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# METHOD OF LOCAL VARIATIONS FOR MIXED NONLOCAL FUNCTIONALS

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**ABSTRACT:** The nonlocal functional is an integral with the integrand depending on the unknown function at different values of arguments. These types of functionals have different applications in physics, engineering and sciences. The Euler type conditions that arise as necessary conditions of extrema of nonlocal functionals are the boundary value problems for functional differential equations. The analytical methods of solving of this type boundary value problems are rather difficult. Therefore a great role play the direct approximate method of solving the problem of extremum for nonlocal functionals. Here we apply the local variation method for approximate solution of variational problems for mixed nonlocal functionals.

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## 1. INTRODUCTION

The nonlocal functional is an integral with the integrand depending on the unknown function at different values of the argument. These types of functionals have different applications in physics, engineering and sciences. The Euler type equations that arise as necessary conditions of extrema of nonlocal functionals are the functional differential equations.

The mixed nonlocal functional is an integral with the integrand depending on the unknown function of two arguments to one of which is applied the operator of differentiation and to the second one is applied the shift operator. The Euler type equations for this type of functionals are the mixed functional differential equations.

Applications of mixed type equations and mixed nonlocal functionals to physics and mechanics were mentioned already in works of J. Bernoulli, L. Euler, J.L. Laplace and other mathematicians of 18-th and 19-th centuries. For description of some of these works and references see Pinney [10].

Many forms of this type equations and functionals appear now in papers dedicated to investigations of different problems of epidemiology, ecology, biology and physics (see Thieme [13], Herod [5], Stewart [12], Hadeler [4], Buerger [3], Britton [2]).

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The survey of the theory of mixed functional differential equations and of mixed nonlocal functionals see in Kamenskii [7].

Here we consider the problem of extremum of the functional

$$J(y) = \int_{t_0}^{t_1} dt \int_{s_0}^{s_1} F(t, s, u(t, s-r), u(t, s), u(t, s+r),$$
$$u'(t, s-r), u'(t, s), u'(t, s+r))\, ds\,. \tag{1}$$

Here $r > 0, s_1 - s_0 > 2r, \quad u, F \in \mathbb{R}^1, \quad u'(t, s) = \frac{\partial u}{\partial t}$. Denote

$$E_0 = \{(t, s) | t_0 \leq t \leq t_1,\ s_0 + r \leq s_0 - r\},$$

$$E_1 = \{(t, s) | t_0 \leq t_1,\ s_1 - r \leq s_1 + r, \}$$

$$G_0 = \{(t, s) | t = t_0, \quad s_0 + r \leq s_0 - r\},$$

$$G_1 = \{(t, s) | t = t_1, \quad s_0 + r \leq s_1 - r\},$$

$$Q = \{(t, s) | t_0 \leq t < t_1,\ s_0 + r < s < s_1 - r\}.$$

On $E_0$ and $E_1$ there are given the boundary function $\varphi$ and $\psi$ and on $G_0$ and $G_1$ there are given the boundary functions $\mu_0(s)$ and $\mu_1(s)$. Functional (1) is considered under the boundary conditions

$$\begin{aligned} u(t, s) &= \varphi(t, s) \quad \text{at} \quad (t, s) \in E_0, \\ u(t, s) &= \psi(t, s) \quad \text{at} \quad (t, s) \in E_1, \\ u(t_0, s) &= \mu_0(s) \quad \text{at} \quad s \in \mu_0, \\ u(t_1, s) &= \mu_1(s) \quad \text{at} \quad s \in \mu_1. \end{aligned} \tag{2}$$

The unknown function supposed to be twice continuously differentiable in $t$, piecewise continuous in $s$ and it must satisfy the restrictions

$$u^-(t, s) \leq u(t, s) \leq u^+(t, s) \quad \text{at} \quad (t, s) \in Q, \tag{3}$$

where $u^-(t, s)$ and $u^+(t, s)$ are some given functions and $u^-(t, s) = u^+(t, s) = \varphi(t, s)$ at $(t, s) \in E_0$, $u^-(t, s) = u^+(t, s) = \psi(t, s)$ at $(t, s) \in E_1, u^-(t_0, s) = u^+(t_0, s) = \mu_0(s)$ at $s \in G_0$, $u^-(t_1, s) = u^+(t_1, s) = \mu_1(s)$ at $s \in G_1$.

If on some set there are no restrictions, then we put on this set $u^-(t, s) = -\infty$ and $u^+(t, s) = \infty$. Conditions (3) include conditions (2). Therefore we shall consider problem (1), (3). Define the function

$$\begin{aligned}
&\Phi(t, s-r, s, s+r, u(t, s-2r), u(t, s-r), u(t, s), u(t, s+r), \\
&u(t, s+2r), u'(t, s-2r), u'(t, s-r), u'(t, s), u'(t, s+r), u'(t, s+2r)) \\
&=: F(t, s-r, u(t, s-2r), u(t, s-r), u(t, s), u'(t, s-2r), \\
&u'(t, s-r), u'(t, s)) + F(t, s, u(t, s-r), u(t, s), u(t, s+r), \\
&u'(t, s-r), u'(t, s), u'(t, s+r)) F(t, s+r, u(t, s), u(t, s+r), \\
&u(t, s+2r), u'(t, s), u'(t, s+r), u'(ts+2r)).
\end{aligned} \tag{4}$$

If

$$\Phi = \Phi(t, s-r, s, s+r, v_1, v_2, \ldots, v_{10}),$$

then

$$\Phi_{u(t,s)} = \frac{\partial \Phi}{\partial v_3}$$

at

$$\begin{aligned}
&v_1 = u(t, s-2r), \quad v_2 = u(t, s-r), \quad v_3 = u(t, s), \\
&v_4 = u(t, s+r), \quad v_5 = u(t, s+2r), \\
&v_6 = u'(t, s-2r), \quad v_7 = u'(t, s-r), \quad v_8 = u'(t, s), \\
&v_9 = u'(t, s+r), \quad v_{10} = u'(t, s+2r).
\end{aligned} \tag{5}$$

In the same way

$$\Phi_{u'(t,s)} = \frac{\partial \Phi}{\partial v_8}$$

at the values (5).

The following theorem was proved in Kamenskii [6].

**Theorem 1.** *If functional* (1) *under conditions* (2) *attains an extremum on the function* $u^*$*, then* $u^*$ *satisfies on* $Q$ *the equation*

$$\Phi_{u(t,s)} - \frac{\Phi_{u'(t,s)}}{dt} = 0. \tag{6}$$

The analytical methods of solving boundary value problem (6), (2) are rather difficult. Therefore, the approximate methods of solving this problem and the direct approximate method of solving the problem of extremum for functionals of the type (1) play a great role.

The finite differences method for solving of linear mixed functional differential equations was worked out in Kopylov [9]. The projective methods for solution of variational problems for mixed nonlocal functionals was developed in Kamenskii and Varfolomejev [8].

Here we apply the local variation method for approximate solution of variational problems for mixed nonlocal functionals (see also Ardova and Kamenskii [1]).

Divide the interval $(t_0, t_1)$ on $m$ equal parts of the length $\Delta t$ and denote

$$i_0 = t_0, i_1 = i_0 + \Delta t, \quad i_k = i_{k-1} + \Delta t, \ldots, i_m = i_0 + m\Delta t = t_1.$$

Divide the interval $(s_0 + r, s_1 - r)$ on $n$ equal parts of the length $\Delta s$ and denote

$$\begin{aligned} j_0 &= s_0 + r, \\ j_1 &= j_0 + \Delta s, \\ &\vdots \\ j_k &= j_{k-1} + \Delta s, \\ &\vdots \\ j_n &= j_0 + n\Delta s = s_1 - r. \end{aligned}$$

Let $r = pm$ and for sufficient small $\Delta s$ we can suppose that $p$ is an integer. We denote then

$$j_{-1} = j_0 - \Delta s, \ldots, j_{-k} = j_{-k+1} - \Delta s, \ldots, j_{-p} = j_0 - n\Delta s,$$

$$j_{n+1} = j_n + \Delta s, \ldots j_k = j_{k-1} + \Delta s, \ldots, j_{n+p} = j_n + p\Delta s.$$

The approximate solution of problem (1), (3) we seek as the function $u(t_i, s_j)$ defined on the above described net, which we denote by $S$. We must find $u(t_i, s_j)$ for $i = 1, \ldots, i = m-1$ and $j = 1, \ldots, j = n-1$. The values of $u(t_i, s_j)$ for $i = 0,\ i = m, j = -p, \ldots, 0,\ j = n, \ldots, n+p$ are known from boundary conditions (2).

The approximate value of (1) is

$$\begin{aligned} J(u) \approx \sum_{i=0}^{m}\sum_{j=0}^{n} & F(t_i, s_j, u(t_i, s_{j-p}), u(t_i, s_j), u(t_i, u_{j+p}), \\ & \frac{u(t_i, s_{j-p+1}) - u(t_i, s_{j-p})}{\Delta s}, \\ & \frac{u(t_i, s_{j+1}) - u(t_i, s_j)}{\Delta s}, \frac{u(t_i, s_j + p + 1) - u(t_i, s_j + p)}{\Delta s})\Delta t \Delta s. \end{aligned} \tag{7}$$

To begin the process we must assign the arbitrary chosen function $\tilde{u}(t_i, s_j)$ on $S$.

Fix the numbers $i$ and $j$ and to $u(t_i, s_j)$ add a number $q$. Denote

$$J_{i,j}(\tilde{u}, q) \tag{8}$$

the value of (7) when instead of $\tilde{u}(t_i, s_j)$ is substituted $\tilde{u}(t_i, s_j) + q$.

Denote

$$\begin{aligned}&F_{i,j}(\tilde{u},q)=F(t_i,s_j,\tilde{u}(t_i,s_{j-p}),\tilde{u}(t_i,s_j)+q,\tilde{u}(t_i,s_{j+p}),\\&\frac{\tilde{u}(t_i,s_{j-p+1})-\tilde{u}(t_i,s_{j-p})}{\Delta s},\\&\frac{\tilde{u}(t_i,s_{j+1})-\tilde{u}(t_i,s_j)-q}{\Delta s},\frac{\tilde{u}(t_i,s_{j+p+1})-\tilde{u}(t_i,s_{j+p})}{\Delta s}),\end{aligned}\tag{9}$$

$$\begin{aligned}&F_{i,j-1}(\tilde{u},q)=F(t_i,s_{j-1},\tilde{u}(t_i,s_{j-p-1}),\tilde{u}(t_i,s_{j-1}),\tilde{u}(t_i,s_{j+p-1}),\\&\frac{\tilde{u}(t_i,s_{j-p+1})-\tilde{u}(t_i,s_{j-p})}{\Delta s},\\&\frac{\tilde{u}(t_i,s_j)+q-\tilde{u}(t_i,s_{j-1})}{\Delta s},\frac{\tilde{u}(t_i,s_{j+p})-\tilde{u}(t_i,s_{j+p-1})}{\Delta s}),\end{aligned}\tag{10}$$

$$\begin{aligned}&F_{i,j-p}(\tilde{u},q)=F(t_i,s_{j-p},\tilde{u}(t_i,s_{j-2p}),\tilde{u}(t_i,s_{j-p}),\tilde{u}(t_i,s_j)+q,\\&\frac{\tilde{u}(t_i,s_{j-2p+1})-\tilde{u}(t_i,s_{j-2p})}{\Delta s},\\&\frac{\tilde{u}(t_i,s_{j-2p+1})-\tilde{u}(t_i,s_{j-2p})}{\Delta s},\frac{\tilde{u}(t_i,s_{j+1})-\tilde{u}(t_i,s_j)-q}{\Delta s}),\end{aligned}\tag{11}$$

$$\begin{aligned}&F_{i,j-p-1}(\tilde{u},q)=F(t_i,s_{j-p-1},\tilde{u}(t_i,s_{j-2p-1}),\tilde{u}(t_i,s_{j-p-1}),\tilde{u}(t_i,s_{j-1}),\\&\frac{\tilde{u}(t_i,s_{j-2p})-\tilde{u}(t_i,s_{j-2p-1})}{\Delta s},\\&\frac{\tilde{u}(t_i,s_{j-2p})-\tilde{u}(t_i,s_{j-2p-1})}{\Delta s},\frac{\tilde{u}(t_i,s_j)-\tilde{u}(t_i,s_{j-1})}{\Delta s}),\end{aligned}\tag{12}$$

$$\begin{aligned}&F_{i,j+p+1}(\tilde{u},q)=F(t_i,s_{j+p+1},\tilde{u}(t_i,s_{j+1}),\tilde{u}(t_i,s_{j+p+1}),\tilde{u}(t_i,s_{j+2p+1}),\\&\frac{\tilde{u}(t_i,s_{j+1})-\tilde{u}(t_i,s_j)-q}{\Delta s},\\&\frac{\tilde{u}(t_i,s_{j+p+1})-\tilde{u}(t_i,s_{j+p})}{\Delta s},\frac{\tilde{u}(t_i,s_{j+2p+1})-\tilde{u}(t_i,s_{j+2p})}{\Delta s}),\end{aligned}\tag{13}$$

$$\begin{aligned}&F_{i,j+p}(\tilde{u},q)=F(t_i,s_{j+p},\tilde{u}(t_i,s_j)+q,\tilde{u}(t_i,s_{j+p}),\tilde{u}(t_i,s_{j+2p}),\\&\frac{\tilde{u}(t_i,s_j)+q-\tilde{u}(t_i,s_{j-1})}{\Delta s},\\&\frac{\tilde{u}(t_i,s_{j+p})-\tilde{u}(t_i,s_{j+p-1})}{\Delta s},\frac{\tilde{u}(t_i,s_{j+2p})-\tilde{u}(t_i,s_{j+2p-1})}{\Delta s}).\end{aligned}\tag{14}$$

Denote

$$\Phi^{(j)}(q)=(11)+(9)+(13)$$

and

$$\Phi^{(j-1)}(q)=(12)+(10)+(14).$$

Then

$$\Delta J_{ij}(\tilde{u},q) = J_{i,j}(\tilde{u},q) - J_{i,j}(\tilde{u},0) = \Phi^{(j)}(q) - \Phi^{(j)}(0) + \Phi^{(j-1)}(q) - \Phi^{(j-1)}(0). \tag{15}$$

The method of local variations is based on formula (15) and it acts as follows. We take the arbitrary chosen $\tilde{u}$ and make successive local variations at the points $t_i, s_j$ successively through all set $S$. If $\Delta J_{ij}(\tilde{u},q) > 0$ at the point $t_i, s_j$, then we substitute $\tilde{u}(t_i, s_j) + q$ for $\tilde{u}(t_i, s_j)$ in $\tilde{u}$ and check condition (3). In case when this condition is satisfied, we obtain the next approximation to the solution. We continue this process several times till we get no new diminution of $\Delta J_{ij}(\tilde{u},q)$. Then we can take $q/2$ instead of $q$ or $\Delta s/2$ instead of $\Delta s$ and continue computations.

Suppose now that the successive approximations converge to some function and problem (1), (3) has a unique solution. We show now that the approximate solutions obtained by the method of local variations at a fixed $\Delta s, \Delta t$ satisfy the finite differences equation, which approximate the generalized Euler equation for considered problem. In Kopylov [9] there are proved sufficient conditions for convergence of solutions of the finite differences equations to the generalized solution of the Euler equation for quadratic mixed nonlocal functionals. It follows that in this case the approximate solutions received by the method of local variations converge to solutions of the generalized Euler equation.

Take the linear part of the Taylor expansion of (15) and receive for a fixed $q$

$$q\left[\frac{1}{2}(\Phi^{i,j-1}_{u(t,s)}(q) + \Phi^{i,j}_{u(t,s)}(q) - \frac{1}{\Delta t}(\Phi^{i,j}_{u'(t,s)}(q) - \Phi^{i,j-1}_{u'(t,s)}(q)\right] + O\left(q^2, (\frac{q}{\Delta t})^2\right) \geq 0. \tag{16}$$

It follows from (16) that if $q \to 0$, then

$$\frac{1}{2}(\Phi^{i,j-1}_{u(t,s)} + \Phi^{i,j}_{u(t,s)}) - \frac{1}{\Delta t}(\Phi^{i,j}_{u'(t,s)} - \Phi^{i,j-1}_{u'(t,s)}) = 0. \tag{17}$$

This equation is the finite differences equation analog to the generalized Euler equation (6).

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