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MULTIVARIATE FRACTIONAL TAYLOR'S FORMULA

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ABSTRACT: Here is established a multivariate fractional Taylor's formula using a suitable definition of fractional derivative. As related results we present that the order of fractional-ordinary partial differentiation is immaterial, we discuss fractional integration by parts, and we estimate the remainder of our multivariate fractional Taylor's formula.

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1. INTRODUCTION

The main motivation here is from Canavati [4], Anastassiou [1] and Anastassiou [2], where there is presented a Taylor's univariate fractional formula by using an appropriate definition of fractional derivative introduced first in Canavati [4].

So we extend this formula to the multivariate fractional case over a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$, for all fractional orders $\nu > 0$.

We give an estimate to the remainder of our multivariate fractional Taylor's formula. We establish under mild and natural assumptions that the order of fractional-ordinary partial differentiation is immaterial. Also we present some fractional integration by parts results. The main ingredient in all here is the *Riemann-Liouville integral.*

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2. RESULTS

We make

Remark 2.1. We follow Anastassiou [2], p. 540, see also Canavati [4], Anastassiou [1]. Let $[a, b] \subseteq \mathbb{R}$. Let $x, x_0 \in [a, b]$ such that $x \geq x_0$, x_0 is fixed. Let $f \in C([a, b])$ and define

$$(J_\nu^{x_0} f)(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} f(t) dt, \; x_0 \leq x \leq b, \tag{2.1}$$

$\nu > 0$, the generalized *Riemann-Liouville integral.* We consider the subspace $C_{x_0}^{\nu}([a, b])$ of $C^n([a, b])$, $n := [\nu]$, $\alpha := \nu - n$ $(0 < \alpha < 1)$:

$$C_{x_0}^{\nu}([a, b]) := \left\{ f \in C^n([a, b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0, b]) \right\}. \tag{2.2}$$

Hence, let $f \in C_{x_0}^{\nu}([a, b])$, we define the *generalized* ν *- fractional derivative of* f *over* $[x_0, b]$, see also Canavati [4], Anastassiou [1] as

$$D_{x_0}^{\nu} f := (J_{1-\alpha}^{x_0} f^{(n)})'. \tag{2.3}$$

Notice that

$$\left(J_{1-\alpha}^{x_0} f^{(n)}\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f^{(n)}(t) dt \tag{2.4}$$

exists for $f \in C_{x_0}^{\nu}([a, b])$.

Let $f_{x_0}(t) := f(x_0 + t)$, $0 \leq t \leq b - x_0$, $x \geq x_0$. By change of variable we obtain

$$(D_0^{\nu} f_{x_0})(x - x_0) = (D_{x_0}^{\nu} f)(x). \tag{2.5}$$

When $\nu \in \mathbb{N}$ then the fractional derivative collapses to the usual one.

We mention the fractional Taylor's formula. See Anastassiou [2], p. 540, Canavati [4] and Anastassiou [1].

Theorem 2.1. *Let* $f \in C_{x_0}^{\nu}([a, b])$, $x_0 \in [a, b]$ *fixed*

(i) If $\nu \geq 1$, *then it holds*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \cdots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!}$$
$$+(J_\nu^{x_0} D_{x_0}^{\nu} f)(x), \quad all \quad x \in [a, b] : x \geq x_0. \tag{2.6}$$

(ii) If $0 < \nu < 1$ *we have*

$$f(x) = (J_\nu^{x_0} D_{x_0}^{\nu} f)(x), \quad all \quad x \in [a, b] : x \geq x_0. \tag{2.7}$$

We transfer Theorem 2.1 to the multivariate case. We make

Remark 2.2. Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$; $z := (z_1, \ldots, z_k)$, $x_0 := (x_{01}, \ldots, x_{0k}) \in Q$. Let $f \in C^n(Q)$, $n \in \mathbb{N}$.

Set

$$g_z(t) := f(x_0 + t(z - x_0)), 0 \leq t \leq 1; g_z(0) = f(x_0), g_z(1) = f(z). \tag{2.8}$$

Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_0 + t(z - x_0)), \tag{2.9}$$

$j = 0, 1, 2, \ldots, n$, and

$$g_z^{(n)}(0) = \left[\left(\sum_{i=1}^{k}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right](x_0). \tag{2.10}$$

If all $f_\alpha(x_0) := \frac{\partial^\alpha f}{\partial x^\alpha}(x_0) = 0$, $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$; $|\alpha| := \sum_{i=1}^k \alpha_i =: l$, then $g_z^{(l)}(0) = 0$, where $l \in \{0, 1, \ldots, n\}$. We quote that

$$g_z'(t) = \sum_{i=1}^{k}(z_i - x_{0i})\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)). \tag{2.11}$$

Let first $1 \le \nu < 2$, then here $n := [\nu] = 1$ and $\alpha = \nu - 1$; $1 - \alpha = 2 - \nu$. Since $0 \le \nu - 1 < 1$, then $n^* := [\nu - 1] = 0$, $\alpha^* = \nu - 1 - n^* = \nu - 1$ and $1 - \alpha^* = 2 - \nu$.

Put

$$(J_\nu g_z)(x) := \frac{1}{\Gamma(\nu)}\int_0^x (x - t)^{\nu - 1} g_z(t)dt, \tag{2.12}$$

$0 \le x \le 1$.

Consider

$$C^\nu([0,1]) := \{g \in C^1([0,1]) : J_{2-\nu}g' \in C^1([0,1])\} \subseteq C^1([0,1]), \quad 1 \le \nu < 2. \tag{2.13}$$

Assume that as function of t : $f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0,1])$, $\quad i = 1, \ldots, k$, then there exists the fractional derivative $g_z^{(\nu)}$, $g_z^{(\nu)} = (J_{2-\nu}g_z')'$. The last comes by using (2.11) to have

$$(J_{2-\nu}g_z')(x) = \frac{1}{\Gamma(2-\nu)}\int_0^x (x - t)^{1-\nu} g_z'(t)dt \tag{2.14}$$

$$= \frac{\sum_{i=1}^k (z_i - x_{0i})}{\Gamma(2-\nu)}\int_0^x (x - t)^{1-\nu}\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0))dt, \tag{2.15}$$

$0 \le x \le 1$.

Hence it holds

$$(J_{2-\nu}g_z'(x))'$$

$$\sum_{i=1}^{k}(z_i - x_{0i})\left(\frac{1}{\Gamma(2-\nu)}\int_0^x (x - t)^{1-\nu}\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0))dt\right)' \tag{2.16}$$

$$= \sum_{i=1}^{k}(z_i - x_{0i})(J_{2-\nu}(f_{x_i}(x_0 + t(z - x_0))))'. \tag{2.17}$$

That is

$$g_z^{(\nu)}(t) = \sum_{i=1}^{k}(z_i - x_{0i})\left(\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0))\right)^{(\nu-1)}, \tag{2.18}$$

$0 \le t \le 1, \quad 1 \le \nu < 2$.

Thus the remainder turns to

$$(J_\nu g_z^{(\nu)})(t) = \sum_{i=1}^{k}(z_i - x_{0i})[J_\nu(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}](t)\,. \tag{2.19}$$

By (2.6) applied on g_z we obtain

$$f(z) = g_z(1) = f(x_0) + (J_\nu g_z^{(\nu)})(1)\,, \tag{2.20}$$

i.e. it holds

$$f(z) = f(x_0) + \sum_{i=1}^{k}(z_i - x_{0i})[J_\nu(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}](1). \tag{2.21}$$

More precisely we get

$$f(z) = f(x_0) + \sum_{i=1}^{k}(z_i - x_{0i})\frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}dt \tag{2.22}$$

From Remark 2.2 we have established the basic multivariate fractional Taylor formula.

Theorem 2.2. *Let $f \in C^1(Q)$, Q compact and convex $\subseteq \mathbb{R}^k$, $k \geq 2$. For fixed $x_0, z \in Q$, assume that as a function of t : $f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0,1])$, $1 \leq \nu < 2$, all $i = 1, \ldots, k$. Then*

(i)

$$f(z_1, \ldots, z_k) = f(x_{01}, \ldots, x_{0k})$$
$$+ \sum_{i=1}^{k}\frac{(z_i - x_{0i})}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}dt. \tag{2.23}$$

(ii) Given $f(x_0) = 0$, then

$$f(z) = \sum_{i=1}^{k}\frac{(z_i - x_{0i})}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1}(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}dt. \tag{2.24}$$

We make

Remark 2.3. Continuing from Remark 2.2. Here $f \in C^2(Q)$, $Q \subseteq \mathbb{R}^2$, we have

$$g_z''(t) = (z_1 - x_{01})^2\,\frac{\partial^2 f}{\partial x_1^2}(x_0 + t(z - x_0)) + 2(z_1 - x_{01})(z_2 - x_{02})$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0 + t(z - x_0)) + (z_2 - x_{02})^2\,\frac{\partial^2 f}{\partial x_2^2}(x_0 + t(z - x_0)). \tag{2.25}$$

Let $2 \leq \nu < 3$, then $n := [\nu] = 2$, $\alpha := \nu - n = \nu - 2$, $1 - \alpha = 3 - \nu$. Set $\nu^* := \nu - 2$, then $n^* := [\nu - 2] = 0$, $\alpha^* = (\nu - 2) - n^* = \nu - 2$, $1 - \alpha^* = 3 - \nu$.

We have $(0 \leq x \leq 1)$

$$(J_{3-\nu}g_z'')(x) = \frac{1}{\Gamma(3-\nu)}\int_0^x (x-t)^{2-\nu}g_z''(t)dt$$

$$= (z_1 - x_{01})^2 \frac{1}{\Gamma(3-\nu)} \int_0^x (x-t)^{2-\nu} f_{x_1x_1}(x_0 + t(z-x_0))dt$$
$$+2(z_1 - x_{01})(z_2 - x_{02}) \frac{1}{\Gamma(3-\nu)} \int_0^x (x-t)^{2-\nu} f_{x_1x_2}(x_0 + t(z-x_0))dt$$
$$+(z_2 - x_{02})^2 \frac{1}{\Gamma(3-\nu)} \int_0^x (x-t)^{2-\nu} f_{x_2x_2}(x_0 + t(z-x_0))dt \,, \tag{2.26}$$

i.e. it holds.

$$(J_{3-\nu} g_z'')(x) = (z_1 - x_{01})^2 (J_{3-\nu}(f_{x_1x_1}(x_0 + t(z-x_0))))(x)$$
$$+2(z_1 - x_{01})(z_2 - x_{02})(J_{3-\nu}(f_{x_1x_2}(x_0 + t(z-x_0))))(x)$$
$$+(z_2 - x_{02})^2 (J_{3-\nu}(f_{x_2x_2}(x_0 + t(z-x_0))))(x). \tag{2.27}$$

Assuming now that $f_{x_1x_1}(x_0+t(z-x_0))$, $f_{x_1x_2}(x_0+t(z-x_0))$, $f_{x_2x_2}(x_0+t(z-x_0))$, as functions of t belong to $C^{(\nu-2)}([0,1])$ we obtain that it exists

$$g_z^{(\nu)}(t) = (z_1 - x_{01})^2 (f_{x_1x_1}(x_0 + t(z-x_0)))^{(\nu-2)}$$
$$+2(z_1 - x_{01})(z_2 - x_{02})(f_{x_1x_2}(x_0 + t(z-x_0)))^{(\nu-2)}$$
$$+(z_2 - x_{02})^2 (f_{x_2x_2}(x_0 + t(z-x_0)))^{(\nu-2)}. \tag{2.28}$$

Next we observe that

$$(J_\nu g_z^{(\nu)})(x) = (z_1 - x_{01})^2 (J_\nu (f_{x_1x_1}(x_0 + t(z-x_0)))^{(\nu-2)})(x)$$
$$+2(z_1 - x_{01})(z_2 - x_{02})(J_\nu (f_{x_1x_2}(x_0 + t(z-x_0)))^{(\nu-2)})(x)$$
$$+(z_2 - x_{02})^2 (J_\nu (f_{x_2x_2}(x_0 + t(z-x_0)))^{(\nu-2)})(x). \tag{2.29}$$

We have proved via (2.6) the next Taylor type result.

Theorem 2.3. *Let $f \in C^2(Q)$, Q compact and convex $\subseteq \mathbb{R}^2$. For fixed $x_0, z \in Q$ assume that as functions of t : $f_{x_1x_1}(x_0+t(z-x_0))$, $f_{x_1x_2}(x_0+t(z-x_0))$, $f_{x_2x_2}(x_0+t(z-x_0)) \in C^{(\nu-2)}([0,1])$, where $2 \le \nu < 3$. Then*

(i)

$$f(z_1,z_2) = f(x_{01},x_{02}) + (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0)$$
$$+(z_1 - x_{01})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f_{x_1x_1}(x_0 + t(z-x_0)))^{(\nu-2)} dt$$
$$+2(z_1 - x_{01})(z_2 - x_{02}) \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f_{x_1x_2}(x_0 + t(z-x_0)))^{(\nu-2)} dt$$
$$+(z_2 - x_{02})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f_{x_2x_2}(x_0 + t(z-x_0)))^{(\nu-2)} dt. \tag{2.30}$$

(ii) When $f(x_0) = \frac{\partial f}{\partial x_1}(x_0) = \frac{\partial f}{\partial x_2}(x_0) = 0$, then

$$f(z_1,z_2) = (z_1 - x_{01})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f_{x_1x_1}(x_0 + t(z-x_0)))^{(\nu-2)} dt$$
$$+2(z_1 - x_{01})(z_2 - x_{02}) \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f_{x_1x_2}(x_0 + t(z-x_0)))^{(\nu-2)} dt$$

$$+(z_2-x_{02})^2\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}(f_{x_2x_2}(x_0+t(z-x_0)))^{(\nu-2)}dt. \tag{2.31}$$

The following general multivariate fractional Taylor formula is valid.

Theorem 2.4. *Let $f \in C^n(Q)$, Q compact and convex $\subseteq \mathbb{R}^k$, $k \geq 2$; here $\nu \geq 1$ such that $n = [\nu]$. For fixed $x_0, z \in Q$ assume that as functions of t : $f_\alpha(x_0 + t(z-x_0)) \in C^{(\nu-n)}([0,1])$, for all $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$; $|\alpha| := \sum_{i=1}^k \alpha_i = n$.*

Then

(i)

$$f(z_1,\ldots,z_k)=f(x_{01},\ldots,x_{0k})+\sum_{i=1}^k(z_i-x_{0i})\frac{\partial f}{\partial x_i}(x_{01},\ldots,x_{0k})$$
$$+\frac{\left[\left(\sum_{i=1}^k(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^2 f\right](x_{01},\ldots,x_{0k})}{2}+\ldots$$
$$\frac{\left[\left(\sum_{i=1}^k(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^{n-1} f\right](x_{01},\ldots,x_{0k})}{(n-1)!}$$
$$+\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}\left\{\left[\left(\sum_{i=1}^k(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)}(x_0+t(z-x_0))\right\}dt. \tag{2.32}$$

(ii) If all $f_\alpha(x_0) = 0$, $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$, $|\alpha| := \sum_{i=1}^k \alpha_i = l$, $l = 0, \ldots, n-1$, then

$$f(z_1,\ldots,z_k)=\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}$$
$$\left\{\left[\left(\sum_{i=1}^k(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)}(x_0+t(z-x_0))\right\}dt. \tag{2.33}$$

Proof. Use of (2.6). □

(Note that fractional differentiation is a linear operation.)

We make

Remark 2.4. Continuing from the previous remarks. Let here $0 < \nu < 1$. Assume that $f(x_0+t(z-x_0)) \in C^\nu([0,1])$ as function of t. Then by $g_z(t) := f(x_0+t(z-x_0))$ we have

$$g_z^{(\nu)}(t)=(f(x_0+t(z-x_0)))^{(\nu)},\ \text{ and }\ (J_\nu\, g_z^{(\nu)})(t)=(J_\nu(f(x_0+t(z-x_0)))^{(\nu)})(t),$$

$t \in [0,1]$. Hence

$$(J_\nu\, g_z^{(\nu)})(1)=\frac{1}{\Gamma(\nu)}\int_0^1(1-t)^{\nu-1}(f(x_0+t(z-x_0)))^{(\nu)}dt. \tag{2.34}$$

We have established the next multivariate fractional Taylor formula when $0 < \nu < 1$.

Theorem 2.5. *Let bounded $f : Q \to \mathbb{R}$, where Q convex $\subseteq \mathbb{R}^k$, $k \geq 2$, such that as a function of $t : f(x_0 + t(z - x_0)) \in C^\nu([0,1])$, $0 < \nu < 1$, $x_0, z \in Q$ being fixed. Then*

$$f(z_1, \ldots, z_k) = \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} (f(x_0 + t(z - x_0)))^{(\nu)} dt. \tag{2.35}$$

Proof. Use of (2.7). □

We make

Remark 2.5. Next we study the ordinary partial derivatives of fractional derivatives. Let $0 < \alpha < 1$, $f \in C^1([0,1]^2)$, $x \in [0,1]$ fixed, and consider

$$\gamma(x,z) := \int_0^x (x-t)^{-\alpha} f(t,z) dt, \tag{2.36}$$

$\forall\, z \in [0,1]$.

We observe that

$$|\gamma(x,z)| \leq \int_0^x (x-t)^{-\alpha} |f(t,z)| dt \leq \|f\|_\infty \int_0^x (x-t)^{-\alpha} dt$$
$$= \|f\|_\infty \frac{x^{1-\alpha}}{1-\alpha} \leq \frac{\|f\|_\infty}{1-\alpha} < +\infty\,,$$

i.e. the function

$$\rho(t) := (x-t)^{-\alpha} f(t,z) \tag{2.37}$$

is Lebesgue integrable in $t \in [0,x]$, $\forall\, z \in [0,1]$. Thus one can consider integration in (2.36) over $[0,x)$, $\forall\, z \in [0,1]$.

Also the function

$$\lambda(z) := (x-t)^{-\alpha} f(t,z) \tag{2.38}$$

is differentiable in $z \in [0,1]$, $\forall \quad t \in [0,x)$, i.e. we have

$$\lambda'(z) = (x-t)^{-\alpha} \frac{\partial f(t,z)}{\partial z}, \forall t \in [0,x). \tag{2.39}$$

Moreover

$$|\lambda'(z)| \leq (x-t)^{-\alpha} \left\| \frac{\partial f}{\partial z} \right\|_\infty, \tag{2.40}$$

$\forall\, (t,z) \in [0,x) \times [0,1]$.

The R.H.S. (2.40) is integrable in $t \in [0,x]$ and nonnegative. Hence by H. Bauer [3], pp. 103-104 we obtain that $(x-t)^{-\alpha} \frac{\partial f(t,z)}{\partial z}$ is integrable in $t \in [0,x)$ and

$$\frac{\partial \gamma(x,z)}{\partial z} = \int_0^x (x-t)^{-\alpha} \frac{\partial f(t,z)}{\partial z} dt, \tag{2.41}$$

$\forall\, z \in [0,1]$.

We have proved

Lemma 2.1. *Let* $0 < \alpha < 1,\ f \in C^1([0,1]^2),\ 0 \leq x \leq 1.$ *Then*

$$\frac{\partial}{\partial z}\left(\int_0^x (x-t)^{-\alpha} f(t,z)dt\right) = \int_0^x (x-t)^{-\alpha} \frac{\partial f(t,z)}{\partial z} dt, \tag{2.42}$$

$\forall\ z \in [0,1].$

We make

Remark 2.6. Assume now $0 < \alpha < 1,\ f \in C^{n+1}([0,1]^2),\ n \in \mathbb{N}$. Then by Lemma 2.1 we get

$$\frac{\partial}{\partial z}\left(\int_0^x (x-t)^{-\alpha} \frac{\partial^n f}{\partial t^n}(t,z)dt\right) = \int_0^x (x-t)^{-\alpha} \frac{\partial^{n+1} f}{\partial t^n \partial z}(t,z)dt. \tag{2.43}$$

Let now $\nu > 0,\ n := [\nu],\ \alpha := \nu - n.$

We suppose existence of

$$g^{(\nu)}(x,z) := \frac{\partial^\nu g(x,z)}{\partial x^\nu} = \frac{\partial}{\partial x}\left(J_{1-\alpha}\left(\frac{\partial^n g}{\partial t^n}(\cdot,z)\right)\right)(x,z). \tag{2.44}$$

We also assume here that $g \in C^{n+1}([0,1]^2)$, and $(g^{(\nu)}(x,z))_z,\ g_z^{(\nu)}(x,z)$ exist and are jointly continuous in $(x,z) \in [0,1]^2,\ [\nu] = n \in \mathbb{N}$.

Then it holds

$$\begin{aligned}(g^{(\nu)}(x,z))_z &= \frac{\partial}{\partial z}(g^{(\nu)}(x,z)) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\left(J_{1-\alpha}\left(\frac{\partial^n g}{\partial t^n}(\cdot,z)\right)\right)\right)(x,z)\\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\left(J_{1-\alpha}\left(\frac{\partial^n g}{\partial t^n}(\cdot,z)\right)\right)\right)(x,z)(\text{by } (2.43))\\ &= \frac{\partial}{\partial x}\left(J_{1-\alpha}\left(\frac{\partial^{n+1}}{\partial z \partial t^n} g(\cdot,z)\right)\right)(x,z)\\ &= \frac{\partial}{\partial x}\left(J_{1-\alpha}\left(\frac{\partial^n}{\partial t^n} g_z(\cdot,z)\right)\right) = g_z^{(\nu)}(x,z).\end{aligned}$$

That is

$$(g^{(\nu)}(x,z))_z = g_z^{(\nu)}(x,z), \quad \forall (x,z) \in [0,1]^2,\ \nu > 0. \tag{2.45}$$

In brief, it holds

$$(g^{(\nu)})_z = (g_z)^{(\nu)}. \tag{2.46}$$

Under more similar suitable assumptions one obtains

$$(g^{(\nu)})_{zz} = (g_{zz})^{(\nu)},\ (g^{(\nu)})_{z_1 z_2} = (g_{z_1 z_2})^{(\nu)},\ (g^{(\nu)})_{z_1 z_2 z_3} = (g_{z_1 z_2 z_3})^{(\nu)}, \text{etc.} \tag{2.47}$$

We have established that the order of fractional-ordinary partial differentiation is immaterial.

Theorem 2.6. *Let* $g \in C^{n+1}([0,1]^2)$*, and* $\nu > 0$ *such that* $[\nu] = n \in \mathbb{N}$*. Assume the existence of* $g^{(\nu)}(x,z)$*, and* $(g^{(\nu)}(x,z))_z,\ g_z^{(\nu)}(x,z)$ *both exist and are jointly continuous in* $(x,z) \in [0,1]^2$.

Then

$$(g^{(\nu)}(x,z))_z = g_z^{(\nu)}(x,z), \tag{2.48}$$

$\forall (x,z) \in [0,1]^2$.

We make

Remark 2.7. Next comes fractional integration by parts. Let $f, g \in C^{\nu}([0,1])$, $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$. Here

$$g^{(\nu)} = \frac{d(J_{1-\alpha} g^{(n)})}{dx}, \quad f^{(\nu)} = \frac{d\,(J_{1-\alpha} f^{(n)})}{dx}.$$

That is $\quad d\,(J_{1-\alpha} g^{(n)}) = g^{(\nu)}dx, \quad d\,(J_{1-\alpha} f^{(n)}) = f^{(\nu)}dx.$

We observe that

$$\int_0^1 (J_{1-\alpha} f^{(n)})(x) g^{(\nu)}(x)dx = \int_0^1 (J_{1-\alpha} f^{(n)})(x) d\,(J_{1-\alpha} g^{(n)})(x)$$

$$= (J_{1-\alpha} f^{(n)})(1)(J_{1-\alpha} g^{(n)})(1) - \int_0^1 (J_{1-\alpha} g^{(n)})(x) d\,(J_{1-\alpha} f^{(n)})(x) =$$

$$(J_{1-\alpha}\, f^{(n)})(1)(J_{1-\alpha} g^{(n)})(1) - \int_0^1 (J_{1-\alpha} g^{(n)})(x) f^{(\nu)}(x)dx. \tag{2.49}$$

Next let us take $g \in C^{\nu}([0,1]),\ 1 \le \nu < 2, \quad n := [\nu] = 1,\ \alpha := \nu - n = \nu - 1,\ 1 - \alpha = 2 - \nu,$ and $f \in C^1([0,1])$.

Then $g^{(\nu)}(x) = \frac{d\,(J_{2-\nu}\, g')(x)}{dx}$, *i.e.* $\quad d\,(J_{2-\nu}\, g')(x) = g^{(\nu)}(x)dx$. Hence

$$\int_0^1 f(x) g^{(\nu)}(x)dx = \int_0^1 f(x) d(J_{2-\nu}\, g')(x)$$

$$= f(1)(J_{2-\nu}\, g')(1) - \int_0^1 (J_{2-\nu}\, g')(x) f'(x)dx. \tag{2.50}$$

We have established the following fractional integration by parts formulae.

Theorem 2.7. *(i) Let $f, g \in C^{\nu}([0,1]),\ \nu > 0,\ n := [\nu],\ \alpha := \nu - n$. Then*

$$\int_0^1 (J_{1-\alpha}\, f^{(n)})(x) g^{(\nu)}(x)dx$$

$$= (J_{1-\alpha}\, f^{(n)})(1)(J_{1-\alpha}\, g^{(n)})(1) - \int_0^1 (J_{1-\alpha}\, g^{(n)})(x) f^{(\nu)}(x)dx. \tag{2.51}$$

(ii) Let $g \in C^{\nu}([0,1]),\ 1 \le \nu < 2,\ f \in C^1([0,1])$. Then

$$\int_0^1 f(x) g^{(\nu)}(x)dx = f(1)(J_{2-\nu}\, g')(1) - \int_0^1 (J_{2-\nu}\, g')(x) f'(x)dx. \tag{2.52}$$

We make the last

Remark 2.8. Here we estimate the Remainder (2.32). By definition in this article, see (2.2), (2.3), the fractional derivatives are continuous functions. So the function

$$G_{\nu}(t) := \left\{ \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right]^{(\nu - n)} (x_0 + t(z - x_0)) \right\}, \tag{2.53}$$

$t \in [0,1]$, that appears in the remainder of (2.32), is continuous in t. We write the remainder (2.32) as

$$R_\nu$$
$$:= \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left\{ \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right]^{(\nu-n)} (x_0 + t(z - x_0)) \right\} dt$$
$$= \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} G_\nu(t) dt, \quad \nu \geq 1. \tag{2.54}$$

We obtain

$$|R_\nu| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} |G_\nu(t)| dt \leq \frac{1}{\Gamma(\nu)} \int_0^1 |G_\nu(t)| dt$$
$$= \frac{1}{\Gamma(\nu)} \|G_\nu\|_{L_1([0,1])}. \tag{2.55}$$

Also for $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we get

$$|R_\nu| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} |G_\nu(t)| dt \leq$$
$$\frac{1}{\Gamma(\nu)} \left(\int_0^1 ((1-t)^{\nu-1})^p dt \right)^{1/p} \left(\int_0^1 |G_\nu(t)|^q dt \right)^{1/q}$$
$$= \frac{1}{\Gamma(\nu)} \frac{1}{(p(\nu-1)+1)^{1/p}} \|G_\nu\|_{L_q([0,1])}. \tag{2.56}$$

In case $p = q = 2$ we have

$$|R_\nu| \leq \frac{1}{\Gamma(\nu)} \frac{1}{\sqrt{2\nu - 1}} \|G_\nu\|_{L_2([0,1])}. \tag{2.57}$$

Finally we get that

$$|R_\nu| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} |G_\nu(t)| dt \leq \frac{\|G_\nu\|_\infty}{\Gamma(\nu+1)}. \tag{2.58}$$

We have established the following remainder estimate.

Theorem 2.8. *All here as in Theorem 2.4. Let R_ν be the remainder in (2.32), see (2.54), and G_ν as in (2.53). Then*

$$|R_\nu| \leq \min \left\{ \frac{\|G_\nu\|_{L_1([0,1])}}{\Gamma(\nu)}, \frac{\|G_\nu\|_{L_q([0,1])}}{\Gamma(\nu)(p(\nu-1)+1)^{1/p}}, \right.$$
$$\left. \frac{\|G_\nu\|_{L_2([0,1])}}{\Gamma(\nu)\sqrt{2\nu-1}}, \frac{\|G_\nu\|_\infty}{\Gamma(\nu+1)} \right\}, \tag{2.59}$$

where $p, q > 1 \ : \ \frac{1}{p} + \frac{1}{q} = 1$.

Comment. The Chain Rule is not possible in fractional differentiation. That limits us a lot from using the multivariate fractional Taylor formula, as we employ the usual one involving only ordinary partial derivatives of functions.

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