Communications in Applied Analysis, **11** (2007), no. 1, 23–29

SPECTRUM OF ZERO POINT FIELD AND STATISTICAL STABILITY

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Communicated by S. Nenov

ABSTRACT: The source of zero point field (ZPF) as an aggregate of random electromagnetic fields generated by charged particles through out the universe is shown to be inconsistent with Lorentz invariance. Here, we have used the idea of statistically stable distributions in analyzing the spectral density of ZPF.

AMS (MOS) Subject Classification: 35B35

1. INTRODUCTION

The idea of real zero point fluctuations was introduced by Plank [10] in connection with his second form of the Planck distribution, which differed from the first one, aside from conceptual matters, just by the presence of an extra term that assigns an energy $1/2hw$ to the fluctuations of each oscillator. It was Nernst [7] who took the matter seriously and even generalized the idea by applying it to the oscillators of the field. The idea of real zero point fluctuation has been put forward time and again from different considerations (see Sakharov [12], or Misner et al [6]). Weisskopf [13] and Welton [14] considered the vacuum field of quantum electrodynamics (QED) as real field. Welton explained Lamb shift by considering the effect of fluctuation of this vacuum field to the hydrogen atom. Braffort et al [1] came to the conclusion that in absorber theory of Wheeler and Feynman there must exist a remnant fluctuating field, due to the highly irregular motion of the atoms of the absorber and concluded that the spectral density of the vacuum field should be of the form

$$\rho(w) = aw^3 ,$$

Received August 28, 2006

where w denotes the frequency. The fundamental properties of the zero point field are:

(1) It is homogeneous.
(2) It is isotropic.
(3) It is stationary field with a Lorentz invariant spectrum.

Homogeneity and isotropy are required to guarantee that no position and no direction in space are privileged. Lorentz invariance is required to guarantee that no inertial frame is preferred. In a recent paper Braffort [2], Ibison observed that the w^3 spectrum is the only spectrum that is self-consistent and self-maintaining in a expanding (or contracting) cosmology with an FRW metric, in addition to Lorentz Invariance. Ibison and Haisch also discussed the quantum and classical statistics of the ZPF. They emphasized on the connection between the probability distributions of the stochastic variables in the classical field, and those of the Fourier amplitudes of the quantum field theory. In this paper we investigate the source of ZPF in the light of statistically stable distributions. This shows that 'random aggregate' of minute sources cannot possibly give rise to the ZPF. One may have to sacrifice Lorentz invariance of ZPF under such a 'random aggregate' model. In Section 2, we shall briefly discuss the characteristics of the ZPF for our convenience. Then we shall study the stability of distributions in Section 3. Finally we shall indicate its consequences in Section 4.

2. ZERO POINT FIELD

Let us consider the radiation field in a cubic box of side L with perfectly conducting walls. For simplicity let us describe these in coulomb gauge. Then a decomposition of the vector potential into plane waves with propagation vector $\vec{k}$ and frequency $w(= ck),\ k = \ \vec{k}$, gives

$$\vec{A} = \frac{1}{L^{3/2}} \sum_{k,\lambda} \hat{\epsilon}_{k\lambda} [c_{k\lambda} e^{i(\vec{k}.\vec{r}-wt)} + c^*_{k\lambda} e^{-i(\vec{k}.\vec{r}-wt)}] \,, \tag{1}$$

where $\hat{\epsilon}_{k\lambda}$ are the polarization vectors $\lambda = 1, 2,$ which satisfy the transversality condition

$$\vec{k}.\hat{\epsilon}_{k\lambda} = 0 \quad \text{and the orthogonality condition:}$$

$$\hat{\epsilon}_{k\lambda}.\hat{\epsilon}_{k\lambda'} = \delta_{\lambda\lambda'} \,.$$

In Coulomb gauge

$$\vec{E}(\vec{r}, t) = -\frac{1}{c}\frac{\partial \vec{A}}{\partial\, t}, \qquad \vec{B}(\vec{r}, t) \ = \nabla \ \times \vec{A} \,. \tag{2}$$

Then the fields can also be decomposed as

$$\vec{E}(\vec{x}, t) = \sqrt{\frac{4\pi}{L^3}} \sum_{n,\lambda} \hat{\epsilon}_{n\lambda} [p_{n\lambda} \cos(\vec{k}_n.\vec{x}) + w_n q_{n\lambda} \sin(\vec{k}_n.\vec{x})] \,, \tag{3}$$

$$\vec{B}(\vec{x},t) = \sqrt{\frac{4\pi c^2}{L^3}} \sum_{n,\lambda} (\vec{k}_n x \hat{\epsilon}_{n\lambda})[q_{n\lambda} \sin(\vec{k}_n.\vec{x}) + \frac{p_{n\lambda}}{w_n} \cos(\vec{k}_n.\vec{x})] \,, \tag{4}$$

where $\dot{q}_{n\lambda} = p_{n\lambda}$, $\dot{p}_{n\lambda} = -w_n^2 q_{n\lambda}$.

Here $q_{n\lambda}$ and $p_{n\lambda}$ are random variables. Since equation (3) and (4) refer to a vacuum field that averages to zero we have

$$< q_{n\lambda} >= 0, \qquad < p_{n\lambda} >= 0 \,.$$

These are the statistical properties of amplitudes q and p. Again, the idea that the field is in a maximum state of randomness leads us to assume that each Fourier component in equation (1) fluctuates independently of the others, i.e. they are not correlated. Mathematically, we can write (see Ibison [4])

$$\begin{aligned} &< q_{n\lambda} p_{n'\lambda'} >= 0 \,, \\ &< q_{n\lambda} q_{n'\lambda'} >= \frac{h}{2w_n} \delta_{nn'} \delta_{\lambda\lambda'} \,, \\ &< p_{n\lambda} p_{n'\lambda'} >= \frac{1}{2} h w_n \delta_{nn'} \delta_{\lambda\lambda'} \,, \end{aligned} \tag{5}$$

which is equivalent to assuming that the amplitudes q, p have a Gaussian distribution. This follows from a general theorem which establishes that a random stationary field with statistically independent Fourier components, is Gaussian.

Now we calculate the autocorrelation function of the electric field at two different points (see Ibison [4]) in space-time. We have

$$\begin{aligned} < E_i(\vec{x},t) E_j(\vec{x}',t') >= &\sum_{n,n'} \sum_{\lambda,\lambda'} f(w_n) f(w_{n'}) \hat{\epsilon}_{(n\lambda)_i} \hat{\epsilon}_{(n'\lambda')_j} \\ &\times \Big[< a_{n\lambda} a^*_{n'\lambda'} e^{i(\vec{k}_n.\vec{x} - w_n t) - i(\vec{k}_{n'}.\vec{x}' - w_{n'} t')} > \\ &+ < a^*_{n\lambda} a_{n'\lambda'} e^{-i(\vec{k}_n.\vec{x} - w_n t) + i(\vec{k}_{n'}.\vec{x}' - w_{n'} t')} > \Big] \,. \end{aligned} \tag{6}$$

After simplification we get

$$\begin{aligned} &< E_i(\vec{x},t) E_j(\vec{x}',t') >=< B_i(\vec{x},t) B_j(\vec{x}',t') > \\ &= \frac{2\pi h}{L^3} \sum_n w_n (\delta_{ij} - \frac{k_{n_i} k_{n_j}}{k_n^2}) \cos[\vec{k}_n.(\vec{x} - \vec{x}') - w_n(t - t')] \,. \end{aligned} \tag{7}$$

Let us use the above results to calculate the energy density of the field. For this purpose it is convenient to take the limit $L \to \infty$, which is achieved with the substitution

$$\frac{1}{L^3} \sum_n \longrightarrow \frac{1}{8\pi^3} \int d^3k \,.$$

We then obtain

$$< E_i(\vec{x},t)E_j(\vec{x}',t') >= \frac{h}{4\pi^2}\int(\delta_{ij}-\frac{k_ik_j}{k^2})w\cos[\vec{k}.(\vec{x}-\vec{x}')-w(t-t')]d^3k\,, \quad (8)$$

$$\text{again } \oint(\delta_{ij}-\frac{k_ik_j}{k^2})d\,\Omega_k=\frac{8\pi}{3}\delta_{ij}\,. \quad (9)$$

Changing the variable $w = ck$; one can get

$$< E_i(\vec{x},t)E_j(\vec{x}',t') >= \frac{2h}{3\pi c^3}\delta_{ij}\int_0^\infty w^3\cos\left[(t-t'-\frac{|\vec{x}-\vec{x}'|}{c})w\right]dw\,. \quad (10)$$

For many applications, the effects of the retardation (at least when $|\vec{x}-\vec{x}'|$ is not too large) are negligible, in those cases we can approximate (10) by its value at $\vec{x}'=\vec{x}$. So,

$$< E_i(\vec{x},t)E_j(\vec{x}',t') >= \frac{4\pi}{3}\delta_{ij}\int_0^\infty \rho(w)\cos w(t-t')dw = \frac{4\pi}{3}\delta_{ij}\int_0^\infty \rho(w)dw, \quad (11)$$

taking $t \simeq t'$, where $\rho(w)=\frac{h}{2\pi^2c^3}w^3$.

This indicates that the variance of $\vec{E}$ in zero point field (ZPF) is infinite.

3. STABILITY OF DISTRIBUTION

The limiting distribution of standardized sum of independent and identically distributed (iid) random variables is normal provided the variance is finite (or the truncated variance is a slowly varying function). This concept is generalized by the class of stable distribution when the truncated variance has growth higher than that already stated above i.e., the variance is finite or the truncated variance is a slowly varying function. This is especially relevant in view of the fact that in ZPF the spectral density is proportional to w^3 resulting to the infinite variance of the electromagnetic variable. We intend to show that under appropriate assumption, $\vec{E}$ (electric field) falls under stable domain. We briefly discuss the properties of stable distributions Ibison and Haisch [5] in the followings.

Let $X_1,\cdots,X_n$ be independent and identically distributed normal random variables with mean 0 and variance 1, then

$$X_1+\cdots+X_n=\sqrt{n}(\sqrt{n}\,\overline{X}_n)\stackrel{\mathcal{D}}{=}\sqrt{n}\,X\,, \quad \text{where} \quad X\sim N(0,1)\,.$$

Now let $X, X_1,\cdots,X_n$ be iid random variable distributed as F, then:

(1) Iff, $X_1+\cdots+X_n\stackrel{\mathcal{D}}{=}c_nX+\gamma_n$ for some c_n and γ_n, then X is said to be a stable distribution.

(2) If $\gamma_n=0$, then the distribution is said to be strictly stable.

Every stable distribution with exponent $\alpha\neq 1$, can be centered so as to become strictly stable. For $\gamma=1$, the centering constant is $\gamma_n=\gamma\; n\;\log\; n$. It can be shown that the scaling constant $n^{1/\alpha}$ is possibility for c_n, only when $0 < \alpha \leq 2$. Note that for small α, c_n is much higher than $n^{1/2}$, the scaling constant of central limit theorem. The stable distribution for $\alpha = 2$, corresponds to normal distribution. Now if we relax the above condition slightly and only require that Ibison and Haisch [5]

$$\frac{X_1 + \cdots + X_n}{c_n} - \gamma_n \xrightarrow{\mathcal{D}} G\,,$$

then we say that X belongs to domain of attraction of G. It can be shown that then G is stable. A necessary condition that F falls in the domain of a stable distribution $G = G(\alpha)$ is that

$$U(x) = \int_{-x}^{x} y^2 dF(y) \sim x^{(2-\alpha)} L(x),\ 0 < \alpha \leq 2$$

where L is a slowly varying function.

Now note that the truncated variance of E up to frequency w_0 (say) becomes

$$< E_i(w)E_i(w) >|_{w_0} = \frac{4\pi}{3} \int_0^{w_0} \rho(w)dw \quad \propto \quad w_0^{2-\alpha} \tag{12}$$

if $\rho(w) \ \propto \ w^{(1-\alpha)}$.

So, if the spectral density

$$\rho(w) \ \propto \ w^{(1-\alpha)},\ \text{with} \ \ 0 < \alpha \leq 2\,,$$

then E falls in the domain of attraction of stable distribution of exponent α.

The statistical meaning of

$$\rho(w) \not\propto w^{(1-\alpha)}, \ \ 0 < \alpha \leq 2$$

is that E cannot be represented as sum of several random independent component variables of equal magnitude. In such a sum representation, contribution from a few particular component then becomes exceedingly large and no normalization is possible. For example, consider $0 < \alpha < 1$, then

$$n^{-1}(X_1 + \cdots + X_n) \sim X_1 n^{[(-1)+\frac{1}{\alpha}]}\,,$$

i.e., average is larger (in distribution) than any given component which means that $\max(X_1, \cdots, X_n)$ grows at an exceedingly large rate to have a tremendous effect on the sum to blow it up. Incidentally for α-stable laws, moments of order $\geq \alpha$ may not exist as

$$1 - G(x) + G(-x) \ \sim \ \frac{(2-\alpha)}{\alpha} x^{-\alpha} L(x)\,. \tag{13}$$

The tail probabilities of these distribution are larger, as a result higher moments do not exist.

So we see that statistical stability is *not* in conformity with Lorentz invariance of ZPF, where we require $\rho(w) \propto w^3$.

Lorentz invariance may not follow and in fact contradictory to a equally appealing assumption of the source of ZPF as ‘random aggregate’ of randomly varying motion of charged particles, see de La Pena [9]. These particles may also be thought as the source of random radiations all over the universe. The same idea is also used to derive the gravitational field of stars. This is called Holtzmark distribution. We follow the same idea here which leads to a stable distribution of exponent 3/2. Let us assume that E_λ, the x component of the electric field with density λ has the following property:

Two independent aggregate of densities λ and μ may be combined into a single aggregate of densities $(\lambda + \mu)$. This amounts to assuming that

$$E_\lambda + E_\mu = E_{\lambda+\mu} \,. \tag{14}$$

Now the change of density from 1 to λ amounts to a change of unit length of the box (in which the field is confined) from 1 to $1/\lambda^{(1/3)}$. Also the electric force varies inversely with the square of the distance. So E_λ must have same distribution as $\lambda^{(2/3)} E_1$. This implies that E has a symmetric stable distribution with exponent 2/3, since for stable α distribution X,

$$\lambda^{1/\alpha} X_1 + \mu^{1/\alpha} X_2 = (\lambda + \mu)^{1/\alpha} X \,.$$

It may be worth mentioning Feller [3] sketched a calculation supposedly giving the result that the ZPF is self-consistently sourced by direct-action interactions.

4. CONCLUSIONS

It is evident from the above analysis that *the source of ZPF as an aggregate of random electromagnetic fields generated by charged particles throughout the universe is inconsistent with Lorentz invariance.*

One can also imagine that the frequencies w_n appearing in expression (7) may be either:

(1) of unequal weighting, or

(2) of a dissipative nature where a higher frequency dampens by a contraction factor depending on the frequency.

In both the situation an extra term say $g(w)$ appears which may be considered either as unequal weighting or as a contraction factor in (7). Therefore (11) reduces to

$$< E_i(\vec{X}, t) E_j(\vec{X}, t) >= \frac{4\pi}{3} \delta_{ij} \int_0^\infty \rho(w) dw \,, \tag{15}$$

with

$$\rho(w) = \frac{h}{2\pi^2 c^3} w^3 g(w) \,.$$

Now if

$$g(w) \propto w^{-(2+\alpha)} \quad \text{for} \quad 0 < \alpha \leq 2 \,,$$

then E belongs to domain of attraction of a α stable law, $\alpha = 2$ indicates that the distribution belongs to normal domain. One may also take

$$g(w) \ll \exp(-\epsilon w), \quad \epsilon > 0\,,$$

as advocated in equilibrium of electromagnetic radiations with relativistic particle distribution leading E to the normal domain of attraction.

ACKNOWLEDGMENTS

One of the authors (Sisir Roy) is greatly indebted to College of Science, George Mason University, Fairfax, VI, USA, for the financial support during certain part of this work.

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