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OSCILLATION OF A CLASS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT DEPENDING ON THE UNKNOWN FUNCTION

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ABSTRACT: In the present paper the equivalence of the oscillation of the equations

$$[x(t) - x(t-\tau)]^{(n)} + q(t)x(g(t)) = 0 \quad \text{and} \quad x^{(n+1)}(t) + \frac{q(t)}{\tau}x(t) = 0$$

is established, where $q(t) \geq 0$, $n \geq 1$ is an odd integer, $\tau > 0$ and $t - \sigma \leq g(t) \leq t + \sigma$, $t \geq T$ for some $\sigma > 0$ and $T \geq 0$.

As a consequence some new oscillation criteria for the equation

$$[x(t) - x(t-\tau)]^{(n)} + q(t)x(\Delta(t, x(t))) = 0\,,$$

are obtained, where $\Delta(t,x) \geq t - \sigma$, $t \geq T$ for some $\sigma > 0$ and $T \geq 0$.

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1. INTRODUCTION

Consider the neutral differential equations with deviating arguments

$$[x(t) - x(t-\tau)]^{(n)} + q(t)x(g(t)) = 0\,, \qquad t \in J\,, \tag{1}$$

$$[x(t) - x(t-\tau)]^{(n)} + q(t)x(\Delta(t, x(t))) = 0\,, \qquad t \in J\,, \tag{2}$$

and the ordinary differential equation

$$x^{(n+1)}(t) + \frac{q(t)}{\tau}x(t) = 0\,, \qquad t \in J\,, \tag{3}$$

where $\tau > 0$, $n \geq 1$ is an odd integer and $q(t) \geq 0$ for $t \in J = [\alpha, +\infty) \subseteq [0, +\infty) = \mathbb{R}_+$.

As is customary, a solution of equation (1) (or (2)) is said to be *proper*, if it is defined on some interval $[T_x, +\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for each $T \geq T_x$. A proper solution of equation (1) is said to be *oscillatory* if it is neither eventually positive nor

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eventually negative. If every proper solution of equation (1) is oscillatory, equation (1) itself is said to be oscillatory; otherwise equation (1) is said to be nonoscillatory.

It is proved in the main Theorem 1 that equation (1) is oscillatory if and only if equation (3) is oscillatory. As a consequence of Theorem 1 some new oscillation criteria for equation (2) are found.

The obtained results extend the results of B.G. Zhang and Bo Yang [5] who consider equation (1) in the case $g(t) = t - \sigma$ ($\sigma \in \mathbb{R}$).

2. PRELIMINARY NOTES

Introduce the following conditions:

H1. $q \in C(J, \mathbb{R}_+)$ and $\sup\{q(t) : t \geq T\} > 0$ for each $T \in J$.

H2. $g \in C(J, \mathbb{R})$ and there exist constants $T \in J$ and $\sigma > 0$ such that $t - \sigma \leq g(t) \leq t + \sigma$, $t \geq T$.

H3. $\Delta \in C(J \times \mathbb{R}, \mathbb{R})$ and there exist constants $T \in J$ and $\sigma > 0$ such that $\Delta(t, x) \geq t - \sigma$, $t \geq T$, $x \in \mathbb{R}$.

H4. $\Delta \in C(J \times \mathbb{R}, \mathbb{R})$ and there exist constants $T \in J$ and $\sigma > 0$ such that $t - \sigma \leq \Delta(t, x) \leq t + \sigma$, $t \geq T$, $x \in \mathbb{R}$.

We need the following lemmas.

Lemma 1. *Let $x(t)$ be an n times differentialble function on J of constant sign, $x^{(n)}(t)$ be of constant sign and not identically zero in any interval $[t_*, +\infty) \subseteq J$.*

Then there exist a $t_k \geq t_*$ and an integer k, $0 \leq k < n$ with $n + k$ even for $x(t)x^{(n)}(t)$ nonnegative and $n + k$ odd for $x(t)x^{(n)}(t)$ nonpositive such that for every $t \geq t_k$

$$x(t)x^{(i)}(t) > 0\,, \qquad i = 0, 1, \ldots, k\,,$$
$$(-1)^{k+i} x(t)x^{(i)}(t) > 0\,, \qquad i = k, k+1, \ldots, n-1\,.$$

The proof of Lemma 1 is given in Kiguradze [2] and [4], Lemma 5.2.1 and Lemma 5.2.2.

Lemma 2. *Assume that conditions H1 and H3 hold, $\tau \in (0, +\infty)$, $n \geq 1$ is an odd integer, $p \in C(J, \mathbb{R}_+)$ and $0 \leq p(t) \leq 1$, $t \in J$. Let $x(t)$ be an eventually positive solution of the inequality*

$$[x(t) - p(t)x(t-\tau)]^{(n)} + q(t)x(\Delta(t, x(t))) \leq 0 \tag{4}$$

and set

$$y(t) = x(t) - p(t)x(t-\tau)\,. \tag{5}$$

Then $y(t) > 0$ eventually.

The proof is quite easy and similar to that of Erbe et al [1], Lemma 5.1.4 and we omit it.

Lemma 3. (see Zhand and Yang [5]) *Let* $m \geq 2$ *be an even integer and* $Q \in C(J, \mathbb{R}_+)$. *Then the equation*

$$x^{(m)}(t) + Q(t)x(t) = 0 \tag{6}$$

is oscillatory if and only if the inequality

$$x^{(m)}(t) + Q(t)x(t) \leq 0 \tag{7}$$

has no eventually positive solution.

Lemma 4. *Assume that conditions H1 and H2 hold,* $\tau \in (0, +\infty)$ *and* $n \geq 1$ *is an odd integer. Then the equation*

$$[x(t) - x(t-\tau)]^{(n)} + q(t)x(g(t)) = 0 \tag{8}$$

is oscillatory if and only if the inequality

$$[x(t) - x(t-\tau)]^{(n)} + q(t)x(g(t)) \leq 0 \tag{9}$$

has no eventually positive solution.

The proof of Lemma 4 is quite similar to that of Erbe et al [1], Theorem 5.5.1; only a slight modification is needed and we omit it.

3. MAIN RESULTS

Theorem 1. *Assume that conditions H1 and H2 hold,* $\tau \in (0, +\infty)$ *and* $n \geq 1$ *is an odd integer. Then equation* (1) *is oscillatory if and only if equation* (3) *is oscillatory.*

Proof. Without loss of generality we assume $n = 3$. That is, we will prove that the oscillation of the equations

$$[x(t) - x(t-\tau)]''' + q(t)x(g(t)) = 0 \tag{10}$$

and

$$x''''(t) + \frac{q(t)}{\tau}x(t) = 0 \tag{11}$$

is equivalent.

Sufficiency. Let equation (11) be oscillatory. We will prove that equation (10) is oscillatory. Suppose to the contrary that equation (10) has an eventually positive solution $x(t)$. Set $y(t) = x(t) - x(t-\tau)$. Then from (10) and Lemma 2 we have that $y'''(t) \leq 0$ and $y(t) > 0$ eventually. It follows from Lemma 1 that there exists a $T_0 \geq T$ such that either

$$x(t) > 0\,, \quad y(t) > 0\,, \quad y'(t) < 0\,, \quad y''(t) > 0\,, \quad t \geq T_0 - \tau\,, \tag{12}$$

or

$$x(t) > 0\,, \quad y(t) > 0\,, \quad y'(t) > 0\,, \quad y''(t) > 0\,, \quad t \geq T_0 - \tau\,. \tag{13}$$

Let conditions (12) hold. Then we obtain by induction that

$$x(t) = y(t) + y(t-\tau) + \cdots + y(t-n\tau) + x(t-n\tau-\tau) \tag{14}$$

for $T_0 + nT \le t \le T_0 + n\tau + \tau$, $n = 0, 1, 2, \dots$. Since the function $y(t)$ is decreasing for $t \ge T_0 - \tau$ and $y(t) \ge \frac{1}{\tau}\int_t^{t+\tau} y(s)ds$, $t \ge T_0$ it follows from (14) that

$$x(t) \ge \frac{1}{\tau}\int_{T_0+\tau}^{t} y(s)ds\,, \qquad t \ge T_0 + \tau\,.$$

Then condition H2 implies

$$\begin{aligned} x(g(t)) &\ge \frac{1}{\tau}\int_{T_0+\tau}^{g(t)} y(s)ds \ge \frac{1}{\tau}\int_{T_0+\tau}^{t-\sigma} y(s)ds \\ &\ge \frac{1}{\tau}\int_{T_*}^{t} y(s)ds \qquad \text{for} \quad t \ge T_* = T_0 + \tau + \sigma\,. \end{aligned} \tag{15}$$

From (15) and (10) we get

$$y'''(t) + \frac{q(t)}{\tau}\left(\int_{T_*}^{t} y(s)ds\right) \le 0\,, \qquad t \ge T_*\,.$$

Then the function $z(t) = \int_{T_*}^{t} y(s)ds$, $t \ge T_*$ is a positive solution of the inequality

$$z''''(t) + \frac{q(t)}{\tau} z(t) \le 0\,. \tag{16}$$

By Lemma 3 equation (11) has a nonoscillatory solution, which is a contradiction.

Let conditions (13) hold. Since $y(t)$ is increasing and $y(t) \ge \frac{1}{\tau}\int_{t-\tau}^{t} y(s)ds$, it follows from (14) that

$$x(t) \ge \frac{1}{\tau}\int_{T}^{t} y(s)ds\,, \qquad t \ge T_* = T + \tau\,.$$

Then

$$\begin{aligned} x(g(t)) &\ge \frac{1}{\tau}\int_{T}^{g(t)} y(s)ds \ge \frac{1}{\tau}\int_{T}^{t-\sigma} y(s)ds \\ &= \frac{1}{\tau}\left(\int_{T}^{t} y(s)ds - \int_{t-\sigma}^{t} y(s)ds\right), \qquad t \ge T_*\,. \end{aligned} \tag{17}$$

From $y'' > 0$, $y'''(t) \le 0$ we conclude that there exists the limit $\lim_{t\to+\infty} y''(t) = k \in \mathbb{R}_+$.

In the following we will distinguish three cases.

Case 1. $\lim_{t\to+\infty} y''(t) = k > 0$. Then

$$y'(t) = kt + o(t)\,, \quad y(t) = \frac{kt^2}{2} + o(t^2) \quad \text{and} \quad \int_T^t y(s)ds = \frac{kt^3}{6} + o(t^3)\,,$$

as $t \to +\infty$. This implies

$$\int_{t-\sigma}^{t} y(s)ds \le k\sigma t^2\,, \qquad \int_{T}^{t} y(s)ds > k\sigma t^2 \tag{18}$$

for $t \geq T_1$, where $T_1 \geq T_*$ is sufficiently large.

From (17) and (18) it follows that

$$x(g(t)) \geq \frac{1}{\tau}\bigg(\int_T^t y(s)ds - k\sigma t^2\bigg) > 0\,, \qquad t \geq T_1\,.$$

Then the function $z(t) = \int_T^t y(s)ds - k\sigma t^2$, $t \geq T_1$ is a positive solution of inequality (16), and applying Lemma 3 we get a contradiction.

Case 2. $\lim_{t\to+\infty} y''(t) = 0$, $\lim_{t\to+\infty} y'(t) = k > 0$. Then

$$y(t) = kt + o(t)\,, \qquad \int_T^t y(s)ds = \frac{kt^2}{2} + o(t^2) \qquad \text{as} \qquad t \to +\infty$$

and

$$\int_{t-\sigma}^t y(s)ds < 2\sigma kt \qquad \text{eventually.}$$

Then

$$x(g(t)) \geq \frac{1}{\tau}\bigg(\int_T^t y(s)ds - 2\sigma kt\bigg) > 0 \qquad \text{eventually}$$

and the function $z(t) = \int_T^t y(s)ds - 2\sigma kt$ is an eventually positive solution of inequality (16). Applying Lemma 3 we get a contradiction.

Case 3. $\lim_{t\to+\infty} y''(t) = 0$, $\lim_{t\to+\infty} y'(t) = +\infty$. Then

$$y'(t) = o(t)\,, \qquad y(t) = o(t^2)\,, \qquad t = o(y(t))\,,$$
$$\int_T^t y(s)ds = o(t^3)\,, \qquad t^2 = o\bigg(\int_T^t y(s)ds\bigg)$$

as $t \to +\infty$. So we have

$$\int_{t-\sigma}^t y(s)ds < t^2 \qquad \text{eventually.}$$

Then

$$x(g(t)) \geq \frac{1}{\tau}\bigg(\int_T^t y(s)ds - t^2\bigg) > 0 \qquad \text{eventually}$$

and the function $z(t) = \int_T^t y(s)ds - t^2$ is an eventually positive solution of inequlity (16), which leads to a contradiction as above.

The proof of the sufficiency is complete.

Necessity. That is, the oscillation of equation (10) implies that for equation (11). Suppose to the contrary that equation (11) has an eventually positive solution y. Then $y''''(t) \leq 0$ eventually. From Lemma 1 there exists a $T_* \geq T$ such that either

$$y'(t) > 0\,, \quad y''(t) < 0\,, \quad y'''(t) > 0\,, \quad t \geq T_*\,, \tag{19}$$

or

$$y'(t) > 0\,, \quad y''(t) > 0\,, \quad y'''(t) > 0\,, \quad t \geq T_*\,. \tag{20}$$

Let conditions (19) hold. Since $y(t)$ is increasing and $y'(t)$ is decreasing for $t \geq T_*$, there exists a $T^* \geq T_*$ such that $y(t) > M$ and $y'(t) < \frac{M}{1+\sigma}$ for $t \geq T^*$.

Set $T_0 = T^* + \tau + \sigma$, $T_k = T_0 + k\tau$, $k = -1, 0, 1, 2 \ldots$, and define the functions

$$\lambda(t) = \frac{y'(T_0)}{\tau}(t - T_0 + \tau)$$

and

$$z(t) = \begin{cases} 0\,, & t \le T_{-1}\,, \\ \lambda(t)\,, & t \in [T_{-1}, T_0]\,, \\ \lambda(t - k\tau) + \sum_{j=0}^{k-1} y'(t - j\tau)\,, & t \in [T_{k-1}, T_k]\,,\ k = 1, 2, \ldots\,. \end{cases}$$

It is easy to verify that $z \in C(\mathbb{R}, \mathbb{R}_+)$, $z(t) > 0$ for $t > T_{-1}$ and

$$z(t) - z(t - \tau) = y'(t) \qquad \text{for} \qquad t \ge T_0\,. \tag{21}$$

Let $m_1 = \max_{[T_{-1}, T_0]} \lambda(t)$. Then $m_1 = y'(T_0) \in (0, \frac{M}{1+\sigma})$.

Since $y'(t)$ is decreasing for $t \ge T_{-1}$ we have for $t \in [T_{k-1}, T_k]$, $k = 1, 2, \ldots$ that

$$\begin{aligned} z(t) \le m_1 + y'(t - (k-1)\tau) + \cdots + y'(t) \le m_1 + \int_{t-k\tau}^{t} y'(s)ds \\ = m_1 + y(t) - y(t - k\tau) \le m_1 + y(t) - M\,. \end{aligned}$$

Since $g(t) \le t + \sigma$, $t \ge T$, $y(t)$ is increasing and $y'(t)$ is decreasing for $t \ge T_*$ we obtain that for $t \ge T_0 + \sigma$

$$\begin{aligned} z(g(t)) \le m_1 - M + y(g(t)) \le m_1 - M + y(t + \sigma) = m_1 - M + y(t) + \int_t^{t+\sigma} y'(s)ds \\ \le m_1 - M + y(t) + y'(t)\sigma \le \frac{M}{1+\sigma} - M + y(t) + \frac{M\sigma}{1+M} = y(t)\,. \end{aligned}$$

Substituting the above inequality and (21) into (11) we get

$$[z(t) - z(t-\tau)]''' + q(t)z(g(t)) \le 0\,, \qquad t \ge T_0 + \sigma\,.$$

Then by Lemma 4 equation (10) has an eventually positive solution, which leads to a contradiction.

Let conditions (20) hold. Then $\lim_{t\to+\infty} y'''(t) = k \in \mathbb{R}_+$. Define the functions $\lambda(t)$ and $z(t)$ as above and set $m_1 = \max_{[T_{-1}, T_0]} \lambda(t)$, $m_0 = \max_{[T_{-1}, T_0]} y(t)$. Now $y'(t)$ is increasing for $t \ge T_{-1}$. Then we have for $t \in [T_{k-1}, T_k]$, $k = 1, 2, \ldots$

$$\begin{aligned} z(t) = \lambda(t - k\tau) + y'(t - (k-1)\tau) + \cdots + y'(t) \le m_1 + \int_{t-(k-1)\tau}^{t+\tau} y'(s)ds \\ \le m_1 + y(t+\tau) - y(t - (k-1)\tau) \le m_1 + y(t + \tau)\,, \\ z(t) \ge \int_{t-k\tau}^{t} y'(s)ds = y(t) - y(t - k\tau) \ge y(t) - m_0\,, \end{aligned}$$

that is,

$$y(t) - m_0 \le z(t) \le m_1 + y(t + \tau)\,, \qquad t \ge T_0\,. \tag{22}$$

Hence

$$z(g(t)) \le m_1 + y(g(t)+\tau) \le m_1 + y(t+\tau+\sigma) = m_1 + y(t) + \int_t^{t+\tau+\sigma} y'(s)ds$$
$$\le m_1 + y(t) + y'(t+\tau+\sigma)\sigma\,, \qquad t \ge T_0+\sigma\,. \quad (23)$$

In the following, we will distinguish three cases.

Case 1. Let $k>0$. Then

$$y''(t) = kt + o(t)\,, \qquad y'(t) = \frac{k}{2}t^2 + o(t^2) \quad \text{and} \quad y(t) = \frac{kt^3}{6} + o(t^3) \quad \text{as} \quad t \to +\infty\,.$$

From (22) it follows that $z(t) = \frac{k}{6}t^3 + o(t^3)$ as $t \to +\infty$. This and (23) imply that for all sufficiently large t

$$z(g(t)) \le m_1 + y(t) + k\sigma t^2\,.$$

Let

$$u(t) = z(t) - k\sigma(t+\sigma)^2 - m_1\,.$$

Then $u(t) > 0$ eventually, $y(t) \ge u(g(t))$ and

$$y''''(t) = [z(t) - z(t-\tau)]''' = [u(t) - u(t-\tau)]'''\,.$$

Therefore it follows from (11) that

$$[u(t) - u(t-\tau)]''' + q(t)u(g(t)) \le 0 \qquad \text{eventually.}$$

From Lemma 4 it follows that equation (10) has an eventually positive solution, which is a contradiction.

Case 2. Let $k=0$ and $\lim_{t\to+\infty} y''(t) = \lambda > 0$. Then

$$y'(t) = \lambda t + o(t) \qquad \text{and} \qquad y(t) = \frac{\lambda t^2}{2} + o(t^2) \qquad \text{as} \qquad t \to +\infty\,.$$

Obviously $z(t) = \frac{\lambda}{2}t^2 + o(t^2)$ as $t \to +\infty$. Hence for all sufficiently large t

$$z(g(t)) \le m_1 + y(t) + 2\lambda\sigma t\,.$$

Let

$$u(t) = z(t) - m_1 - 2\lambda\sigma(t+\sigma)\,.$$

Then $u(t) > 0$ eventually and $y(t) \ge u(g(t))$. Repeating the same arguments as in Case 1, we get a contradiction.

Case 3. Let $k=0$ and $\lim_{t\to+\infty} y''(t) = +\infty$. Then

$$y''(t) = o(t)\,, \quad y'(t) = o(t^2)\,, \quad t = o(y'(t))\,, \quad y(t) = o(t^3)\,, \quad t^2 = o(y(t))$$

as $t \to +\infty$.

Obviously, $z(t) = o(t^3)$ and $t^2 = o(z(t))$ as $t \to +\infty$. Hence

$$z(g(t)) \le m_1 + y(t) + \sigma t^2 \qquad \text{eventually.}$$

Let

$$u(t) = z(t) - m_1 - \sigma(t+\sigma)^2\,.$$

Then $u(t) > 0$ eventually and $y(t) \geq u(g(t))$. Repeating the same arguments as in Case 1, we get a contradiction. □

Proceeding as in the proof of Theorem 1 and using the function $\Delta(t, x(t))$ instead of $g(t)$ one can prove the following two theorems.

Theorem 2. *Assume that:*

1. *Conditions H1 and H3 hold,* $\tau \in (0, +\infty)$ *and* $n \geq 1$ *is an odd integer.*
2. *Equation* (3) *is oscillatory.*

Then equation (2) *is oscillatory.*

Theorem 3. *Assume that:*

1. *Conditions H1 and H4 hold,* $\tau \in (0, +\infty)$ *and* $n \geq 1$ *is an odd integer.*
2. *Equation* (3) *has an eventually positive solution.*

Then the inequality

$$[x(t) - x(t-\tau)]^{(n)} + g(t)x(\Delta(t, x(t))) \leq 0 \tag{24}$$

has an eventually positive solution.

Remark 1. Let conditions H1 and H4 hold. In order to prove that the oscillations of equations (2) and (3) are equivalent it remains to prove that equation (2) has an eventually positive solution if inequality (24) has such a solution . This is an open problem for now.

Consider the equations

$$x^{(m)}(t) + q(t)x(\Delta(t, x(t))) = 0 \tag{25}$$

and

$$x^{(m)}(t) + q(t)x(t) = 0\,. \tag{26}$$

Theorem 4. *Assume that:*

1. *Conditions H1 and H3 hold and* $m \geq 2$ *is an even integer.*
2. *Equation* (26) *is oscillatory.*

Then equation (25) *is oscillatory.*

Proof. Assume the opposite. Then equation (25) has an eventually positive solution $x(t)$ and $x^{(m)}(t) \leq 0$ eventually. Since $m \geq 2$ is even, then by Lemma 1 $x'(t) > 0$ eventually and the function $x(t)$ is increasing. Therefore $x(\Delta(t, x(t))) \geq x(t-\sigma)$ and the inequality

$$x^{(m)}(t) + q(t)x(t-\sigma) \leq 0$$

has an eventually positive solution. By Zhang and Yang [5], Theorem 2.5, equation (26) also has an eventually positive solution, which is a contradiction. □

Let H_m denote the maximum of $P(x) = x(1-x)\ldots(m-1-x)$ on $(0, 1)$. The following lemma is known.

Lemma 5. (see Kiguradze and Chanturia [3]) *Let $m \geq 2$ be even and $q \in C(J, \mathbb{R}_+)$. Then equation* (26) *is oscillatory if one of the following conditions is fulfillled:*

$$\lim_{t\to+\infty} \inf t \int_t^{\infty} s^{m-2} q(s) ds > H_m \,, \tag{27}$$

or

$$\lim_{t\to+\infty} \sup t \int_t^{\infty} s^{m-2} q(s) ds > (m-1)! \,. \tag{28}$$

Combining Theorems 2 and 4 with Lemma 5 we get the following theorems.

Theorem 5. *Let conditions H1 and H3 hold, $\tau \in (0, +\infty)$ and $n \geq 1$ be an odd integer. Then equation* (2) *is oscillatory if one of the following conditions is fulfilled:*

$$\lim_{t\to+\infty} \inf t \int_t^{\infty} s^{n-1} q(s) ds > \tau H_{n+1} \,, \tag{29}$$

or

$$\lim_{t\to+\infty} \sup t \int_t^{\infty} s^{n-1} q(s) ds > \tau n! \,. \tag{30}$$

Theorem 6. *Let conditions H1 and H3 hold and $m \geq 2$ be an even integer. Then equation* (25) *is oscillatory if one of the conditions* (27) *or* (28) *is fulfilled.*

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