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ON WEIGHTED L-CONJUGATE MEANS

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ABSTRACT: Some Jensen's type inequalities strongly connected with weighted $\boldsymbol{L}$-conjugate means of $n \geq 2$ variables are given. Comparison and equality problems for weighted $\boldsymbol{L}$-conjugate means are solved.

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1. INTRODUCTION

The use of the averages in statistics and probability is well known, and the oldest concept — the arithmetic mean of two numbers — is also very old since it was known and used by the Babylonians in 7000 B.C. The *arithmetic mean* with equal weights which is for an n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ defined as

$$A_n(\boldsymbol{x}) = \frac{x_1 + \ldots + x_n}{n},$$

is the simplest mean and the most common concept for averaging a set of numbers.

Two other means of a very elementary nature are *geometric* and *harmonic* means which are for a positive n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ defined as

$$G_n(\boldsymbol{x}) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}, \quad H_n(\boldsymbol{x}) = n\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^{-1}.$$

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It is well known that for any positive n-tuple $\boldsymbol{x}$ the following, so called HGA inequalities (see Cauchy, 1821), hold

$$H_n(\boldsymbol{x}) \leq \; G_n(\boldsymbol{x}) \leq \; A_n(\boldsymbol{x}).$$

But what properties should a function M have for it to be considered as a mean?

Clearly we require M to be continuous and symmetric, but a little less obvious is the crucial property of internality that justifies the very name of mean.

Definition 1. Let $I \subset \mathbb{R}$ be an open interval and $n \geq 2$. A function $M : I^n \to I$ is called a mean *of n variables* on I if it has the following properties:

(In) If $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $x_k \neq x_l$ for some $k, l \in \{1, \ldots, n\}$, then

$$\min_{i \in \{1,\ldots,n\}} \{x_i\} < M(\boldsymbol{x}) < \max_{i \in \{1,\ldots,n\}} \{x_i\}.$$

(Sy) M is symmetric in the sense that for $\boldsymbol{x} = (x_1, \ldots, x_n)$

$$M(\widetilde{\boldsymbol{x}}) = M(\boldsymbol{x})$$

whenever $\widetilde{\boldsymbol{x}} = (x_{i_1}, \ldots, x_{i_n})$, where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$.

(Co) M is continuous on I^n.

A very natural extension of Definition 1 is obtained when some of the arguments in $\boldsymbol{x} = (x_1, ..., x_n)$ occur more then once, or when (from the practical point of view) some of the arguments in $\boldsymbol{x} = (x_1, ..., x_n)$ are considered to be more important then others. In this case we introduce an n-tuple of weights $\boldsymbol{w} = (w_1, ..., w_n)$ and we say that M is a *weighted mean.*

Condition (Sy) is now slightly changed: while $M(\boldsymbol{x}; \boldsymbol{w})$ may depend on the order of $x_1, ..., x_n$, it must be symmetric in the following way: for $\boldsymbol{x} = (x_1, ..., x_n)$ and $\boldsymbol{w} = (w_1, ..., w_n)$

$$M(\widetilde{\boldsymbol{x}}; \widetilde{\boldsymbol{w}}) = M(\boldsymbol{x}; \boldsymbol{w})$$

whenever $\widetilde{\boldsymbol{x}} = (x_{i_1}, ..., x_{i_n})$ and $\widetilde{\boldsymbol{w}} = (w_{i_1}, ..., w_{i_n})$, where $(i_1, ..., i_n)$ is any permutation of $(1, ..., n)$.

Means can have many special properties, but we will be interested in only one: the property of homogeneity which is defined as follows.

Definition 2. Let $M : \mathbb{R}_+^n \to \mathbb{R}_+$ be a mean of $n \geq 2$ variables. We say that M is homogenous if

$$M(t\boldsymbol{x}; \boldsymbol{w}) = tM(\boldsymbol{x}; \boldsymbol{w})$$

holds for all $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{R}_+^n$ and all $t \in \mathbb{R}_+$.

It can be easily seen that H_n, G_n and A_n are all homogenous.

The first classical generalization of the arithmetic, geometric and harmonic means are the r-th *power means*, which are defined as

$$M_n^{[r]}(\boldsymbol{x};\boldsymbol{w}) = \begin{cases} \left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i^r\right)^{\frac{1}{r}}, & r \in \mathbb{R}\setminus\{0\}, \\ G_n(\boldsymbol{x};\boldsymbol{w}), & r = 0, \\ \max \boldsymbol{x}, & r = \infty, \\ \min \boldsymbol{x}, & r = -\infty, \end{cases}$$

where $\boldsymbol{x}$ and $\boldsymbol{w}$ are positive n-tuples, $r \in \overline{\mathbb{R}}$ and $W_n = \sum_{i=1}^n w_i$. It is known that if $r \leq s$ the following inequality holds

$$M_n^{[r]}(\boldsymbol{x};\boldsymbol{w}) \leq M_n^{[s]}(\boldsymbol{x};\boldsymbol{w}).$$

This inequality, which is the basic generalization of HGA inequalities, is called the *power mean inequality* (see Schlomilch, 1858) or the (r,s)-inequality.

It can be easily seen that the r-th power means are also homogenous.

The power means are defined using the power, logarithmic and exponential functions. Further generalizations are obtained using arbitrary continuous strictly monotonic functions, and we call them *quasi-arithmetic means*. They are defined in the following way.

Let I be an open, nonempty interval in $\mathbb{R}$, $\boldsymbol{x} \in I^n$, $\boldsymbol{w}$ a positive n-tuple and $\varphi : I \to \mathbb{R}$ a continuous, strictly monotonic function. The quasi-arithmetic φ-mean of $\boldsymbol{x}$ with weights $\boldsymbol{w}$ is defined as

$$M_\varphi(\boldsymbol{x};\boldsymbol{w}) = \varphi^{-1}(A_n(\varphi(\boldsymbol{x});\boldsymbol{w})),$$

where $\varphi(\boldsymbol{x})$ denotes $(\varphi(x_1),\ldots,\varphi(x_1))$, and we say that the function φ *generates* the mean M_φ.

It is known that the r-th power means are the only homogenous quasi-arithmetic means (see Jessen, 1931). The corresponding generalization (see Bonferoni, 1927) of the power mean inequality is stated as follows.

Let φ and ψ be two continuous, strictly monotonic functions on I and let $\varepsilon_\psi = 1$ if ψ is increasing and $\varepsilon_\psi = -1$ if ψ is decreasing. The inequality

$$\varphi^{-1}(A_n(\varphi(\boldsymbol{x});\boldsymbol{w})) \leq \psi^{-1}(A_n(\psi(\boldsymbol{x});\boldsymbol{w})) \tag{1}$$

holds for all $\boldsymbol{x} \in I^n$ and all positive n-tuples $\boldsymbol{w}$ if and only if $\varepsilon_\psi \psi \circ \varphi^{-1}$ is convex. The reverse inequality (1) holds for all $\boldsymbol{x} \in I^n$ and all positive n-tuples $\boldsymbol{w}$ if and only if $\varepsilon_\psi \psi \circ \varphi^{-1}$ is concave.

The interested reader can find all these classical and many of the latest results in Bullen [1].

In paper Daróczy and Páles [2] introduced the notion of L-conjugate means of $n \geq 2$ variables. In the following we denote by I an open interval in $\mathbb{R}$ and by $CM(I)$ the set of all continuous and strictly monotonic real functions defined on I.

Definition 3. Let $L : I^n \to I$ be a fixed mean of $n \geq 2$ variables on I. A mean $M : I^n \to I$ is called L-conjugate mean of n variables on I if there exists a function $\varphi \in CM(I)$ for which

$$M(\boldsymbol{x}) = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} \varphi(x_i) - \varphi(L(\boldsymbol{x}))}{n-1}\right) =: L_{\varphi}^{*}(\boldsymbol{x})$$

for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$. Then the function φ is called the generating function of the L-conjugate mean L_{φ}^{*} of n variables.

In the same paper Z. Daróczy and Z. Páles investigated the problem of comparison in this class of means and they proved the following theorem.

Theorem 4. *Let $\varphi, \psi \in CM(I)$. Inequality*

$$L_{\varphi}^{*}(\boldsymbol{x}) \leq L_{\psi}^{*}(\boldsymbol{x}) \tag{2}$$

holds for all $\boldsymbol{x} \in I^n$ if and only if $\varepsilon_{\psi}\psi \circ \varphi^{-1}$ is convex, where $\varepsilon_{\psi} = 1$ if ψ is increasing and $\varepsilon_{\psi} = -1$ if ψ is decreasing.

Remark 5. Observe that the necessary and sufficient conditions stated in Theorem 4 are also necessary and sufficient conditions for the comparison of the quasi-arithmetic means generated by φ and ψ respectively.

It can be easily seen that the problem of comparison is closely related to the notion of convexity and to the one of the most important inequalities in mathematics and statistics: Jensen's inequality for convex functions Pečarić et al [4], p. 43 which is stated like this:

Let I be an interval in $\mathbb{R}$. If function $f : I \to \mathbb{R}$ is convex on I and $\boldsymbol{p}$ is a nonnegative n-tuple, then inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{3}$$

holds for all $\boldsymbol{x} \in I^n$ $(n \geq 2)$, where $P_n = \sum_{i=1}^{n} p_i \neq 0$. If f is strictly convex, then (3) is strict unless $x_1 = ... = x_n$.

If we allow negative weights in a way that we set the following conditions:

$$p_1 > 0, \quad p_i \leq 0 \quad (i = 2, ..., n), \quad P_n > 0$$

and

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in I,$$

then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right). \tag{4}$$

This inequality is known as the reversed Jensen's inequality Pečarić et al [4], p. 83.

The paper is organized as follows: In Section 2 we introduce the notion of $\boldsymbol{L}$-conjugate means of $n \geq 2$ variables which is a generalization of the notion of L-conjugate means. In Section 3 we give two Jensen's type inequalities strongly connected with weighted $\boldsymbol{L}$-conjugate means. In Section 4 we use obtained results to examine the problem of comparison of two weighted $\boldsymbol{L}$-conjugate means. In Section 5 we investigate homogenous $\boldsymbol{L}$-conjugate means and their properties.

2. WEIGHTED $\boldsymbol{L}$-CONJUGATE MEANS

In this section we extend Definition 3 of L-conjugate means in order to obtain a new definition of weighted means conjugated to a fixed m-tuple of means $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$.

Let $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$. In what follows, when we say that a pair $(\boldsymbol{p}, \boldsymbol{w})$ is admissible we mean that for all $i \in \{1, ..., n\}$ inequality

$$p_i \geq \sum_{j=1}^{m} w_j \tag{5}$$

holds.

Definition 6. Let $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on an open real interval I and let $\varphi \in CM(I)$. Let $\boldsymbol{x} \in I^n$ and let $(\boldsymbol{p}, \boldsymbol{w})$ be an admissible pair, where $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$. Weighted $\boldsymbol{L}$-conjugate mean $\boldsymbol{L}_\varphi^*$ of n-tuple $\boldsymbol{x}$ with weights $(\boldsymbol{p}, \boldsymbol{w})$ is defined as

$$\boldsymbol{L}_\varphi^*(\boldsymbol{x}; \boldsymbol{p}, \boldsymbol{w}) = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} p_i \varphi(x_i) - \sum_{j=1}^{m} w_j \varphi(L_j(\boldsymbol{x}))}{P_n - W_m}\right),$$

where

$$P_n = \sum_{i=1}^{n} p_i, \quad W_m = \sum_{j=1}^{m} w_j.$$

In this case, the function φ is called the generating function of the weighted $\boldsymbol{L}$-conjugate mean $\boldsymbol{L}_\varphi^*$ of n variables.

Remark 7. We consider only admissible pairs $(\boldsymbol{p}, \boldsymbol{w})$ because such a condition on weights is sufficient to ensure that $\boldsymbol{L}_\varphi^*$ is a mean (i.e., in order to fulfill the property (In) in Definition 1, because it is obvious that the properties (Sy) and (Co) are fulfilled without this admissibility condition on weights). We prove it here.

Let $(\boldsymbol{p},\boldsymbol{w})$ be a pair of admissible weights and let $\boldsymbol{x}\in I^{n}$ be such that $x_{k}\neq x_{l}$ for some $k,l\in\{1,...,n\}$. Suppose that the function φ is increasing and let

$$\xi=\min\{x_{1},...,x_{n}\},$$
$$\eta=\max\{x_{1},...,x_{n}\}.$$

Since $x_{1},...,x_{n}$ are not all equal, we know that $\xi\neq\eta$. We have

$$p_{1}\varphi(x_{1})+...+p_{n}\varphi(x_{n})\leq(P_{n}-p_{i})\varphi(\eta)+p_{i}\varphi(\xi)$$

and since L_{j} is for every $j\in\{1,...,m\}$ a mean on I, we obtain

$$\begin{aligned}&\sum_{i=1}^{n}p_{i}\varphi(x_{i})-\sum_{j=1}^{m}w_{j}\varphi(L_{j}(\boldsymbol{x}))\\&\leq(P_{n}-p_{i})\varphi(\eta)+p_{i}\varphi(\xi)-\sum_{j=1}^{m}w_{j}\varphi(L_{j}(\boldsymbol{x}))\\&<(P_{n}-p_{i})\varphi(\eta)+p_{i}\varphi(\xi)-\sum_{i=1}^{m}w_{j}\varphi(\xi)\\&=(P_{n}-p_{i})\varphi(\eta)+(p_{i}-W_{m})\varphi(\xi).\end{aligned}$$

Since the pair $(\boldsymbol{p},\boldsymbol{w})$ is admissible we know that $p_{i}-W_{m}\geq0$, so we have

$$\begin{aligned}&\sum_{i=1}^{n}p_{i}\varphi(x_{i})-\sum_{j=1}^{m}w_{j}\varphi(L_{j}(\boldsymbol{x}))\\&<(P_{n}-p_{i})\varphi(\eta)+(p_{i}-W_{m})\varphi(\eta)\\&=(P_{n}-W_{m})\varphi(\eta),\end{aligned}$$

and because of $P_{n}-W_{m}>0$, we obtain

$$\frac{\sum_{i=1}^{n}p_{i}\varphi(x_{i})-\sum_{j=1}^{m}w_{j}\varphi(L_{j}(\boldsymbol{x}))}{P_{n}-W_{m}}<\varphi(\eta),$$

i.e.,

$$\varphi^{-1}\left(\frac{\sum_{i=1}^{n}p_{i}\varphi(x_{i})-\sum_{j=1}^{m}w_{j}\varphi(L_{j}(\boldsymbol{x}))}{P_{n}-W_{m}}\right)<\max\{x_{1},...,x_{n}\}.$$

The other side of (In) can be proved in a similar way. If φ is decreasing we proceed analogously.

It can be proved that if $P_{n}-W_{m}>0$ holds, the admissibility condition on $(\boldsymbol{p},\boldsymbol{w})$ is also the necessary condition for the property (In).

3. JENSEN'S TYPE INEQUALITIES RELATED TO THE WEIGHTED $\boldsymbol{L}$-CONJUGATE MEANS

Next we give two Jensen's type inequalities related to the weighted $\boldsymbol{L}$-conjugate means.

Theorem 8. *Let $f : I \to \mathbb{R}$ be a function and let $\boldsymbol{M} = (M_1, \ldots, M_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on I. Inequality*

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m} \tag{6}$$

holds for all $\boldsymbol{x} \in I^n$ and all admissible pairs $(\boldsymbol{p}, \boldsymbol{w})$, where $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$, if and only if f is convex on I.

If f is concave on I, then (6) *is reversed.*

Proof. First we prove the sufficiency of the condition. Let $f : I \to \mathbb{R}$ be a convex function on I. If $\boldsymbol{x} \in I^n$ and $x_1 = ... = x_n$ then (6) clearly holds as an equality, so we can suppose that that $x_k \neq x_l$ for some $k, l \in \{1, ..., n\}$. From Introduction we know that

$$\min\{x_1, ..., x_n\} < \frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m} < \max\{x_1, ..., x_n\},$$

so the left hand side of (6) is well defined. Furthermore, since for every $j \in \{1, ..., m\}$ M_j is a mean, we know that

$$\min_{i\in\{1,\ldots,n\}} \{x_i\} < M_j(\boldsymbol{x}) < \max_{i\in\{1,\ldots,n\}} \{x_i\},$$

so for all $j \in \{1, ..., m\}$ there are some $\boldsymbol{\lambda}^{(j)} \in [0,1]^n$ such that $\sum_{i=1}^{n} \lambda_i^{(j)} = 1$ and

$$M_j(\boldsymbol{x}) = \sum_{i=1}^{n} \lambda_i^{(j)} x_i.$$

Since f is convex on I, for every $j \in \{1, ..., m\}$ the inequality

$$f(M_j(\boldsymbol{x})) = f\left(\sum_{i=1}^{n} \lambda_i^{(j)} x_i\right) \leq \sum_{i=1}^{n} \lambda_i^{(j)} f(x_i)$$

holds. Now we can write

$$\begin{aligned} f&\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \\ &= f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j \sum_{i=1}^{n} \lambda_i^{(j)} x_i}{P_n - W_m}\right) \\ &= f\left(\frac{1}{P_n - W_m} \sum_{i=1}^{n} \left(p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)}\right) x_i\right). \end{aligned}$$

We can easily check that

$$\frac{1}{P_n - W_m}\sum_{i=1}^{n}\left(p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)}\right) = 1,$$

and since for all $i \in \{1, ..., n\}$

$$p_i \geq W_m \geq \sum_{j=1}^{m} w_j \lambda_i^{(j)},$$

we also have

$$\frac{1}{P_n - W_m}\left(p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)}\right) \geq 0, \quad i = 1, ..., n.$$

Now we use the convexity of f to obtain from (3) the following:

$$\begin{aligned}
f&\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \\
&\leq \frac{1}{P_n - W_m}\sum_{i=1}^{n}\left(p_i - \sum_{j=1}^{m} w_j \lambda_i^{(j)}\right) f(x_i) \\
&= \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} w_j \lambda_i^{(j)} f(x_i)}{P_n - p_0} \\
&= \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j \sum_{i=1}^{n} \lambda_i^{(j)} f(x_i)}{P_n - p_0} \\
&\leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m}.
\end{aligned}$$

It can be easily seen that if f is concave the inequality (6) is reversed. To prove the necessity of the condition we simply take the limits $w_j \to 0$ for all $j \in \{1, ..., m\}$ in (6) to obtain that the inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i),$$

holds for all $\boldsymbol{x} \in I^n$ and all $\boldsymbol{p} \in \mathbb{R}_+^n$, that is f is convex. Analogously, if reversed (6) holds we obtain that f is concave. $\square$

The following theorem gives a converse inequality of Jensen's type related to the weighted $\boldsymbol{L}$-conjugate means.

Theorem 9. *Let $f : I \to \mathbb{R}$ be a function and let $\boldsymbol{M} = (M_1, \ldots, M_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on I. Let $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$ be such that*

$P_n - W_m > 0$. *If f is convex on I then inequality*

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \geq \frac{P_n f(\overline{x}) - W_m f\left(\overline{M}\right)}{P_n - W_m}$$

$$\geq \frac{P_n f(\overline{x}) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m} \tag{7}$$

holds for all $\boldsymbol{x} \in I^n$ such that

$$\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m} \in I, \tag{8}$$

where

$$\overline{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \quad \overline{M} = \frac{1}{W_m} \sum_{j=1}^{m} w_j M_j(\boldsymbol{x}).$$

If f is concave on I, then (7) is reversed.

Proof. From (4) and then (3) we immediately obtain

$$f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - W_m\left(\frac{1}{W_m}\sum_{j=1}^{m} w_j M_j(\boldsymbol{x})\right)}{P_n - W_m}\right)$$

$$\geq \frac{P_n f(\overline{x}) - W_m f\left(\overline{M}\right)}{P_n - W_m} \geq \frac{P_n f(\overline{x}) - W_m \frac{1}{W_m}\sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m}$$

$$\geq \frac{P_n f(\overline{x}) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m}.$$

□

Remark 10. From Introduction we know that for any admissible pair $(\boldsymbol{p}, \boldsymbol{w})$ and all $\boldsymbol{x} \in I^n$ the condition (8) holds, so in Theorem 9 this condition, as well as the condition $P_n - W_m > 0$, can be omitted if $(\boldsymbol{p}, \boldsymbol{w})$ is an admissible pair. In this case the inequality (6) from Theorem 8 can be extended in the following way

$$\frac{P_n f(\overline{x}) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m} \leq \frac{P_n f(\overline{x}) - W_m f\left(\overline{M}\right)}{P_n - W_m}$$

$$\leq f\left(\frac{\sum_{i=1}^{n} p_i x_i - \sum_{j=1}^{m} w_j M_j(\boldsymbol{x})}{P_n - W_m}\right)$$

$$\leq \frac{\sum_{i=1}^{n} p_i f(x_i) - \sum_{j=1}^{m} w_j f(M_j(\boldsymbol{x}))}{P_n - W_m}.$$

The following theorem gives AG-type inequalities related to this class of means.

Theorem 11. *Let $\boldsymbol{M} = (M_1, \ldots, M_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on $I \subset \mathbb{R}_+$ and let $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$ be such that $P_n - W_m > 0$. Then for all $\boldsymbol{x} \in I^n$ such that*

$$\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j M_j(\boldsymbol{x})}{P_n - W_m} \in \mathbb{R}_+ \tag{9}$$

inequality

$$\begin{aligned}
&\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j M_j(\boldsymbol{x})}{P_n - W_m} \\
&\leq \left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^{P_n}\left(\frac{1}{W_m}\sum_{j=1}^m w_j M_j(\boldsymbol{x})\right)^{-W_m}\right]^{\frac{1}{P_n - W_m}}
\end{aligned} \tag{10}$$

holds.

Proof. Since function $\ln : \mathbb{R}_+ \to \mathbb{R}$ is concave and strictly increasing on $\mathbb{R}_+$, from (4) we obtain that for any $\boldsymbol{p} \in \mathbb{R}_+^n$ and $\boldsymbol{w} \in \mathbb{R}_+^m$ such that $P_n - W_m > 0$ and all $\boldsymbol{x} \in I^n$ such that (9) is fulfilled the following inequality holds

$$\begin{aligned}
&\ln\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \\
\leq\ &\frac{P_n \ln\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) - W_m \ln\left(\frac{1}{W_m}\sum_{j=1}^m w_j M_j(\boldsymbol{x})\right)}{P_n - W_m} \\
=\ &\ln\left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)^{P_n}\left(\frac{1}{W_m}\sum_{j=1}^m w_j M_j(\boldsymbol{x})\right)^{-W_m}\right]^{\frac{1}{P_n - W_m}},
\end{aligned}$$

and from this (10) immediately follows. □

Remark 12. Similarly as in the previous remark, the conditions (9) and $P_n - W_m > 0$ can be replaced with the condition that $(\boldsymbol{p}, \boldsymbol{w})$ is an admissible pair. In this case, for any positive function f convex on $I \subset \mathbb{R}_+$ the inequality (6) can be extended as

$$\begin{aligned}
&f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \\
&\leq \frac{\sum_{i=1}^n p_i f(x_i) - \sum_{j=1}^m w_j f(M_j(\boldsymbol{x}))}{P_n - W_m} \\
&\leq \left[\left(\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)\right)^{P_n}\left(\frac{1}{W_m}\sum_{j=1}^m w_j f(M_j(\boldsymbol{x}))\right)^{-W_m}\right]^{\frac{1}{P_n - W_m}},
\end{aligned}$$

if the middle term is a positive number, which will hold true whenever f is a strictly monotonic function.

4. COMPARISON

Let $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on an open real interval J, let $(\boldsymbol{p}, \boldsymbol{w})$ be an admissible pair and let φ and ψ be two given functions from $CM(J)$. We want to determine what conditions are sufficient if the inequality

$$\boldsymbol{L}^*_\varphi(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \leq \boldsymbol{L}^*_\psi(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w})$$

is to hold for all $\boldsymbol{\xi} \in J^n$. These conditions are given in the following theorem.

Theorem 13. *Let $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on an open real interval J, $\varphi,\psi \in CM(J)$ and let $\varepsilon_\psi = 1$ if ψ is increasing and $\varepsilon_\psi = -1$ if ψ is decreasing. Inequality*

$$\boldsymbol{L}^*_\varphi(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \leq \boldsymbol{L}^*_\psi(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \tag{11}$$

holds for all $\boldsymbol{\xi} \in J^n$ and all admissible pairs $(\boldsymbol{p}, \boldsymbol{w})$, where $\boldsymbol{p} \in \mathbb{R}^n_+$ and $\boldsymbol{w} \in \mathbb{R}^m_+$, if and only if $\varepsilon_\psi \psi \circ \varphi^{-1}$ is convex.

The reversed (11) *holds for all $\boldsymbol{\xi} \in J^n$ and all admissible pairs $(\boldsymbol{p}, \boldsymbol{w})$ if and only if $\varepsilon_\psi \psi \circ \varphi^{-1}$ is concave.*

Proof. We shall prove Theorem 13 only for $\psi \circ \varphi^{-1}$ convex and ψ increasing, since the the remaining cases are similar. Let $\boldsymbol{\xi} \in J^n$. We denote

$$\varphi(\xi_i) = x_i \in \varphi(J) = I, \quad i \in \{1, ..., n\},$$
$$\varphi\left(L_j\left(\varphi^{-1}(\boldsymbol{x})\right)\right) = M_j(\boldsymbol{x}), \quad j \in \{1, ..., m\},$$

and

$$\psi \circ \varphi^{-1} = f \in CM(I).$$

It is easy to see that for all $j \in \{1, ..., m\}$ the functions M_j are means of n variables on I. From Theorem 8 we know that the inequality

$$f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j M_j(\boldsymbol{x})}{P_n - W_m}\right) \leq \frac{\sum_{i=1}^n p_i f(x_i) - \sum_{j=1}^m w_j f(M_j(\boldsymbol{x}))}{P_n - W_m}$$

holds for all $\boldsymbol{x} \in I^n$ and all admissible pairs $(\boldsymbol{p}, \boldsymbol{w})$ if and only if f is convex. After the adequate substitutions and using the fact that ψ is strictly increasing on J, the assertions immediately follow. $\square$

Corollary 14. *Let $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$, be an m-tuple of fixed means of $n \geq 2$ variables on an open real interval J and let $\varphi,\psi \in CM(J)$. The equality*

$$\boldsymbol{L}^*_\varphi(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{L}^*_\psi(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \tag{12}$$

holds for all $\boldsymbol{\xi} \in J^n$ and for all admissible pairs $(\boldsymbol{p}, \boldsymbol{w})$, where $\boldsymbol{p} \in \mathbb{R}^n_+$ and $\boldsymbol{w} \in \mathbb{R}^m_+$, if and only if there exist real constants $\alpha \neq 0$ and β such that

$$\psi(x) = \alpha\varphi(x) + \beta \tag{13}$$

for all $x \in J$.

Proof. It can be easily checked that if (13) holds we have (12). Conversely, suppose that (12) holds. In that case we have $\boldsymbol{L}_{\varphi}^{*} \leq \boldsymbol{L}_{\psi}^{*}$ and $\boldsymbol{L}_{\psi}^{*} \leq \boldsymbol{L}_{\varphi}^{*}$, so by Theorem 13 the function $f =: \psi \circ \varphi^{-1}$ is both convex and concave on $\varphi(I) = J$. This means that for all $u, v \in J$ and any $\lambda \in (0,1)$

$$f(\lambda u + (1-\lambda) v) = \lambda f(u) + (1-\lambda) f(v),$$

i.e., $f(u) = \alpha u + \beta$, $u \in J$, for some constants $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. Now, if we let $u = \varphi(x)$, $x \in I$, we obtain (13). □

As usual, when we say that two functions φ and ψ are equivalent ($\varphi \sim \psi$) we mean that the equality (13) holds for all $x \in J$.

5. HOMOGENOUS WEIGHTED $\boldsymbol{L}$-CONJUGATE MEANS

In this section we want to determine for which $\varphi \in CM(J)$ an $\boldsymbol{L}$-conjugate mean $\boldsymbol{L}_{\varphi}^{*}$ is homogenous. The answer is given in the following theorem.

Theorem 15. *Let J be an open interval in $\mathbb{R}_{+}$ and $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$, an m-tuple of fixed homogenous means of $n \geq 2$ variables on J. An $\boldsymbol{L}$-conjugate mean $\boldsymbol{L}_{\varphi}^{*} : J^n \to J$ of $n \geq 2$ variables is homogenous if and only if for some $r \in \mathbb{R}$*

$$\varphi \sim l_r$$

for all $x \in J$, where

$$l_r(x) = \begin{cases} x^r, & r \neq 0, \\ \log x, & r = 0. \end{cases}$$

Proof. If for some $r \in \mathbb{R}$ we have that

$$\varphi \sim l_r$$

for all $x \in J$, then the corresponding $\boldsymbol{L}_{\varphi}^{*}$ mean is obviously homogenous. Suppose now that

$$\boldsymbol{L}_{\varphi}^{*}(t\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) = t\boldsymbol{L}_{\varphi}^{*}(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \tag{14}$$

for all $\boldsymbol{\xi} \in J^n$ and all $t \in \mathbb{R}_{+}$. We define the function $\psi_t : J \to \mathbb{R}_{+}$ with

$$\psi_t(x) = \varphi(tx)$$

for all $x \in J$. It can be easily seen that $\psi_t \in CM(J)$. By (14) we have

$$\begin{aligned} \boldsymbol{L}_{\psi_t}(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) &= \frac{1}{t}\varphi^{-1}\left(\frac{\sum_{i=1}^{n} p_i \varphi(t\xi_i) - \sum_{j=1}^{m} w_j \varphi(L_j(t\boldsymbol{\xi}))}{P_n - W_m}\right) \\ &= \frac{1}{t}\boldsymbol{L}_{\varphi}^{*}(t\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) = \frac{1}{t}t\boldsymbol{L}_{\varphi}^{*}(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{L}_{\varphi}^{*}(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}). \end{aligned}$$

By Corollary 14 there exist two real numbers $\alpha(t) \neq 0$ and $\beta(t)$ such that

$$\psi_t(x) = \varphi(tx) = \alpha(t)\varphi(x) + \beta(t) \tag{15}$$

for all $x \in J$ and all $t \in \mathbb{R}_+$. The functional equation (15) and its solutions are known (see for example Hardy et al [3], p. 69). □

Theorem 16. *Let J be an open interval in $\mathbb{R}_+$ and $\boldsymbol{L} = (L_1, \ldots, L_m)$, $m \geq 1$, an m-tuple of fixed homogenous means of $n \geq 2$ variables on J. $\boldsymbol{L}$-conjugate mean $\boldsymbol{L}_\varphi^* : J^n \to J$ of $n \geq 2$ variables is homogenous if and only if there exists a real number r such that*

$$\boldsymbol{L}_\varphi^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) =: \boldsymbol{L}_r^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w})$$

$$= \begin{cases} \left(\frac{\sum_{i=1}^n p_i \xi_i^r - \sum_{j=1}^m w_j L_j(\boldsymbol{\xi})^r}{P_n - W_m}\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\frac{\xi_1^{p_1} \ldots \xi_n^{p_n}}{L_1(\boldsymbol{\xi})^{w_1} \ldots L_m(\boldsymbol{\xi})^{w_m}}\right)^{\frac{1}{P_n - W_m}}, & r = 0, \end{cases}$$

for all $\boldsymbol{\xi} \in J^n$. This family of means of n variables is increasing in r, that is, if $r_1 \leq r_2$ then

$$\boldsymbol{L}_{r_1}^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \leq \boldsymbol{L}_{r_2}^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \tag{16}$$

for all $\boldsymbol{\xi} \in J^n$.

Proof. Directly from Theorem 15 we obtain that

$$\boldsymbol{L}_\varphi^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{L}_r^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w})$$

for all $\boldsymbol{\xi} \in J^n$ if and only if $\boldsymbol{L}_\varphi^* : J^n \to J$ is homogenous. Let now $r_1 \leq r_2$. We will consider the case $r_1, r_2 \geq 0$. We define functions

$$\varphi(x) = \begin{cases} x^{r_1}, & r_1 \neq 0, \\ \log x, & r_1 = 0, \end{cases}$$

and

$$\psi(x) = \begin{cases} x^{r_2}, & r_2 \neq 0, \\ \log x, & r_2 = 0, \end{cases}$$

for all $x \in J$. If $r_1 = r_2 = 0$ we have $\boldsymbol{L}_{r_1}^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) = \boldsymbol{L}_{r_2}^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w})$ for all $\boldsymbol{\xi} \in J^n$, so (16) is true. If $r_1, r_2 > 0$, then the function ψ is strictly increasing on J and the function $\psi \circ \varphi^{-1}$ defined with

$$\left(\psi \circ \varphi^{-1}\right)(x) = x^{\frac{r_2}{r_1}}$$

for all $x \in J$, is convex on J. Using Theorem 13 we obtain

$$\boldsymbol{L}_\varphi^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \leq \boldsymbol{L}_\psi^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}),$$

i.e.,

$$\boldsymbol{L}_{r_1}^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \leq \boldsymbol{L}_{r_2}^*(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w})$$

for all $\boldsymbol{\xi} \in J^n$. Finally, if $r_1 = 0$ and $r_2 > 0$ then ψ is strictly increasing on J and the function $\psi \circ \varphi^{-1}$ defined with

$$\left(\psi \circ \varphi^{-1}\right)(x) = e^{r_2 x}$$

for all $x \in J$, is convex on J. Again, using Theorem 13 we obtain

$$\boldsymbol{L}^*_{r_1}(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w}) \leq \boldsymbol{L}^*_{r_2}(\boldsymbol{\xi}; \boldsymbol{p}, \boldsymbol{w})$$

for all $\boldsymbol{\xi} \in J^n$, i.e. (16) holds for all nonnegative r_1 and r_2. In other cases we proceed similarly. $\square$

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