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EXISTENCE OF NONOSCILLATORY SOLUTIONS TENDING TO ZERO AT ∞ FOR DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS DEPENDING ON THE UNKNOWN FUNCTION

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ABSTRACT: In this paper differential equations of the type

$$(-1)^n D_r^{(n)} x(t) = F(t, x(\Delta_1(t, x(t))), \ldots, x(\Delta_m(t, x(t)))) \tag{N}$$

and

$$(-1)^n D_r^{(n)} x(t) = p(t) x(\Delta(t, x(t))) \tag{L}$$

are considered, where $n \geq 1$ and the retarded arguments $\Delta_1, \ldots, \Delta_m$ and Δ depend on the independent variable t as well as on the unknown function x.

Sufficient conditions are found under which equation (N) (or (L)) has a positive solution x such that $\lim_{t\to+\infty} D_r^{(k)} x(t) = 0$, $k = 0, \ldots, n-1$ monotonically.

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1. INTRODUCTION

In this paper we consider the n-th order differential equations

$$(-1)^n D_r^{(n)} x(t) = F(t, x(\Delta_1(t, x(t))), \ldots, x(\Delta_m(t, x(t)))) \tag{1}$$

and

$$(-1)^n D_r^{(n)} x(t) = p(t) x(\Delta(t, x(t))) \tag{2}$$

with retarded arguments $\Delta_1, \ldots, \Delta_m$ and Δ which depend on the independent variable t as well as on the unknown function x.

Here $n \geq 1$ is an integer, $t \in J = [\alpha, +\infty) \subseteq \mathbb{R}_+ = [0, +\infty)$,

$$D_r^{(0)} x(t) = x(t)\,, \qquad D_r^{(i)} x(t) = r_i(t)(D_r^{(i-1)} x(t))'\,, \qquad i = 1, \ldots, n,$$

where $r_i : J \to (0, +\infty)$, $i = 1, \ldots, n$.

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The oscillatory and asymptotic behavior of the solutions of such type equations have been studied in the papers of Bainov et al [1], Markova and Simeonov [7], [8], [9].

The purpose here is to find sufficient conditions under which equation (1) (or (2)) possesses positive solutions tending monotonically to zero at infinity together with their first $n-1$ r-derivatives.

The main results obtained in this paper generalize similar results of Lovelady [6], Sficas [13] and Philos [11], where the case is considered when $\Delta_1, \ldots, \Delta_m$ and Δ do not depend on $x : \Delta_j = \sigma_j(t)$, $j = 1, \ldots, m$, $\Delta = \sigma(t)$. For related results the reader is referred to the papers of Kusano and Onose [3], [4], Philos [10], Philos and Staikos [12], Sficas [14].

2. PRELIMINARY NOTES

Introduce the following conditions:

H1. $r_i \in C(J, (0, +\infty))$, $i = 1, \ldots, n-1$ and $r_n(t) \equiv 1$, $t \in J$.

H2. $F \in C(J \times \mathbb{R}^m_+, \mathbb{R})$ and

$$x_1 F(t, x_1, \ldots, x_m) > 0 \quad \text{for} \quad t \in J,\ x_1 x_j > 0,\ j = 1, \ldots, m.$$

H3. $F(t, x_1, \ldots, x_m) \leq F(t, y_1, \ldots, y_m)$ provided that $0 < x_j \leq y_j$, $j = 1, \ldots, m$.

H4. $\Delta_j \in C(J \times \mathbb{R}, \mathbb{R})$, $j = 1, \ldots, m$ and there exist $\sigma_j \in C(J, \mathbb{R})$, $j = 1, \ldots, m$ and $T \in J$ such that

$$\lim_{t \to +\infty} \sigma_j(t) = +\infty\,, \qquad \sigma_j(t) \leq \Delta_j(t, x) < t\,, \qquad j = 1, \ldots, m\,, \quad t \geq T\,, \quad x \in \mathbb{R}\,.$$

H5. $\displaystyle\int^{+\infty} \frac{dt}{r_i(t)} = +\infty$, $i = 1, \ldots, n-1$.

H6. $\Delta \in C(J \times \mathbb{R}, \mathbb{R})$ and there exist $\sigma \in (J, \mathbb{R})$ and $T \in J$ such that

$$\lim_{t \to +\infty} \sigma(t) = +\infty\,, \qquad \sigma(t) \leq \Delta(t, x) < t\,, \qquad t \geq T\,, \quad x \in \mathbb{R}\,.$$

H7. $p \in C(J, (0, +\infty))$.

The domain $\mathcal{D}$ of $D_r^{(n)}$ is defined to be the set of all functions $x : [t_x, +\infty) \to \mathbb{R}$ such that the r-derivatives $D_r^{(k)} x(t)$, $k = 1, \ldots, n$ exist and are continuous on interval $[t_x, +\infty) \subseteq J$. By a *proper* solution of equation (1) is meant a function $x \in \mathcal{D}$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq t_x$. We assume that equation (1) do possess proper solutions. A proper solution $x : [t_x, +\infty) \to \mathbb{R}$ is called *positive* if there exists $t \geq t_x$ such that $x(t) > 0$ for $t \geq T$.

We will need the following lemma which is a generalization of the well-known lemma of Kiguradze [2] and can be proved similarly.

Lemma 1. *Suppose conditions H1 and H5 hold and the functions $D_r^{(n)} x$ and $x \in \mathcal{D}$ are of constant sign and not identically zero for $t \geq t_* \geq \alpha$.*

Then there exist a $t_k \geq t_*$ and an integer k, $0 \leq k \leq n$ with $n+k$ even for $x(t)D_r^{(n)}x(t)$ nonnegative and $n+k$ odd for $x(t)D_r^{(n)}x(t)$ nonpositive and such that for every $t \geq t_k$

$$x(t)D_r^{(i)}x(t) > 0\,, \qquad i = 0, 1, \ldots, k\,,$$
$$(-1)^{k+i}x(t)D_r^{(i)}x(t) > 0\,, \qquad i = k, k+1, \ldots, n-1\,.$$

3. MAIN RESULTS

Theorem 1. *Assume conditions H1-H4 hold and y is a positive and strictly decreasing solution of the integral inequality*

$$y(t) \;\geq\; \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^\infty F(s, y(\sigma_1(s)), \ldots, y(\sigma_m(s)))ds \ldots ds_1\,, \qquad t \geq T\,. \tag{3}$$

Then there exists a positive solution x of differential equation (1) *such that $x(t) \leq y(t)$ for t sufficiently large and*

$$\lim_{t\to+\infty} D_r^{(i)}x(t) = 0 \qquad \textit{monotonically}\,, \quad i = 0, \ldots, n-1\,.$$

Proof. Let y be a positive and strictly decreasing solution of integral inequality (3) on the interval $[\tau, +\infty) \subseteq J$. From condition H4 it follows that there exists a $T > \tau$ such that $\Delta_j(t,x) \geq \sigma_j(t) \geq \tau$ for $t \geq T$, $x \in \mathbb{R}$, $j = 1, \ldots, m$.

Consider the set

$$X = \{x \in C([T, +\infty), \mathbb{R}_+) : x(t) \leq y(t)\,,\, t \geq T\}$$

with the norm $\|x\| = \sup\{|x(t)| : t \geq T\}$ of $x \in X$.

For any $x \in X$ we set

$$\widetilde{x}(t) = \begin{cases} x(t)\,, & t \geq T\,, \\ x(T) + y(t) - y(T)\,, & \tau \leq t \leq T \end{cases}$$

and define the operator $S : X \to C([T, +\infty), \mathbb{R}_+)$ by the formula

$$Sx(t) = \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^\infty F(s, \widetilde{x}(\Delta_1(s, x(s))), \ldots, \widetilde{x}(\Delta_m(s, x(s))))ds \ldots ds_1\,, \quad t \geq T\,. \tag{4}$$

From (4), (3) and conditions H2-H4 we obtain

$$0 \leq Sx(t) \leq \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots$$

$$\int_{s_{n-1}}^{\infty} F(s, y(\sigma_1(s)), \dots, y(\sigma_m(s)))ds \dots ds_1 \le y(t)\,, \quad t \ge T\,,$$

which means that $SX \subseteq X$.

It is standard to verify that the other conditions of the Schauder's Second Fixed Point Theorem [5] are fulfilled and therefore there exists $x \in X$ such that $x = Sx$, that is, for every $t \ge T$

$$x(t) = \int_t^{\infty} \frac{1}{r_1(s_1)} \int_{s_1}^{\infty} \frac{1}{r_2(s_2)} \dots \int_{s_{n-1}}^{\infty} F(s, \widetilde{x}(\Delta_1(s, x(s))), \dots, \widetilde{x}(\Delta_m(s, x(s))))ds \dots ds_1\,. \tag{5}$$

This implies that

$$\lim_{t\to+\infty} D_r^{(i)} x(t) = 0\,, \quad \text{monotonically}\,, \quad i = 0, 1, \dots, n-1 \tag{6}$$

and

$$(-1)^n D_r^{(n)} x(t) = F(t, \widetilde{x}(\Delta_1(t, x(t))), \dots, \widetilde{x}(\Delta_m(t, x(t))))\,, \qquad t \ge T\,. \tag{7}$$

From condition H4 there exists a $T_1 \ge T$ such that $\Delta_j(t, x) \ge \sigma_j(t) \ge T$ for $t \ge T_1$, $j = 1, \dots, m$ and hence the function $x(t)$ is a solution of equation (1) for $t \ge T_1$. We have that $x(t) \le y(t)$, $t \ge T$ and $\widetilde{x}(t) > 0$, $\tau \le t < T$. We prove that $\widetilde{x}(t) > 0$ for $t \ge T$. Assume the opposite. Then there exists a $T_* \ge T$ such that $\widetilde{x}(t) > 0$, $\tau \le t < T_*$ and $x(T_*) = \widetilde{x}(T_*) = 0$. Since $\tau \le \sigma_j(T_*) \le \Delta_j(T_*, x(T_*)) < T_*$, $j = 1, \dots, m$ (by condition H4), then $\widetilde{x}(\Delta_j(T_*, x(T_*))) > 0$ and

$$F(T_*, \widetilde{x}(\Delta_1(T_*, x(T_*))), \dots, \widetilde{x}(\Delta_m(T_*, x(T_*)))) > 0\,.$$

Hence by (7) we obtain $D_r^{(n)} x(T_*) \ne 0$. Furthermore, we have that x is nonnegative and strictly decreasing on $[T, +\infty)$. Hence $x(t) = 0$ for $t \ge T_*$ since $x(T_*) = 0$. This implies $D_r^{(n)} x(T_*) = 0$. which is a contradiction. Therefore $\widetilde{x}(t) > 0$ for $t \ge T$ and $x(t)$ is a positive solution of equation (1) for $t \ge T_1$. □

Corollary 1. *Assume conditions H1-H5 hold and* y *is a positive bounded solution of the differential inequality*

$$(-1)^n D_r^{(n)} y(t) \ge F(t, y(\sigma_1(t)), \dots, y(\sigma_m(t)))\,. \tag{8}$$

Then there exists a positive solution x *of differential equation* (1) *such that* $x(t) \le y(t)$ *for* t *sufficiently large and* $\lim_{t\to+\infty} D_r^{(i)} x(t) = 0$ *monotonically,* $i = 0, 1, \dots, n-1$.

Proof. Let y be a positive bounded solution of inequality (8) on an interval $[\tau, +\infty) \subseteq J$ and $T > \tau$ be chosen so that

$$\Delta_j(t, x) \ge \sigma_j(t) \ge \tau \qquad \text{for} \qquad t \ge T\,, \quad j = 1, \dots, m\,.$$

From (8) and condition H2 it follows that $(-1)^n D_r^{(n)} y(t) > 0$ for $t \geq \tau$. Then by Lemma 1 there exist a $t_k \geq \tau$ and an integer k, $0 \leq k \leq n$ which is even such that for $t \geq t_k$

$$\begin{aligned} D_r^{(i)} y(t) > 0\,, &\qquad i = 0, 1, \ldots, k\,, \\ (-1)^{k+i} D_r^{(i)} y(t) > 0\,, &\qquad i = k, k+1, \ldots, n-1\,. \end{aligned} \tag{9}$$

Since $y(t)$ is bounded the case $k \geq 2$ is impossible. Hence $k = 0$ and $D_r^{(1)} y(t) < 0$, $t \geq t_k$, that is, $y(t)$ is strictly decreasing on $[t_k, +\infty)$. Moreover, it follows from (8) and (9) with $k = 0$, that

$$y(t) \geq \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \cdots \int_{s_{n-1}}^\infty F(s, y(\sigma_1(s)), \ldots, y(\sigma_m(s))) ds \ldots ds_1\,, \quad t \geq t_k\,.$$

Hence Corollary 1 follows from Theorem 1. □

Corollary 2. *Let conditions H1, H6 and H7 hold and*

$$\sup_{t \geq T} \int_{\sigma(t)}^t p(s) ds \leq \begin{cases} \dfrac{1}{e}\,, & \text{if} \quad n = 1\,, \\ \dfrac{n}{e} \left(\displaystyle\prod_{i=1}^{n-1} Q_i \right)^{\frac{1}{n}}, & \text{if} \quad n > 1\,, \end{cases} \tag{10}$$

where $Q_i = \inf_{t \geq T} \{p(t) r_i(t)\} > 0$, $i = 1, \ldots, n-1$ *and* $T \geq \alpha$ *is such that* $\sigma(t) \geq \alpha$ *for* $t \geq T$.

Then there exists a positive solution x *of differential equation* (2) *such that* $\lim_{t \to +\infty} D_r^{(i)} x(t) = 0$ *monotonically,* $i = 0, \ldots, n-1$.

Proof. Set

$$M_T = \sup_{t \geq T} \int_{\sigma(t)}^t p(s) ds \qquad \text{and}$$

$$y(t) = \exp\left(-\frac{n}{M_T} \int_\alpha^t p(s) ds \right) \qquad \text{for} \quad t \geq \alpha\,.$$

For every $t \geq T$ we have

$$\begin{aligned} y(\sigma(t)) &= \exp\left(-\frac{n}{M_T} \int_\alpha^{\sigma(t)} p(s) ds \right) \\ &= \exp\left(\frac{n}{M_T} \int_{\sigma(t)}^t p(s) ds \right) \exp\left(-\frac{n}{M_T} \int_\alpha^t p(s) ds \right) \leq e^n \exp\left(-\frac{n}{M_T} \int_\alpha^t p(s) ds \right). \end{aligned}$$

For $n = 1$ and $t \geq T$ we have $eM_T \leq 1$ and

$$\begin{aligned} \int_t^\infty p(s) y(\sigma(s)) ds &\leq e \int_t^\infty p(s) \exp\left(-\frac{1}{M_T} \int_\alpha^s p(u) du \right) ds \\ &= eM_T \exp\left(-\frac{1}{M_T} \int_\alpha^t p(s) ds \right) = eM_T y(t) \leq y(t)\,. \end{aligned}$$

For $n>1$ and $t\geq T$ we have $e^n(\frac{M_T}{n})^n(\prod_{i=1}^{n-1}Q_i)^{-1}\leq 1$ and

$$\begin{aligned}
&\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty p(s)y(\sigma(s))ds ds_{n-1}\dots ds_1\\
&\leq \int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^\infty p(s)e^n\\
&\qquad\qquad \times \exp\bigg(-\frac{n}{M_T}\int_\alpha^s p(u)du\bigg)ds ds_{n-1}\dots ds_1\\
&=\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\\
&\quad\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}\int_{sn-1}^\infty e^n\bigg(-\frac{M_T}{n}\bigg)d\exp\bigg(-\frac{n}{M_T}\int_\alpha^s p(u)du\bigg)ds_{n-1}\dots ds_1\\
&=\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\\
&\quad\int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})}e^n\frac{M_T}{n}\exp\bigg(-\frac{n}{M_T}\int_\alpha^{s_{n-1}} p(u)du\bigg)ds_{n-1}\dots ds_1\\
&\leq\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\\
&\quad\int_{s_{n-2}}^\infty \frac{e^nM_T}{nQ_{n-1}}p(s_{n-1})\exp\bigg(-\frac{n}{M_T}\int_\alpha^{s_{n-1}} p(u)du\bigg)ds_{n-1}\dots ds_1\\
&\quad\dots\\
&\leq\frac{e^n(\frac{M_T}{n})^n}{Q_{n-1}\dots Q_1}y(t)\leq y(t)\,.
\end{aligned}$$

Then for every $t\geq T$ we obtain

$$y(t)\geq\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-1}}^\infty p(s)y(\sigma(s))ds\dots ds_1$$

and hence Corollary 2 follows from Theorem 1. □

Remark 1. Corollary 2 generalizes Proposition 1′ from [11] where equation (2) is considered in the case $\Delta=\sigma(t)$ and instead of condition (10) the following condition

$$\sup_{t\geq T}\int_{\sigma(t)}^t p(s)ds\leq\frac{n}{e}\bigg(P_T^{n-1}\prod_{i=1}^{n-1}R_{i,T}\bigg)^{\frac{1}{n}} \tag{11}$$

is assumed, where

$$P_T=\inf_{t\geq T}p(t)>0\quad\text{and}\quad R_{i,T}=\inf_{t\geq T}r_i(t)>0\,,\quad i=1,\dots,n-1\,.$$

We note that condition (10) of Corollary 2 is better than condition (11) since $P_TR_{i,T}\leq Q_i$, $i=1,\dots,n-1$.

Example 1. Consider the equation

$$(tx'(t))' = \frac{1}{t}x(t-1)\,, \qquad t > 0\,. \tag{12}$$

Here

$$n = 2\,, \quad r_1(t) = t\,, \quad p(t) = \frac{1}{t}\,, \quad \sigma(t) = (t-1)\,,$$

$$M_T = \sup_{t\geq T}\int_{\sigma(t)}^{t} p(s)ds = \ln\frac{T}{T-1}\,, \qquad \text{for} \quad T > 1\,,$$

$$Q_1 = \inf_{t\geq T}\{p(t)r_1(t)\} = 1\,, \qquad \text{for} \quad T > 1\,,$$

$$P_T = \inf_{t\geq T} p(t) = 0\,, \qquad R_{1,T} = \inf_{t\geq T} r_1(t) = T\,.$$

Hence Proposition 1′ from [11] does not work in this case since $P_T = 0$ and $M_T > \frac{2}{e}(P_T R_{1,T})^{\frac{1}{2}} = 0$. On the other hand condition (10) of Corollary 2 is satisfied for T sufficiently large:

$$M_T = \ln\frac{T}{T-1} \leq \frac{2}{e} = \frac{2}{e}(Q_1)^{\frac{1}{2}}\,.$$

Consequently by Corollary 2 equation (12) has a positive solution $x(t)$ such that

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} tx'(t) = 0 \qquad \text{monotonically.}$$

Corollary 3. *Let conditions H1, H6 and H7 hold and*

$$\sup_{t\geq T} t\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-1}}^\infty \frac{p(s)}{\sigma(s)}ds\ldots ds_1 \leq 1\,, \tag{13}$$

where $t \geq \alpha$ is such that $\sigma(t) > 0$ for $t \geq T$.

Then there exists a positive solution x of differential equation (2) such that $\lim_{t\to+\infty} D_r^{(i)}x(t) = 0$ *monotonically, $i = 0, \ldots, n-1$.*

Proof. If we set $y = \frac{1}{t}$ for $t > 0$, then for every $t \geq T$ we have

$$y(t) \geq \int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-1}}^\infty p(s)y(\sigma(s))ds\ldots ds_1$$

and Corollary 3 follows from Theorem 1. □

Remark 2. Condition (13) is satisfied, if

$$\lim_{t\to+\infty}\sup t\int_t^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-1}}^\infty \frac{p(s)}{\sigma(s)}ds\ldots ds_1 < 1\,. \tag{14}$$

Corollary 4. *Let conditions H1, H6 and H7 hold and*

$$\int_\alpha^\infty \frac{1}{r_1(s_1)}\int_{s_1}^\infty \frac{1}{r_2(s_2)}\cdots\int_{s_{n-1}}^\infty p(s)ds\ldots ds_1 < +\infty\,. \tag{15}$$

Then there exists a positive solution x of differential equation (2) *such that* $\lim_{t\to+\infty} D_r^{(i)}x(t) = 0$ *monotonically, $i = 0, \ldots, n-1$.*

Proof. Let $y(t) = 1 + \frac{1}{t}$ for $t \geq 1$ and $T = \max\{\alpha, 1\}$ be such that $\sigma(t) \geq 1$ for $t \geq T$ and

$$\int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-1}}^\infty p(s) ds \ldots ds_1 \leq \frac{1}{2}\,.$$

Then for every $t \geq T$ we have

$$\begin{aligned}
&\int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-1}}^\infty p(s) y(\sigma(s)) ds \ldots ds_1 \\
&\qquad = \int_t^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-1}}^\infty p(s) \left[1 + \frac{1}{\sigma(s)}\right] ds \ldots ds_1 \\
&\qquad\qquad \leq 2 \int_T^\infty \frac{1}{r_1(s_1)} \int_{s_1}^\infty \frac{1}{r_2(s_2)} \ldots \int_{s_{n-1}}^\infty p(s) ds \ldots ds_1 \leq 1 \leq y(t)\,.
\end{aligned}$$

Hence Corollary 4 follows from Theorem 1. □

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