}(t)Py_0 + \tilde{T}(t)Py_0 - Py(t) = 0. \quad (2.12)$$

Subtract (2.6) from (2.5), and employ condition (iii) and the inequality $\|\tilde{T}(t)\| \leq M_2 e^{\omega_2 t}$ to find

$$\|y(t) - \tilde{y}(t)\| \leq M_2 e^{\omega_2 T}\delta + LM_2\|I-P\|e^{\omega_2 T}\int_0^t \|y(s) - \tilde{y}(s)\|ds. \quad (2.13)$$

From Gronwall inequality, the consequence of Theorem 2.1 are now obtained, and (2.3) is the sliding equation of system (1.1). The proof is complete. □

It can be seen from Theorem 2.1 that after all the unfavorable factors are considered, the actual sliding model state $y(t)$ can arbitrarily close to ideal sliding model state $\tilde{y}$, and the conditions in Theorem 2.1 are more general than that in Orlov and Utkin [11] and Yueming and Qijie [18].

Under the condition of Theorem 2.1, if the initial states for system (1.1) are on the sliding mode $S = Cy = 0$, and the state deviates a little from sliding model, can it go back to the sliding mode by discontinuous control $u(y,t)$? This is just a problem of stability for sliding mode. Since the problems in this respect has been studied in Orlov and Utkin [11] and Yueming and Qijie [18], we shall not repeat here.

3. VARIABLE STRUCTURE CONTROL FOR A SYSTEM OF HEAT PROCESS

In this section we consider the following variable structure control problem for heat process.

$$\begin{cases} \dfrac{\partial q}{\partial t} = E\dfrac{\partial^2 q}{\partial t^2} + Fq + Bu, \\ \dfrac{\partial q}{\partial t}(0,t) = \dfrac{\partial q}{\partial t}(l,t) = 0, \quad t \geq 0, \\ \dfrac{\partial q}{\partial t}(x,0) = q_0(x), \quad 0 \leq x \leq l, \end{cases} \quad (3.1)$$

F, B is a constant matrix respectively, $E = \text{diag}(\lambda_i)$, $\lambda_i > 0$ $(i = 1, 2, \cdots, n)$ is the heat physics coefficient, $q(x,t) \in R^n$ is the temperature distribution, and $u(q,t) \in R^m$ is the temperature control.

We define an operator by $A = E\partial^2/\partial x^2$, $D(A) = \{q \in H^1(0,1) | q'(0) = q'(1) = 0\}$, where "$'$" denotes partial derivative with respect to spatial variable x, and $H^1(0,1)$ is a Sobolev space. Then the system (3.1) is equivalent to the following evolution equation:

$$\begin{cases} \dfrac{\partial q}{\partial t} = Aq + Fq + Bu(q,t), \\ q(0) = q_0. \end{cases} \tag{3.2}$$

Lemma 3.1. *Operator A is an infinitesimal generator of an analytic semigroup $T(t), t \geq 0$, on $H^1(0,1)$.*

Proof. By verifying directly we can find that A is a dissipative and self-adjoint operator in $H^1(0,1)$. In fact, applying integration by parts with the definition of A and the boundary conditions of q_0 we have

$$\begin{aligned} \langle Aq, q\rangle &= \int_0^l Eq''(x)\overline{q(x)} \\ &= [Eq'(x)\overline{q(x)}]\Big|_0^l - \int_0^l Eq'(x)\overline{q'(x)}dx \\ &= -\int_0^l E\big|q'(x)\big|^2 \leq 0\,, \end{aligned} \tag{3.3}$$

for any $q \in D(A)$. Therefore, A is a symmetric operator (see Hou and Tsui [9]) and A is dissipative. In order to show that A is self-adjoint, it suffices to show that there is a constant c such that $\|Aq\| \geq c\|f\|$, $f \in D(A)$, see Balaskrishman [2]. Actually, we see from (3.3) that

$$\|Aq\|\ \|q\| \geq \big|\langle Aq, q\rangle\big| = \|E\| \int_0^l \big|q'(x)\big|^2 dx \geq c\,\|q'\|^2\,, \tag{3.4}$$

where $c = \min\limits_{1\leq i\leq n}\{\lambda_i\}$ (since $E = \mathrm{diag}\{\lambda_i\}$ and $\lambda_i > 0$, $i = 1, 2, \cdots, n, c$ exists), and $c > 0$.

Applying the boundary condition of q in $D(A)$, we can get inequality, see Desoer [5],

$$\int_0^l \big|q(x)\big|^2 dx \leq \frac{l^2}{2}\int_0^l \big|q'(x)\big|^2 dx,$$

that is,

$$\|q\|^2 \leq \frac{l^2}{2}\|q'\|^2 \quad \text{or} \quad \|q'\|^2 \geq \frac{2}{l^2}\|q\|^2. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\|Aq\|\ \|q\| \geq c\|q'\|^2 \geq \frac{2c}{l^2}\|q\|^2,$$

and

$$\|Aq\| \geq \frac{2c}{l^2}\|q\|.$$

Thus, A is self-adjoint.

Let $r > 0$, then $A - rI$ is also a dissipative self-adjoint operator in $H^1(0,1)$, and $0 \in \rho(A - rI)$. In view of Lummer-Phillips Theorem (see Pazy [12]), it can be known that $A - rI$ generates uniformly bounded C_0-semigroup on $H^1(0,1)$. Taking an arbitrary complex number $\lambda = \sigma + i\tau$, (σ and τ are real numbers and $\tau \neq 0$), then $\lambda \in \rho(A - rI)$, we have from the Pazy [12], Theorem 2.5.2 that

$$\|[\lambda - (A - rI)]^{-1}\| \leq \text{dist}(\lambda, \sigma(A - \tau I)) \leq 1/\|\tau\|.$$

Therefore, we can conclude from that $A - rI$ generates an analytic semigroup on $H^1(0,1)$. Thus, in view of the perturbation theory for analytic semigroup (see Pazy [12]), $A = (A - rI) + rI$ is an infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ on $H^1(0,1)$, and the proof is complete. □

Next, let us turn to discuss the stability of solution of (3.1) under the variable structure control. The main results is obtained as the following theorem.

Theorem 3.1. *Assume that system (3.1) and sliding mode $S = Cq = 0$ satisfy the following conditions:*

1. $(CB)^{-1}$ exists, where C is a finite dimensional constant matrix;

2. $EP = PE$, where $P = B(CB)^{-1}C$;

3. $u(q,t)$ is bounded in any bounded region, and the solution of (3.1) is unique and bounded in the bounded layer $\|S(q)\| \leq \delta$.

Then there exists the appropriate sliding mode $S = Cq = 0$ such that the solution of (3.1) (or (3.2)) is exponentially stable under the sliding mode control.

Proof. We see from Lemma 3.1 that A in (3.2) satisfies condition (i) of Theorem 2.1. Since P is a finite dimension constant matrix, PA is a closed operator. Therefore, the condition (ii) of Theorem 2.1 is satisfied. It is easy to see that the other conditions of Theorem 2.1 are also met, hence the equivalent control method is valid for system (3.1) or (3.2) by means of Theorem 2.1, and the sliding equation is as follows:

$$\frac{\partial q}{\partial t} = (I - P)Aq + (I - P)Fq\,.$$

Since $S'' = 0$ and $EP = PE$, above sliding equation yields

$$\frac{\partial q}{\partial t} = E\frac{\partial^2 q}{\partial x^2} + Nq\,, \tag{3.6}$$

where $N = (I - P)F$.

Let $C = (C_1, C_2)$, where C_2 is an $m \times m$ invertible matrix, $q = (q_1, q_2)^T$, $q_1 \in R^{n-m}$, $q_2 \in R^m$, since $S = C_1q_1 + C_2q_2 = 0$, $q_2 = -C_2^{-1}C_1q_1$. Now, we define E, N by

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}.$$

Then, we have

$$\frac{\partial q_1}{\partial t} = E_1 \frac{\partial^2 q_1}{\partial x^2} + (N_1 - N_2 C_2^{-1} C_1) q_1 = E_1 q_1'' + N_0 q_1 \,, \tag{3.7}$$

where $N_0 = N_1 - N_2 C_2^{-1} C_1$. It is obvious that stability of solution of (3.6) is equivalent to stability of solution of (3.7). Choosing C so that all the eigenvalues of $(N_0 + N_0^*)$ all negative, then we can show that the solution of (3.7) is exponentially stable.

In fact, let $A_1 = E_1 \partial^2/\partial x^2$, then (3.7) becomes

$$\frac{\partial q_1}{\partial t} = A_1 q_1 + N_0 q_1 = (A_1 + N_0) q_1 \,.$$

We can see from Lemma 3.1 that A_1 is an infinitesimal generator of an analytic semigroup $T_1(t)$ on $H^1(0,1)$, and so is $A_1 + N_0$. Since the spectrum of $A_1 + N_0$ consists of its isolated eigenvalues, and for any eigenvalue λ and the corresponding eigenvector ψ of $A_1 + N_0$, we see that

$$A_1 \psi + (\mathrm{Re} N_0)\psi + i(\mathrm{Im} N_0)\psi = \lambda \psi \tag{3.8}$$

Take inner product with ψ on the two sides of (3.8) to find

$$\langle A_1 \psi, \psi \rangle + ((\mathrm{Re} N_0)\psi, \psi) + i\langle (\mathrm{Im} N_0)\psi, \psi \rangle = \lambda \langle \psi, \psi \rangle \,. \tag{3.9}$$

Noting that all the eigenvalues of $(N_0 + N_0^*)$ are negative, we can infer that $\langle (\mathrm{Re} N_0)\psi, \psi \rangle \le -r_0 \langle \psi, \psi \rangle$, where $-r_0$ is the maximum eigenvalue of $\mathrm{Re} N_0, r_0 > 0$.

It is clear that $\langle A_1 \psi, \psi \rangle \le 0$. In view of the real part of two sides of (3.9), we obtain that $\mathrm{Re}\lambda \langle \psi, \psi \rangle \le -r_0 \langle \psi, \psi \rangle$, and hence $\mathrm{Re}\lambda \le -r_0$, thus

$$\sup\{\mathrm{Re}\lambda | \lambda \in \sigma(A_1 + N_0)\} < 0.$$

It follows from Pazy [12], Theorem 4.4.3 that there exist constants $M \ge 1$ and $\mu > 0$ such that

$$\|(T_1(t)\| \le M e^{-\mu t} \quad (t \ge 0) \,. \tag{3.10}$$

Since the solution of (3.7) can be expressed as

$$q_1(x,t) = T_1(t) q_{10}(x), \tag{3.11}$$

estimating the norm of two sides of the above expression in terms of (3.10) we have

$$\|q_1(x,t)\| \le M \|q_{10}(x)\| e^{-\mu t} \quad (t \ge 0). \tag{3.12}$$

Since the right-hand of (3.12) tends to zero as $n \to \infty$ in exponential decay, the solution of (3.7) as well as the solution of (3.6) are exponentially stable, which implies that the solution of system (3.1) is exponentially stable under the variable structure control, and the proof of Theorem 3.1 is complete. □

Remark. The condition $EP = PE$ in Theorem 3.1 is weaker than the condition $AP = PA$ in Yueming and Qijie [18].

Actually, if we choose in R^2 that

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \lambda_1 \neq \lambda_2, \quad \lambda_1 > 0, \quad \lambda_2 > 0,$$

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad q \in D(A), \quad q_2'' \neq 0,$$

then $P^2 = P$, $PE = EP$. However, by definition, we find that

$$APq = E(Pq)'' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right]'' = \begin{pmatrix} \lambda_1 q_1'' + \lambda_1 q_2'' \end{pmatrix} ,$$

$$PAq = P(Eq)'' = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right]'' = \begin{pmatrix} \lambda_1 q_1'' + \lambda_1 q_2'' \end{pmatrix} .$$

Since $\lambda_1 \neq \lambda_2$, $q'' \neq 0$, thus $APq \neq PAq$, and so $AP \neq PA$.

Conversely, if $AP = PA$, in this case for every $q \in D(A)$, it is clear that $APq = PAq$, namely, $EPq'' = PEq''$, and hence $EP = PE$.

4. VARIABLE STRUCTURE CONTROL FOR FLEXIBLE ROBOT SYSTEM

We start this section with the following flexible robot system (see Hou and Tsui [8], Hou and Tsui [9]):

$$\begin{cases} \ddot{y}(t,x) + 2\delta EI\rho^{-1}y''''(t,x) + EI\rho^{-1}y''''(t,x) = -x\ddot{\theta}(t), \\ \ddot{\varphi}(t,x) - 2\delta(GJ/\rho k^2)\dot{\varphi}''(t,x) - (GJ/\rho k^2)\varphi''(t,x) = 0, \\ m[(l+c)\ddot{\theta}(t) + \ddot{y}(t,l) + c\ddot{y}'(t,l) + e\ddot{\varphi}(t,l)] \\ \qquad = EIy''''(t,l) - 2\delta EI\dot{y}''''(t,l)\,, \\ m[(l+c)\ddot{\theta}(t) + \ddot{y}(t,l) + c\ddot{y}'(t,l) + e\ddot{\varphi}(t,l)] + J_0[\ddot{\theta}(t,l)] \\ \qquad = -EI\ddot{y}(t,l) - 2\delta EI\dot{y}''(t,l)\,, \\ me[(l+c)\ddot{\theta} + \ddot{y}(t,l) + c\ddot{y}'(t,l) + e\ddot{\varphi}(t,l)] + J_T\ddot{\varphi}(t,l) \\ \qquad = -GJ\varphi'(t,l) - 2\delta GJ\dot{\varphi}(t,l)\,, \\ y(t,0) = 0, \quad \dot{y}(t,0) = 0; \quad (t,0) = 0, \end{cases} \tag{4.1}$$

where the meaning of the symbols used above in system (4.1) are the same as in Hou and Tsui [8] and Hou and Tsui [9].

We shall choose the space $H^1 = L^2(0,l) \times L^2(0,l) \times R^3$ as the state space of system (4.1), which is a Hilbert space equipped with inner product

$$\langle w, v\rangle = \rho \int_0^l [w_1(x)\overline{v_1(x)} w_2(x)\overline{v_2(x)}]dx + \sum_{i=3}^{5} w_i \overline{v}_i,$$

where $w = (w_1, \cdots, w_5)^T$, $v = (v_1, \cdots, v_5)^T$.

Define an operator $\Lambda : H \to H$ as follows

$$\Lambda w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M \end{pmatrix} w,$$

$$M = \begin{pmatrix} m & mc & me \\ mc & J_0 + mc^2 & mce \\ me & mce & J_T + me^2 \end{pmatrix},$$

It is not difficult to verify that Λ is the positive definite linear operator on H'.

Let $B_1 : D(B_1) \to H$:

$$B_1 w = \mathrm{diag}\left(\frac{EI}{\rho}\frac{d^4}{dx^4}, -\frac{GJ}{\rho K^2}\frac{d^2}{dx^2}, -EI\frac{d^3}{dx^3}, EI\frac{d}{dx}, GJ\frac{d}{dx}\right) w,$$

$$D(B_1) = \left\{ w | w = (w_1, \cdots, w_5)^T, w_1''(\cdot) \in H^2(0,l), w_2'(\cdot) \in \tilde{H}(0,l) \right\},$$

$$\Omega = -(x, 0, m(l+c), J_0 + mc(l+c), me(l+c))^T.$$

After defining the previous operators, we can now write again the system (4.1) as follows:

$$\begin{cases} \ddot{w}(t) + 2\delta A_1 \dot{w}(t) + A_1 w(t) = \Lambda^{-1}\Omega\ddot{\theta}(t)\,, \\ w(0) = w_0, \dot{w}(0) = w_{10}, \end{cases} \tag{4.2}$$

here $A_1 = \Lambda^{-1}B_1, D(A_1) = D(B_1); W = (w_1, \cdots, w_05)^T, w_1 = y(t,_2) = \varphi(t,l) = w_3 = y(t,l), w_4 = -y;(t,l), w_5 = y(t,l); w_0, w_{10}$ are the initial values of the system (4.1).

Now, we turn to consider the variable structural control in system (4.2), namely

$$\begin{cases} \ddot{w}(t) + 2\delta A_1 \dot{w}(t) + A_1 w(t) = Cu(t, w, \dot{w}) + \Lambda^{-1}\Omega\ddot{\theta}(t), \\ w(0) = w_0, \quad \dot{w}(0) = w_1 0\,, \\ S = G_1 w + G_2 \dot{w} = 0, \end{cases} \tag{4.3}$$

where C, G_1, G_2 are bounded linear operators on H, u is the control system (4.3).

Set $v = (v_1, v_2)^T$, $v_1 = w$, $v_2 = \dot{w}$

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_1 & -2\delta A_1 \end{pmatrix}, \quad D(\mathcal{A}) = D(A_1) \times D(A_1), \quad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

$$\mathcal{F}_1(t, v) = (0, \Lambda^{-1}\ddot{\theta}(t))^T, \mathcal{L}(G_1, G_2), \tag{4.4}$$

then system (4.3) is equivalent to the following first order evolution equation in $\mathcal{H} = H_1 \times H_1$

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}u(t, v) + \mathcal{F}(t, v)\,, \\ v(0) = v_0\,, \\ S = \mathcal{L}v = 0\,, \end{cases} \tag{4.5}$$

where $v_0 = (w_0, w_{10})^T$.

To study further the stability of the system (4.5), we shall introduce the following useful lemmas, see Hou and Tsui [8].

Lemma 4.1. *The Operator $\mathcal{A}$ in (4.5) is the infinitesimal generator of a C_0-semigroup $T(t)$ and Hilbert Space $\mathcal{H}$ and there exist the constants $M > 0$ and $\omega > 0$ such that*

$$\|T(t)\| \le Me^{-\omega t} \quad (t \ge 0)\,, \tag{4.6}$$

where $\omega = \sup\{\mathrm{Re}\lambda | \lambda \in \sigma(\mathcal{A})\} < 0$.

Lemma 4.2. *The eigenvectors of $\mathcal{A}$ constitute a Riesz bases, see Curtain and Zwart [4], of $\mathcal{H}$.*

Lemma 4.3. *Let $C[0, +\infty)$ be a Banach space consisted of all uniformly continuous functions on $[0, +\infty)$ with norm in the form*

$$\|g\|_m = \max_{t \ge 0} |g(t)|. \tag{4.7}$$

Let

$$(Kg)(t) = \int_0^t e^{-\omega(t-s)} g(s) ds, \quad g \in C[0, +\infty),$$

then K is a bounded linear operator, and $\|K\| \le 1/\omega$.

Theorem 4.1. *In the system (4.5), let the following conditions be satisfied:*

(a) $(\mathcal{L}\mathcal{B})^{-1}$ is existent and compact:

(b) $\mathcal{A}P = P\mathcal{A}$, where $P = \mathcal{B}(\mathcal{L}\mathcal{B})^{-1}\mathcal{L}$;

(c) $\mathcal{F}(t, v)$ satisfies Lipschitz condition in v, and Lipschitz constant $L < \omega/M\|I - P\|$ (ω and M are the same as that in (4.6)).

Then solution $v(t)$ of robot control system (4.5) is exponentially stable.

In fact, we see from Lemma 4.2 that the eigenvectors of $\mathcal{A}$ constitute the Riesz basis. Let $\{e_{n_k}|n=1,2,\cdots;k=1,2,\cdots,n_k\}$ be all eigenvectors corresponding to eigenvalue λ_n of $\mathcal{A}$. Then $\mathcal{A}e_{n_k}=\lambda_n e_{n_k}$, $\mathcal{P}\mathcal{A}=\mathcal{A}\mathcal{P}$, $\mathcal{P}^2=\mathcal{P}$, $(I-\mathcal{P})^2=I-\mathcal{P}$, we have

$$(I-\mathcal{P})\mathcal{A}(I-\mathcal{P})e_{n_k}=\lambda_n(I-\mathcal{P})e_{n_k}. \quad n=1,2,\cdots,$$

write $\varphi_{n_k}=(I-\mathcal{P})e_{n_k}$, then $(I-\mathcal{P})\mathcal{A}\varphi_{n_k}=\lambda_n\varphi_{n_k}$, and hence $\{\varphi_{n_k}|n=1,2,\cdots,k=1,2,\cdots,n_k\}_{n=1}^{\infty}$ are all the eigenvectors for eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of $(I-\mathcal{P})\mathcal{A}$. It follows that $w=\sum_{n=1}^{\infty}\sum_{k=1}^{n_k}a_{n_k}e_{n_k}$ for every $w\in D(\mathcal{A})$, and so

$$(I-\mathcal{P})\mathcal{A}w=(I-\mathcal{P})\mathcal{A}\sum_{n=1}^{\infty}a_ne_n=\sum_{n=1}^{\infty}a_{n_k}\lambda_n(I-\mathcal{P})e_n=\sum_{n=1}^{\infty}\beta_{n_k}\varphi_n\,,$$

where $\beta_{n_k}=a_{n_k}\lambda_n$. This implies that $\{\varphi_n\}_{n=1}^{\infty}$ constitute Riesz basis of $\mathcal{H}_0$, and

$$\sigma_p((I-\mathcal{P})\mathcal{A})=\sigma_p(\mathcal{A})=\{\lambda_n\}_{n=1}^{\infty}\,,$$

where $\sigma_p(\cdot)$ is the point spectrum of operator.

Applying Pazy [12], to the operator $(I-\mathcal{P})\mathcal{A}$, we see that $(I-\mathcal{P})\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup $\hat{T}(t)$ on $\mathcal{H}_0$, and there is a constant N such that $\|\hat{T}(t)\|\le Ne^{-\omega t}$. By means of functional analysis (Pazy [12]), every $\hat{T}(t)$ can be extended so as to be an operator acting on $\mathcal{H}$, maintaining the norm of $T(t)$, still is written as $\hat{T}(t)$, and then the conclusion of Lemma 2 is obtained.

Proof. We know from Lemma 4.1 that $\mathcal{A}$ is an infinitesimal generator of C_0-semigroup $T(t)$. By using the equivalent control method in Section 2, we can get the following equivalent control equation:

$$\begin{cases}\dot{v}=(I-P_1)\mathcal{A}v+(I-P_1)(F)(t,v),\\ v(0)=v_0\,.\end{cases} \tag{4.8}$$

It can be shown that $(I-P)\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup $T_1(t)$, and there is constant M_1 satisfying

$$\|T_1(t)\|\le M_1e^{-\omega t}\,. \tag{4.9}$$

In fact, we see from the condition (a) that P is a compact operator on $\mathcal{H}$, and its range is closed in $\mathcal{H}$, see Taylor and Lay [14].

Since $\mathcal{A}$ is a closed linear operator with dense domain $D(\mathcal{A})$, its range $\mathcal{A}\mathcal{H}=\mathcal{H}$, see Taylor and Lay [14]. Hence the range of $(I-P)\mathcal{A}$ is $(I-P)\mathcal{A}\mathcal{H}=(I-P)\mathcal{H}$, which is a complete subspace of $\mathcal{H}$, written by $\mathcal{H}_0$. Now, we shall prove that all eigenvectors of $(I-P)\mathcal{A}$ constitute the Riesz basis of $\mathcal{H}_0$.

It is easy to see that the solution of (4.8) can be resolved into the following form:

$$v(t)=\varphi(t)+\psi(t)\,, \tag{4.10}$$

here $\varphi(t)$ and $\psi(t)$, satisfy the following equations, respectively

$$\begin{cases} \dot{\varphi} = (I-P)\mathcal{A}\varphi, \\ \varphi(0) = v_0\,, \end{cases} \tag{4.11}$$

$$\begin{cases} \dot{\psi} = (I-P)\mathcal{A}\psi + (I-P)\mathcal{F}(t,v) \\ \psi_0 = v_0. \end{cases} \tag{4.12}$$

In view of the theory of operator semigroup, the solution of (4.12) can be expressed as follows

$$\psi(t) = \int_0^t T(t-s)(I-P)\mathcal{F}(s,v(s))dx.$$

Combining (4.9) with condition (c), we have

$$\|\psi(t)\| \le M\|I-P\|L\int_0^t e^{-\omega(t-s)}(\|\varphi(s)\| + \|\psi(s)\|ds\,. \tag{4.13}$$

Analogously, the solution of (4.11) can be expressed as follows

$$\varphi(t) = T(t)v_0$$

and

$$\|\varphi(t)\| \le M\|v_0\|e^{-\omega t}\,. \tag{4.14}$$

Substitute (4.14) into (4.13), it following from Lemma 4.3 that

$$\|\psi(t)\| \le M^2\|I-P_{\|}L\|v_0\|te^{-\omega t} + M\|I-P\|LK(\|\psi\|).$$

Arranging the above inequality, we have

$$(I - M\|I-P\|LK)(\|\psi\|) \le M^2\|I-P\|L\|v_0\|te^{-\omega t}. \tag{4.15}$$

With the assumption (c), it is clear that $\|M\|I-p\|LK\| < 1$, and therefore, the inverse of $I - M_1\|I-P_1\|LK$ is existent, and

$$(I-M)\|I-P\|LK)^{-1} = \sum_{n=0}^{\infty}(M\|I-P\|L)^nK^n. \tag{4.16}$$

Since K is a monotonically increasing operator, applying (4.16) onto the both sides of (4.15), in terms of the inducible result

$$K^n(te^{-\omega t}) = e^{-\omega_1 t}t^{n+1}/(n+1)!,$$

we have

$$\|\psi\| \le M\|v_0\|\sum_0^{\infty}(M\|I-p\|L)^{n+1}(t^{n+1}/(n+1)!)e^{-\omega t} = M_1\|v_0\|e^{-\omega - M}\|I-p\|L)t.$$

Write $\mu = \omega - M\|I-p\|L$, then we know from the condition (c) that $\mu > 0$, and

$$\|\psi\| \le M\|v_0\|e^{-\mu t}\,. \tag{4.17}$$

It follows from (4.14), (4.17) and $\mu < \omega$ that

$$\|v(t)\| \leq \|\varphi(t)\| + \|\psi(t)\| \leq M\|v_0\|e^{-\mu t} .$$

This concludes that the solution to the robot control system (4.5) is exponential stable, and the proof is complete. □

5. CONCLUSION

In this paper, the variable structure control problem of distributed parameter system was investigated with frequent domain approach. Under the conditions more general than in Orlov and Utkin [11] and Yueming and Qijie [18], an equivalent control theorem was proposed and proved. Finally, the equivalent control theorem was applied to a heat system and a robot system, and it was shown that both the heat and robot systems are exponentially stable under the variable structure control.

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