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FRACTIONAL MULTIVARIATE OPIAL TYPE INEQUALITIES OVER SPHERICAL SHELLS

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ABSTRACT: Here is introduced the concept of multivariate fractional differentiation especially of the fractional radial differentiation, by extending the univariate definition of Canavati [11]. Then we produce Opial type inequalities over compact and convex subsets of $\mathbb{R}^N, N \geq 2$, mainly over spherical shells, studying the problem in all possibilities. Our results involve one, or two, or more functions.

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1. INTRODUCTION

This work is motivated by the articles of Opial [12], Beesack [10], and Anastassiou [3]-[9].

We would like to mention

Theorem 1.1. (see Opial [12], 1960) *Let $c > 0$, and $y(x)$ be real, continuously differentiable on $[0, c]$, with $y(0) = y(c) = 0$. Then*

$$\int_0^c |y(x)y'(x)|dx \leq \frac{c}{4}\int_0^c (y'(x))^2 dx. \tag{1.1}$$

Equality holds for the function $y(x) = x$ on $[0, c/2]$, and $y(x) = c - x$ on $[c/2, c]$.

The next result implies Theorem 1.1 and is used a lot in applications.

Theorem 1.2. (see Beesack [10], 1962) *Let $b > 0$, If $y(x)$ is real, continuously differentiable on $[0, b]$, and $y(0) = 0$ then*

$$\int_0^b |y(x)y'(x)|dx \leq \frac{b}{2}\int_0^b (y'(x))^2 dx. \tag{1.2}$$

Equality holds only for $y = mx$, where m is a constant.

We describe here our specific multivariate setting. Let the balls $B(0, R_1)$, $B(0, R_2)$; $0 < R_1 < R_2$. Here $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and

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the sphere $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let $\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)}$. For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$, and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Let the *spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$. We have that $\text{Vol}(A) = \frac{\omega_N (R_2^N - R_1^N)}{N}$. Indeed $\bar{A} = [R_1, R_2] \times S^{N-1}$.

For $F \in C(\bar{A})$ it holds

$$\int_A F(x)dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) d\omega, \tag{1.3.}$$

we exploit a lot this formula here.

In this article we present a series of various fractional multivariate Opial type inequalities over spherical shells and arbitrary domains. Opial type inequalities find applications in establishing uniqueness of solution of initial value problems for differential equations and their systems, see Willett [13].

2. RESULTS

We make

Remark 2.1. We introduce here the *partial fractional derivatives.* Let $f : [0,1]^2 \to \mathbb{R}$. Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $0 < \alpha < 1$; $\mu > 0$, $m := [\mu]$, $\beta := \mu - m$, $0 < \beta < 1$. Assume $\exists\ \frac{\partial^{m+n} f(t,s)}{\partial x^n \partial y^m} \in C([0,1]^2)$, then $(x-t)^{-\alpha}(y-s)^{-\beta} \frac{\partial^{m+n} f(t,s)}{\partial x^n \partial y^m}$ is integrable over $[0,x] \times [0,y]; x, y \in [0,1]$, that is

$$F(x,y) := \int_0^x \int_0^y (x-t)^{-\alpha}(y-s)^{-\beta} \frac{\partial^{m+n} f(t,s)}{\partial x^n \partial y^m} dt\, ds \tag{2.1}$$

is real valued.

Thus, by Fubini's Theorem, the order of integration in (2.1) does not matter.

Let now $g \in C([0,1])$, we define the *Riemann-Liouville integral*, Γ is the gamma function: $\Gamma(\nu) := \int_0^\infty e^{-t} t^{\nu-1} dt$, as

$$(\mathcal{J}_\nu g)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} g(t) dt, \quad 0 \le x \le 1. \tag{2.2}$$

We consider here the space

$$C^\nu([0,1]) := \{g \in C^n([0,1]) : \mathcal{J}_{1-\alpha}\, g^{(n)} \in C^1([0,1])\}, \tag{2.3}$$

then the *ν- fractional derivative of g* is defined by $g^{(\nu)} := (\mathcal{J}_{1-\alpha}\, g^{(n)})'$, see Canavati [11].

We assume here $f(\cdot, y) \in C^\nu([0,1])$, $\forall\, y \in [0,1]$, then we define the *ν-partial fractional derivative* of f with respect to x : $\frac{\partial f^\nu(\cdot,y)}{\partial x^\nu}$ as

$$\frac{\partial^\nu f(x,y)}{\partial x^\nu} := \frac{\partial}{\partial x} \left(\mathcal{J}_{1-\alpha} \frac{\partial^n f(x,y)}{\partial x^n} \right), \quad \forall\, (x,y) \in [0,1]^2. \tag{2.4}$$

Also, we assume $f(x,\cdot) \in C^{\mu}([0,1]),\ \forall\, x \in [0,1]$, where

$$C^{\mu}([0,1]) := \{g \in C^{m}([0,1]) : \mathcal{J}_{1-\beta}\, g^{(m)} \in C^{1}([0,1])\}. \tag{2.5}$$

Then we define *the* μ *- partial fractional derivative of* f *with respect to* y*:* $\frac{\partial f^{\mu}}{\partial y^{\mu}}(x,\cdot)$ as

$$\frac{\partial f^{\mu}(x,y)}{\partial y^{\mu}} := \frac{\partial}{\partial y}\left(\mathcal{J}_{1-\beta}\frac{\partial^{m} f}{\partial y^{m}}(x,y)\right), \quad \forall\,(x,y) \in [0,1]^{2}. \tag{2.6}$$

Define the space

$$\begin{aligned} C^{\nu+\mu}([0,1]^{2}) := \big\{ f \in C^{n+m}([0,1]^{2}) :& \\ \mathcal{J}_{1-\alpha}\left(\frac{\partial^{n} f(\cdot,\,y)}{\partial x^{n}}\right) \in C^{1}([0,1]),\ \forall\, y \in [0,1];& \\ \mathcal{J}_{1-\beta}\left(\frac{\partial^{m} f(x,\,\cdot)}{\partial x^{m}}\right) \in C^{1}([0,1]),\ \forall\, x \in [0,1];& \\ \exists\, F_{x},\ F_{y},\ F_{yx} \in C([0,1]^{2})\big\}.& \end{aligned} \tag{2.7}$$

Define *the mixed fractional partial derivative:*

$$\frac{\partial^{\nu+\mu} f(x,y)}{\partial x^{\nu}\partial y^{\mu}}$$

$$:= \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\,\frac{\partial^{2}}{\partial x\partial y}\int_{0}^{x}\int_{0}^{y}(x-t)^{-\alpha}(y-s)^{-\beta}\,\frac{\partial^{n+m} f(t,s)}{\partial x^{n}\partial y^{m}}dt\,ds. \tag{2.8}$$

One can have anchor points $x_0, y_0 \neq 0$, then all above definitions go through for $x \geq x_0,\ y \geq y_0$.

Conclusion 1. Clearly then we have $F_{xy} = F_{yx}$, and

$$\frac{\partial^{\nu+\mu} f}{\partial x^{\nu}\partial y^{\mu}} = \frac{\partial^{\mu+\nu} f}{\partial y^{\mu}\partial x^{\nu}}. \tag{2.9}$$

So the order of fractional differentiation is immaterial.

Here, it is by definition

$$\frac{\partial^{\mu+\nu} f(x,y)}{\partial y^{\mu}\partial x^{\nu}}$$

$$:= \frac{1}{\Gamma(1-\alpha)\,\Gamma(1-\beta)}\,\frac{\partial^{2}}{\partial y\partial x}\int_{0}^{x}\int_{0}^{y}(x-t)^{-\alpha}(y-s)^{-\beta}\,\frac{\partial^{m+n} f(t,s)}{\partial y^{m}\partial x^{n}}dt\,ds. \tag{2.10}$$

Comments. 1) Let $\nu = 0$, then $n = \alpha = 0$, and (2.8) becomes

$$\begin{aligned} \frac{\partial^{\mu} f(x,y)}{\partial y^{\mu}} &= \frac{1}{\Gamma(1-\beta)}\,\frac{\partial^{2}}{\partial x\partial y}\int_{0}^{x}\int_{0}^{y}(y-s)^{-\beta}\,\frac{\partial^{m} f(t,s)}{\partial y^{m}}dt\,ds \\ &= \frac{1}{\Gamma(1-\beta)}\,\frac{\partial^{2}}{\partial y\partial x}\int_{0}^{x}\int_{0}^{y}(y-s)^{-\beta}\,\frac{\partial^{m} f(t,s)}{\partial y^{m}}dt\,ds \\ &= \frac{1}{\Gamma(1-\beta)}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int_{0}^{x}\left(\int_{0}^{y}(y-s)^{-\beta}\,\frac{\partial^{m} f(t,s)}{\partial y^{m}}ds\right)dt\right)\right) =: (\star). \end{aligned} \tag{2.11}$$

Notice for fixed y we have that $(y-s)^{-\beta}\ \frac{\partial^m f(t,s)}{\partial y^m}$ is integrable over $[0,y]$, so the function

$$\varphi(t) := \int_0^y (y-s)^{-\beta}\, \frac{\partial^m f(t,s)}{\partial y^m} ds \tag{2.12}$$

is real valued for any $t\in[0,x]$.

By continuity of $\frac{\partial^m f}{\partial y^m}$ we have true that $\forall\, \varepsilon>0 \quad \exists\, \delta>0$: whenever $|t_1-t_2|<\delta$ we have $\left|\frac{\partial^m f(t_1,s)}{\partial y^m}-\frac{\partial^m f(t_2,s)}{\partial y^m}\right|<\varepsilon$. We further have

$$\varphi(t_1)-\varphi(t_2)=\int_0^y (y-s)^{-\beta}\left(\frac{\partial^m f(t_1,s)}{\partial y^m}-\frac{\partial^m f(t_2,s)}{\partial y^m}\right) ds.$$

Hence

$$|\varphi(t_1)-\varphi(t_2)| \le \int_0^y (y-s)^{-\beta}\left|\frac{\partial^m f(t_1,s)}{\partial y^m}-\frac{\partial^m f(t_2,s)}{\partial y^m}\right| ds$$
$$\le \varepsilon \int_0^y (y-s)^{-\beta} ds = \frac{\varepsilon\, y^{1-\beta}}{1-\beta}, \tag{2.13}$$

proving $\varphi(t)$ is continuous.

Consequently

$$(\star)=\frac{1}{\Gamma(1-\beta)}\left(\frac{\partial}{\partial y}\left(\int_0^y (y-s)^{-\beta}\, \frac{\partial^m f(x,s)}{\partial y^m} ds\right)\right) =: \frac{\partial^\mu f(x,y)}{\partial y^\mu}. \tag{2.14}$$

Conclusion 2. When $\nu=0$, the fractional mixed partial derivative collapses to the single fractional partial derivative.

2) Let $\mu=0$, then $m=\beta=0$, and (2.8) becomes

$$\frac{\partial^\nu f(x,y)}{\partial x^\nu}=\frac{1}{\Gamma(1-\alpha)}\,\frac{\partial^2}{\partial x\partial y}\int_0^x\int_0^y (x-t)^{-\alpha}\,\frac{\partial^n f(t,s)}{\partial x^n} dt\, ds$$
$$=\frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\int_0^y\left(\int_0^x (x-t)^{-\alpha}\,\frac{\partial^n f(t,s)}{\partial x^n} dt\right) ds\right)\right)\right) \tag{2.15}$$

(notice $\int_0^x (x-t)^{-\alpha}\,\frac{\partial^n f(t,s)}{\partial x^n} dt$ is continuous in $s\in[0,y]$)

$$=\frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x}\left(\int_0^x (x-t)^{-\alpha}\,\frac{\partial^n f(t,y)}{\partial x^n} dt\right)\right) =: \frac{\partial^\nu f(x,y)}{\partial x^\nu}. \tag{2.16}$$

Conclusion 3. When $\mu=0$, the mixed fractional derivative collapses again to the single one.

3) Let now $n=\nu\in\mathbb{N}$, i.e. $\alpha=0$, then

$$\frac{\partial^\nu f(x,y)}{\partial x^\nu}=\frac{\partial}{\partial x}\left(\int_0^x \frac{\partial^n f(t,y)}{\partial x^n} dt\right)=\frac{\partial^n f(x,y)}{\partial x^n}, \tag{2.17}$$

the ordinary one.

4) When $m=\mu\in\mathbb{N}$, i.e. $\beta=0$, then

$$\frac{\partial^\mu f(x,y)}{\partial y^\mu}=\frac{\partial}{\partial y}\int_0^y \frac{\partial^m f(x,s)}{\partial y^m} ds=\frac{\partial^m f(x,y)}{\partial y^m}, \tag{2.18}$$

the ordinary one.

5) Furthermore, let finally both $\nu = n \in \mathbb{N}$ and $\mu = m \in \mathbb{N}$, i.e. $\alpha = \beta = 0$. Then

$$\begin{aligned}\frac{\partial^{\nu+\mu} f(x,y)}{\partial x^\nu \partial y^\mu} &= \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \frac{\partial^{n+m} f(t,s)}{\partial x^n \partial y^m} dt\, ds \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\int_0^y \left(\int_0^x \frac{\partial^{n+m} f(t,s)}{\partial x^n \partial y^m} dt\right) ds\right)\right) \\ &= \frac{\partial}{\partial x}\left(\int_0^x \frac{\partial^{n+m} f(t,y)}{\partial x^n \partial y^m} dt\right) = \frac{\partial^{n+m} f(x,y)}{\partial x^n \partial y^m}, \qquad (2.19)\end{aligned}$$

proving the that fractional mixed partial collapses to the ordinary one. Fractional differentiation is a linear operation.

Conclusion 4. The above definitions we gave for the *fractional partial derivatives* are natural extensions of the ordinary positive integer ones.

Having introduced the fractional partial derivatives we are ready to develop our Opial type results.

We make

Remark 2.2. First we consider a general domain. Let Q be a compact and convex subset of $\mathbb{R}^N$, $N \geq 2$; $z := (z_1, \ldots, z_N)$, $x_0 := (x_{01}, \ldots, x_{0N}) \in Q$ be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$. Set $g_z(t) = f(x_0 + t(z - x_0))$, $0 \leq t \leq 1$;

$$g_z(0) = f(x_0),\ g_z(1) = f(z).$$

Then it holds

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_0 + t(z - x_0)), \qquad (2.20)$$

$j = 0, 1, 2, \ldots, n$, and in particular

$$g_z'(t) = \sum_{i=1}^N (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)), \qquad (2.21)$$

$0 \leq t \leq 1$.

Clearly here $g_z \in C^n([0,1])$. Let first $1 \leq \nu < 2$, in that case we take $n := [\nu] = 1$. Following Anastassiou [2] and by assuming that as function of $t : f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0,1])$, $i = 1, \ldots, N$, then there exists $g_z^{(\nu)} = (\mathcal{J}_{2-\nu}\, g_z')'$, and it holds

$$g_z^{(\nu)}(t) = \sum_{i=1}^N (z_i - x_{0i}) \left(\frac{\partial f}{\partial x_i}(x_0 + t(z - x_0))\right)^{(\nu-1)}, \qquad (2.22)$$

$0 \leq t \leq 1$.

Also here we have

$$(\mathcal{J}_{2-\nu}\, g_z')(t) = \frac{\sum_{i=1}^N (z_i - x_{0i})}{\Gamma(2-\nu)} \int_0^t (t-s)^{1-\nu} f_{x_i}(x_0 + s(z - x_0)) ds, \qquad (2.23)$$

$0 \leq t \leq 1$.

Clearly $(\mathcal{J}_{2-\nu}\, g_z')(t) \in C^1([0,1])$ and $(\mathcal{J}_{2-\nu}\, g_z')(0) = 0$. Therefore by (1.2) we get

$$\int_0^s |\mathcal{J}_{2-\nu}\, g_z'(t)|\ |D^\nu g_z(t)|\, dt \ \le\ \frac{s}{2}\int_0^s (D^\nu g_z(t))^2\, dt,\ \forall\, s \in [0,1]. \tag{2.24}$$

We have established the following Opial type result.

Theorem 2.1. *Let Q be a compact and convex subset of $\mathbb{R}^N$, $N \ge 2$; $z, x_0 \in Q$ be fixed; $1 \le \nu < 2$. Let $f \in C^1(Q)$. Assume that as a function of $t : f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0,1])$, $i = 1, \ldots, N$.*

Then

$$\frac{1}{\Gamma(2-\nu)}\int_0^s \left|\sum_{i=1}^N (z_i - x_{0i})\left(\int_0^t (t-s)^{1-\nu} f_{x_i}(x_0 + s(z - x_0))ds\right)\right|$$
$$\left|\sum_{i=1}^N (z_i - x_{0i})(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}\right| dt$$
$$\le \frac{s}{2}\int_0^s \left(\sum_{i=1}^N (z_i - x_{0i})(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}\right)^2 dt, \tag{2.25}$$

$\forall\, s \in [0,1]$.

Remark 2.2. (Continuation) Let here $\nu \ge 2$ and $n := [\nu]$, $\beta := \nu - n$. We assume that as functions of $t : f_\alpha(x_0 + t(z - x_0)) \in C^{(\nu-n)}([0,1])$, for all $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$; $|\alpha| := \sum_{i=1}^N \alpha_i = n$. Clearly then there exists $g_z^{(\nu)} = (\mathcal{J}_{1-\beta}\, g_z^{(n)})'$, and it holds

$$g_z^{(\nu)}(t) = \left[\left(\sum_{i=1}^N (z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^n f\right]^{(\nu-n)} (x_0 + t(z - x_0)), \tag{2.26}$$

all $t \in [0,1]$.

Of course , it holds

$$(\mathcal{J}_{1-\beta}\, g_z^{(n)})(t) \overset{(2.20)}{=} \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}$$
$$\left\{\left[\left(\sum_{i=1}^N (z_i - x_{0i})\,\frac{\partial}{\partial x_i}\right)^n f\right](x_0 + s(z - x_0))\right\} ds. \tag{2.27}$$

Notice $(\mathcal{J}_{1-\beta}\, g_z^{(n)})(0) = 0$. Hence again by (1.2) we get

$$\int_0^s |\mathcal{J}_{1-\beta}\, g_z^{(n)}(t)|\ |D^\nu g_z(t)|dt \ \le\ \frac{s}{2}\int_0^s (D^\nu g_z(t))^2 dt,\ \forall\, s \in [0,1]. \tag{2.28}$$

We have proved the following general Opial type of result.

Theorem 2.2. *Let Q be a compact and convex subset of $\mathbb{R}^N$, $N \ge 2$; $z, x_0 \in Q$ be fixed; $\nu \ge 2$, $n := [\nu]$, $\beta := \nu - n$. Let $f \in C^n(Q)$. Assume that as a function* *of* $t : f_\alpha(x_0 + t(z - x_0)) \in C^{(\nu-n)}([0,1])$, *for all* $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$; $|\alpha| := \sum_{i=1}^N \alpha_i = n$. *Then*

$$\frac{1}{\Gamma(1-\beta)} \int_0^s \left| \int_0^t (t-s)^{-\beta} \left\{ \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + s(z - x_0)) \right\} ds \right|$$
$$\left| \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right]^{(\nu-n)} (x_0 + t(z - x_0)) \right| dt$$
$$\leq \frac{s}{2} \int_0^s \left\{ \left[\left(\sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right]^{(\nu-n)} (x_0 + t(z - x_0)) \right\}^2 dt, \tag{2.29}$$

$\forall\, s \in [0,1]$.

Note. Following the last pattern one can transfer any univariate Opial type inequality (see Agarwal and Pang [1]), into this fractional multivariate general setting. Since no chain rule is valid in the fractional differentiation, inequalities such as (2.28), (2.29) are not revealing themselves, to totally decompose into all of their ingredients. Next, working over *spherical shells* we obtain a series of various Opial type fractional multivariate inequalities that look nice and are very clear.

We give

Definition 2.1. (see Anastassiou [7] and Anastassiou [5], p. 540) In the following we carry earlier notions introduced in Remark 2.1, over to arbitrary $[a,b] \subseteq \mathbb{R}$. Let $x, x_0 \in [a,b]$ such that $x \geq x_0, x_0$ is fixed. Let $f \in C([a,b])$ and define

$$(\mathcal{J}_\nu^{x_0} f)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \quad x_0 \leq x \leq b, \tag{2.30}$$

the generalized Riemann-Liouville integral. We consider the subspace $C_{x_0}^\nu([a,b])$ of $C^n([a,b])$:

$$C_{x_0}^\nu([a,b]) := \{ f \in C^n([a,b]) : \mathcal{J}_{1-\alpha}^{x_0} \quad f^{(n)} \in C^1([x_0,b]) \} \tag{2.31}$$

Hence, let $f \in C_{x_0}^\nu([a,b])$, we define the *generalized ν-fractional derivative of f over* $[x_0,b]$ as

$$D_{x_0}^\nu f := \left(\mathcal{J}_{1-\alpha}^{x_0} f^{(n)} \right)'. \tag{2.32}$$

Notice that

$$\left(\mathcal{J}_{1-\alpha}^{x_0} f^{(n)} \right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^x (x-t)^{-\alpha} f^{(n)}(t)dt \tag{2.33}$$

exists for $f \in C_{x_0}^\nu([a,b])$.

Next we use

Theorem 2.3. (see Anastassiou [8] and Anastassiou [5], p. 567) *Let* $\gamma_i \geq 1$, $\nu \geq 2$ *such that* $\nu - \gamma_i \geq 1$; $i = 1, \ldots, l$ *and* $f \in C_{x_0}^\nu([a,b])$ *with* $f^{(j)}(x_0) = 0$, $j = 0, 1, \ldots, n-1$, $n := [\nu]$. *Here* $x, x_0 \in [a,b] : x \geq x_0$. *Let* $q_1, q_2 > 0$ *continuous* *functions on* $[a,b]$ *and* $r_i > 0 : \sum_{i=1}^{l} r_i = r$. *Let* $s_1, s_1' > 1 : \frac{1}{s_1} + \frac{1}{s_1'} = 1$ *and* $s_2, s_2' > 1 : \frac{1}{s_2} + \frac{1}{s_2'} = 1$ *and* $p > s_2$. *Furthermore suppose that*

$$Q_1 := \left(\int_{x_0}^{x} (q_1(\omega))^{s_1'} d\omega \right)^{1/s_1'} < +\infty \tag{2.34}$$

and

$$Q_2 := \left(\int_{x_0}^{x} (q_2(\omega))^{-s_2'/p} d\omega \right)^{r/s_2'} < +\infty \,. \tag{2.35}$$

Call $\sigma := \frac{p-s_2}{ps_2}$. *Then is holds*

$$\int_{x_0}^{x} q_1(\omega) \prod_{i=1}^{l} |(D_{x_0}^{\gamma_i}(f))(\omega)|^{r_i} d\omega \le Q_1 Q_2$$
$$\cdot \prod_{i=1}^{l} \left\{ \frac{\sigma^{r_i \sigma}}{(\Gamma(\nu-\gamma_i))^{r_i} (\nu - \gamma_i - 1 + \sigma)^{r_i \sigma}} \right\}$$
$$\cdot \frac{(x-x_0)^{\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i + \sigma r + \frac{1}{s_1}\right)}}{\left(\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i s_1\right) + r s_1 \sigma + 1\right)^{1/s_1}}$$
$$\cdot \left(\int_{x_0}^{x} q_2(\omega) |(D_{x_0}^{\nu} f)(\omega)|^p d\omega \right)^{r/p} . \tag{2.36}$$

We next work in the setting of spherical shells introduced in the Introduction.

We need

Definition 2.2. Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $f \in C^n(\bar{A})$, A is a spherical shell. Assume that there exists $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A})$, given by

$$\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r-t)^{-\alpha} \frac{\partial^n f(t\omega)}{\partial r^n} dt \right) , \tag{2.37}$$

where $x \in \bar{A}$, *i.e.* $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We call $\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}$ the *radial fractional derivative of* f *of order* ν .

We need

Lemma 2.1. *Let* $\gamma \ge 1$, $\nu \ge 2$ *such that* $\nu - \gamma \ge 1$. *Let* $f \in C^n(\bar{A})$ *and there exists* $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A})$, $x \in \bar{A}$, A *a spherical shell. Further assume that* $\frac{\partial^j f(R_1 \omega)}{\partial r^j} = 0$, $j = 0, 1, \ldots, n-1$, $n := [\nu]$, $\forall\ \omega \in S^{N-1}$. *Then there exists* $\frac{\partial_{R_1}^{\gamma} f(x)}{\partial r^{\gamma}} \in C(\bar{A})$.

Proof. The assumption implies that $\frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \in C([R_1, R_2]), \forall \omega \in S^{N-1}$, i.e. $f(r\omega) \in C_{R_1}^{\nu}([R_1, R_2], \forall\, \omega \in S^{N-1}$. Following Anastassiou [7], and Anastassiou [5], pp. 544-545, we get that there exists $\frac{\partial_{R_1}^{\gamma} f(r\,\omega)}{\partial r^{\gamma}}$ and is given by

$$\frac{\partial_{R_1}^{\gamma} f(r\,\omega)}{\partial r^{\gamma}} = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \frac{\partial_{R_1}^{\nu} f(t\omega)}{\partial r^{\nu}} dt, \tag{2.38}$$

indeed $f(r\,\omega) \in C^{\gamma}_{R_1}([R_1, R_2]),\ \forall\, \omega \in S^{N-1}$.

Hence

$$\frac{\partial^{\gamma}_{R_1} f(r\,\omega)}{\partial r^{\gamma}} = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{R_2} \mathcal{X}_{[R_1,r]}(t)(r-t)^{\nu-\gamma-1}\, \frac{\partial^{\nu}_{R_1} f(t\,\omega)}{\partial r^{\nu}} dt. \tag{2.39}$$

Let $r_n \to r,\ \omega_n \to \omega$, then $\mathcal{X}_{[R_1,r_n]}(t) \to \mathcal{X}_{[R_1,r]}(t)$, a.e. also $(r_n-t)^{\nu-\gamma-1} \to (r-t)^{\nu-\gamma-1}$, and

$$\frac{\partial^{\nu}_{R_1} f(t\,\omega_n)}{\partial r^{\nu}} \to \frac{\partial^{\nu}_{R_1} f(t\,\omega)}{\partial r^{\nu}}.$$

Furthermore it holds that

$$\mathcal{X}_{[R_1,r_n]}(t)(r_n-t)^{\nu-\gamma-1}\, \frac{\partial^{\nu}_{R_1} f(t\,\omega_n)}{\partial r^{\nu}} \longrightarrow$$

$$\mathcal{X}_{[R_1,r]}(t)(r-t)^{\nu-\gamma-1}\, \frac{\partial^{\nu}_{R_1} f(t\,\omega)}{\partial r^{\nu}}, \text{ a.e. on } [R_1, R_2]. \tag{2.40}$$

However we have

$$\mathcal{X}_{[R_1,r_n]}(t)|r_n-t|^{\nu-\gamma-1}\left|\frac{\partial^{\nu}_{R_1} f(t\,\omega_n)}{\partial r^{\nu}}\right|$$

$$\leq (R_2-R_1)^{\nu-\gamma-1}\left\|\frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}}\right\|_{\infty} < \infty. \tag{2.41}$$

Thus, by the Dominated convergence theorem we obtain

$$\int_{R_1}^{R_2} \mathcal{X}_{[R_1,r_n]}(t)(r_n-t)^{\nu-\gamma-1}\, \frac{\partial^{\nu}_{R_1} f(t\,\omega_n)}{\partial r^{\nu}}\, dt \to$$

$$\int_{R_1}^{R_2} \mathcal{X}_{[R_1,r]}(t)(r-t)^{\nu-\gamma-1}\, \frac{\partial^{\nu}_{R_1} f(t\,\omega)}{\partial r^{\nu}}\, dt, \tag{2.42}$$

proving the claim.

We present the very general result.

Theorem 2.4. *Let* $\gamma_i \geq 1,\ \nu \geq 2,$ *such that* $\nu - \gamma_i \geq 1;\ i = 1, \ldots, l,\ n := [\nu]$. *Let* $f \in C^n(\bar{A})$ *and there exists* $\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}} \in C(\bar{A}),\ x \in \bar{A},\ A$ *is a spherical shell:* $A := B(0,R_2) - \overline{B(0,R_1)} \subseteq \mathbb{R}^N,\ N \geq 2$. *Furthermore assume that* $\frac{\partial^j f}{\partial r^j},\ j = 0, 1, \ldots, n-1,$ *vanish on* $\partial B(0, R_1)$. *Let* $r_i > 0 : \sum_{i=1}^{l} r_i = p$. *Let* $s_1, s_1' > 1 : \frac{1}{s_1} + \frac{1}{s_1'} = 1,$ *and* $s_2, s_2' > 1 : \frac{1}{s_2} + \frac{1}{s_2'} = 1,$ *and* $p > s_2$. *Denote*

$$Q_1 = \left(\frac{R_2^{(N-1)s_1'+1} - R_1^{(N-1)s_1'+1}}{(N-1)s_1'+1}\right)^{1/s_1'}, \tag{2.43}$$

and

$$Q_2 = \left(\frac{R_2^{(1-N)\frac{s_2'}{p}+1} - R_1^{(1-N)\frac{s_2'}{p}+1}}{(1-N)\frac{s_2'}{p}+1}\right)^{p/s_2'}. \tag{2.44}$$

Call $\sigma := \frac{p-s_2}{p\,s_2}$.

Also call

$$C := Q_1\, Q_2 \prod_{i=1}^{l} \left\{ \frac{\sigma^{r_i \sigma}}{(\Gamma(\nu-\gamma_i))^{r_i} (\nu-\gamma_i-1+\sigma)^{r_i\sigma}} \right\}$$

$$\frac{(R_2-R_1)^{(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i+\frac{p}{s_2}+\frac{1}{s_1}-1)}}{\left(\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i s_1\right)+s_1(\frac{p}{s_2}-1)+1\right)^{1/s_1}}. \tag{2.45}$$

Then

$$\int_A \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\gamma_i} f(x)}{\partial r^{\gamma^i}} \right|^{r_i} dx \leq\ C \int_A \left| \frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \right|^{p} dx. \tag{2.46}$$

Proof. The assumption imply that $f(r\,\omega) \in C^n([R_1,R_2])$, $\frac{\partial_{R_1}^{\nu} f(r\,\omega)}{\partial r^{\nu}} \in C([R_1,R_2])$, $\forall\, \omega \in S^{N-1}$. By Theorem 2.3 we have

$$\int_{R_1}^{R_2} r^{N-1} \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\gamma_i} f(r\,\omega)}{\partial r^{\gamma_i}} \right|^{r_i} dr$$

$$\leq C \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\,\omega)}{\partial r^{\nu}} \right|^{p} dr,\ \forall\, \omega \in S^{N-1}. \tag{2.47}$$

Therefore it holds

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\gamma_i} f(r\,\omega)}{\partial r^{\gamma_i}} \right|^{r_i} dr \right) d\,\omega$$

$$\leq C \int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\,\omega)}{\partial r^{\nu}} \right|^{p} dr \right) d\,\omega. \tag{2.48}$$

Using Lemma 2.1 and by (1.3) we derive (2.46). ☐

We mention

Theorem 2.5. (see Anastassiou [8] and Anastassiou [5], p. 573) *Let* $\gamma_i \geq 1,\ \nu \geq 2$ *such that* $\nu-\gamma_i \geq 1;\ i=1,\ldots,l$ *and* $f \in C_{x_0}^{\nu}([a,b])$ *with* $f^{(j)}(x_0)=0,\ j=0,1\ldots,n-1,\ n:=[\nu]$. *Here* $x,x_0 \in [a,b] : x \geq x_0$. *Let* $\tilde{q}(w) \geq 0$ *continuous on [a,b] and* $r_i > 0 : \sum_{i=1}^{l} r_i = r$. *Then it holds*

$$\int_{x_0}^{x} \tilde{q}(w) \cdot \prod_{i=1}^{l} (|D_{x_0}^{\nu} f|(w))^{r_i} dw$$

$$\leq \left\{ \frac{\|\tilde{q}\|_\infty (\|D_{x_0}^{\nu} f\|_\infty)^r}{\prod_{i=1}^{l}(\Gamma(\nu-\gamma_i+1))^{r_i}} \right\} \cdot \left\{ \frac{(x-x_0)^{r\nu-\sum_{i=1}^{l} r_i\gamma_i+1}}{(r\nu-\sum_{i=1}^{l} r_i\gamma_i+1)} \right\}. \tag{2.49}$$

We give

Theorem 2.6. *Let* $\gamma_i \geq 1,\ \nu \geq 2,$ *such that* $\nu-\gamma_i \geq 1;\ i=1,\ldots,l,\ n:=[\nu]$. *Let* $f \in C^n(\bar{A})$ *and there exists* $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A}),\ x \in \bar{A},\ A$ *is a spherical shell:* $A :=$

$B(0,R_2)-\overline{B(0,R_1)} \subseteq \mathbb{R}^N,\ N \geq 2$. *Furthermore assume that* $\frac{\partial^j f}{\partial r^j},\ j=0,1,\ldots,n-1$, *vanish on* $\partial B(0,R_1)$. *Let* $r_i > 0 : \sum_{i=1}^{l} r_i = r$. *Call*

$$M := \frac{R_2^{N-1}(R_2-R_1)^{r\nu-\sum_{i=1}^{l} r_i\gamma_i+1}}{\prod_{i=1}^{l}(\Gamma(\nu-\gamma_i+1))^{r_i}(r\nu-\sum_{i=1}^{l} r_i\gamma_i+1)} > 0. \tag{2.50}$$

Then

$$\int_A \left(\prod_{i=1}^{l}\left|\frac{\partial_{R_1}^{\gamma_i} f(x)}{\partial r^{\gamma_i}}\right|^{r_i}\right) dx \leq\ M\ \frac{2\pi^{N/2}}{\Gamma(N/2)}\left\|\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}\right\|_{\infty,\bar{A}}^{r}. \tag{2.51}$$

Proof. By Theorem 2.5 we get that

$$\int_{R_1}^{R_2} r^{N-1}\left(\prod_{i=1}^{l}\left|\frac{\partial_{R_1}^{\gamma_i} f(r\omega)}{\partial r^{\gamma_i}}\right|^{r_i}\right) dr \leq M\left\|\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}\right\|_{\infty,\bar{A}}^{r}. \tag{2.52}$$

Hence it holds

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left(\prod_{i=1}^{l}\left|\frac{\partial_{R_1}^{\gamma_i} f(r\,\omega)}{\partial r^{\gamma_i}}\right|^{r_i}\right) dr\right) d\,\omega$$

$$\leq M\ \omega_N\left|\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}\right|_{\infty,\bar{A}}^{r}. \tag{2.53}$$

Using (1.3) and Lemma 2.1, we establish the claim. □

We need

Theorem 2.7 (Anastassiou and Goldstein [9]). *Let* $\gamma \geq 1,\quad \nu \geq 2,\quad \nu-\gamma \geq 1,\quad \alpha,\beta > 0,\quad r > \alpha,\quad r > 1$; *let* $p > 0,\quad q \geq 0$ *be continuous functions on* $[a,b]$. *Let* $f \in C_{x_0}^{\nu}([a,b])$ *with* $f^{(i)}(x_0) = 0,\ i = 0,1,\ldots,n-1,\ n := [\nu]$. *Let* $x, x_0 \in [a,b]$ *with* $x \geq x_0$. *Then*

$$\int_{x_0}^{x} q(w)|D_{x_0}^{\gamma} f(w)|^{\beta}|D_{x_0}^{\nu} f(w)|^{\alpha} dw \leq K(p,q,\gamma,\nu,\alpha,\beta,r,x,x_0)$$

$$\cdot\left(\int_{x_0}^{x} p(w)\left|D_{x_0}^{\nu} f(w)\right|^{r} dw\right)^{\left(\frac{\alpha+\beta}{r}\right)}. \tag{2.54}$$

Here

$$K(p,q,\gamma,\nu,\alpha,\beta,r,x,x_0) := \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r}\cdot\frac{1}{(\Gamma(\nu-\gamma))^{\beta}}$$

$$\cdot\left(\int_{x_0}^{x}(q(w))^{r}\cdot(p(w))^{-\alpha})^{\frac{1}{(r-\alpha)}}\cdot(P_1(w))^{\left(\frac{\beta(r-1)}{r-\alpha}\right)}\cdot dw\right)^{\frac{r-\alpha}{r}}, \tag{2.55}$$

with

$$P_1(w) := \int_{x_0}^{w}(p(t))^{-\frac{1}{r-1}}\cdot(w-t)^{(\nu-\gamma-1)\left(\frac{r}{r-1}\right)}\cdot dt. \tag{2.56}$$

We present

Theorem 2.8. *Let* $\gamma \geq 1,\quad \nu \geq 2,\quad n := [\nu],\quad \nu-\gamma \geq 1,\quad \alpha,\beta > 0,\quad \alpha+\beta > 1$. *Let* $f \in C^n(\bar{A})$ *and there exists* $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A}),\ x \in \bar{A},\ A$ *is a spherical shell:*

$A := B(0,R_2) - \overline{B(0,R_1)} \subseteq \mathbb{R}^N$, $N \geq 2$. *Furthermore assume that* $\frac{\partial^j f}{\partial r^j} = 0$, *for* $j = 0,1,\ldots,n-1$, *on* $\partial B(0,R_1)$. *Then*

$$\int_A \left|\frac{\partial^{\gamma}_{R_1} f(x)}{\partial r^{\gamma}}\right|^{\beta} \left|\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}}\right|^{\alpha} dx \leq K \int_A \left|\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}}\right|^{\alpha+\beta} dx. \tag{2.57}$$

Here

$$K = \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/(\alpha+\beta)} \frac{1}{(\Gamma(\nu-\gamma))^{\beta}} \int_{R_1}^{R_2} r^{N-1} \left((P_1(r))^{(a+\beta-1)} dr\right)^{\frac{\beta}{\alpha+\beta}}, \tag{2.58}$$

with

$$P_1(r) := \int_{R_1}^{r} t^{\frac{1-N}{a+\beta-1}} (r-t)^{(\nu-\gamma-1)\left(\frac{\alpha+\beta}{\alpha+\beta-1}\right)} dr. \tag{2.59}$$

Proof. The assumption imply that $f(r\omega) \in C^n([R_1,R_2])$ and $\frac{\partial^{\nu}_{R_1} f(r\,\omega)}{\partial r^{\nu}} \in C([R_1,R_2])$, $\forall\, \omega \in S^{N-1}$. Hence by Theorem 2.7, $\forall\, \omega \in S^{N-1}$ we get that

$$\int_{R_1}^{R_2} r^{N-1} \left|\frac{\partial^{\gamma}_{R_1} f(r\,\omega)}{\partial r^{\gamma}}\right|^{\beta} \left|\frac{\partial^{\nu}_{R_1} f(r\,\omega)}{\partial r^{\nu}}\right|^{\alpha} dr$$

$$\leq K \int_{R_1}^{R_2} r^{N-1} \left|\frac{\partial^{\nu}_{R_1} f(r\,\omega)}{\partial r^{\nu}}\right|^{\alpha+\beta} dr. \tag{2.60}$$

Therefore it holds

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left|\frac{\partial^{\gamma}_{R_1} f(r\,\omega)}{\partial r^{\gamma}}\right|^{\beta} \left|\frac{\partial^{\nu}_{R_1} f(r\,\omega)}{\partial r^{\nu}}\right|^{\alpha} dr\right) d\omega$$

$$= K \left(\int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left|\frac{\partial^{\nu}_{R_1} f(r\,\omega)}{\partial r^{\nu}}\right|^{\alpha+\beta} dr\right) d\,\omega\right). \tag{2.61}$$

Using Lemma 2.1 and by (1.3) we derive (2.57). ☐

We need

Theorem 2.9. (see Anastassiou and Goldstein [9]) *Let* $\nu \geq 2$, $\alpha,\beta > 0$, $r > \alpha$, $r > 1$; *let* $p > 0$, $q \geq 0$ *be continuous functions on* $[a,b]$. *Let* $f \in C^{\nu}_{x_0}([a,b])$ *with* $f^{(i)}(x_0) = 0$, $i = 0,1,\ldots,n-1$, $n := [\nu]$. *Let* $x, x_0 \in [a,b]$ *with* $x \geq x_0$. *Then*

$$\int_{x_0}^{x} g(w)\,|f(w)|^{\beta} \left|D^{\nu}_{x_0} f(x)\right|^{\alpha} dw \leq K^*(p,q,\nu,\alpha,\beta,r,x,x_0)$$

$$\cdot \left(\int_{x_0}^{x} p(w) \left|D^{\nu}_{x_0} f(w)\right|^{r} dw\right)^{\left(\frac{\alpha+\beta}{r}\right)}. \tag{2.62}$$

Here

$$K^*(p,q,\nu,\alpha,\beta,r,x,x_0) := \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha/r} \cdot \frac{1}{(\Gamma(\nu))^{\beta}}$$

$$\cdot \left(\int_{x_0}^{x} ((q(w))^{r} \cdot (p(w))^{-\alpha})^{\frac{1}{(r-\alpha)}} \cdot (P_1^*(w))^{\left(\frac{\beta(r-1)}{r-\alpha}\right)} \cdot dw\right)^{\frac{r-\alpha}{r}}, \tag{2.63}$$

with

$$P_1^*(w) := \int_{x_0}^{w} (p(t))^{-\frac{1}{r-1}} \cdot (w-t)^{(\nu-1)(\frac{r}{r-1})} dt. \tag{2.64}$$

Based on Theorem 2.9 we give similarly:

Theorem 2.10. *Let* $\nu \geq 2, \quad n := [\nu], \quad \alpha, \beta > 0, \quad \alpha + \beta > 1.$ *Let* $f \in C^n(\bar{A})$ *and there exists* $\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \in C(\bar{A}), \quad x \in \bar{A},\ A$ *is a spherical shell:* $A := B(0, R_2) - \overline{B(0, R_1)} \subseteq \mathbb{R}^N,\ N \geq 2.$ *Furthermore assume that* $\frac{\partial^j f}{\partial r^j} = 0,$ *for* $j = 0, 1, \ldots, n-1,$ *on* $\partial B(0, R_1).$ *Then*

$$\int_A |f(x)|^{\beta} \left| \frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \right|^{\alpha} dx \leq K^* \int_A \left| \frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \right|^{\alpha+\beta} dx. \tag{2.65}$$

Here

$$K^* :=$$

$$\left(\frac{\alpha}{\alpha+\beta}\right)^{\left(\frac{\alpha}{\alpha+\beta}\right)} \frac{1}{(\Gamma(\nu))^{\beta}} \left(\int_{R_1}^{R_2} r^{N-1} (P_1^*(r))^{(\alpha+\beta-1)} dr\right)^{\left(\frac{\beta}{\alpha+\beta}\right)}, \tag{2.66}$$

with

$$P_1^*(r) := \int_{R_1}^{r} t^{\left(\frac{1-N}{\alpha+\beta-1}\right)} (r-t)^{\frac{(\alpha+\beta)(\nu-1)}{(\alpha+\beta-1)}} dt. \tag{2.67}$$

Next we present a set of multivariate fractional Opial type inequalities involving two functions over the shell.

We need

Theorem 2.11. (see Anastassiou [4]) *Let* $\nu, \gamma_1, \gamma_2 \geq 1,$ *such that* $\nu - \gamma_1 \geq 1,\ \nu - \gamma_2 \geq 1$ *and* $f_1, f_2 \in C_{x_0}^{\nu}([a, b])$ *with*

$$f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, \quad i = 0, 1, \ldots, n-1, \quad n := [\nu]. \tag{2.68}$$

Here, $x, x_0, \in [a, b] : x \geq x_0.$ *Consider also* $p(t) > 0,$ *and* $q(t) \geq 0$ *continuous functions on* $[x_0, b].$

Let $\lambda_\nu > 0$ *and* $\lambda_\alpha, \lambda_\beta \geq 0,$ *such that* $\lambda_\nu < p,$ *where* $p > 1.$ *Set*

$$P_k(w) := \int_{x_0}^{w} (w-t)^{(\nu-\gamma_k-1)p/(p-1)} (p(t))^{-1/(p-1)} dt,$$

$$k = 1, 2, x_0 \leq w \leq b, \tag{2.69}$$

$$A(w) := \frac{q(w) \cdot (P_1(w))^{\lambda_\alpha((p-1)/p)} \cdot (P_2(w))^{\lambda_\beta((p-1)/p)} (p(w))^{-\lambda_\nu/p}}{(\Gamma(\nu-\gamma_1))^{\lambda_\alpha} \cdot (\Gamma(\nu-\gamma_2))^{\lambda_\beta}}, \tag{2.70}$$

$$A_0(x) := \left(\int_{x_0}^{x} A(w)^{p/(p-\lambda_\nu)} dw\right)^{(p-\lambda_\nu)/p}, \tag{2.71}$$

and

$$\delta_1 := \begin{cases} 2^{1-((\lambda_\alpha+\lambda_\nu)/p)}, & \text{if} \quad \lambda_\alpha + \lambda_\nu \leq p, \\ 1, & \text{if} \quad \lambda_\alpha + \lambda_\nu \geq p. \end{cases} \tag{2.72}$$

If $\lambda_\beta = 0$, *we obtain that,*

$$\int_{x_0}^{x} q(w)\left[\left|(D_{x_0}^{\gamma_1} f_1)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu} + \right.$$

$$\left.\left|\left(D_{x_0}^{\lambda_1} f_2\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{\lambda_\nu}\right] dw$$

$$\leq \left(A_0(x)\left|_{\lambda_\beta=0}\right.\right) \cdot \left(\frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu}\right)^{\lambda_\nu/p} \cdot \delta_1$$

$$\cdot \left[\int_{x_0}^{x} p(w)\left[\left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^p + \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^p\right] dw\right]^{((\lambda_\alpha+\lambda_\nu)/p)}. \tag{2.73}$$

Similarly, by (2.73), we derive

Theorem 2.12. *Let* $\nu, \gamma_1, \gamma_2 \geq 1$, *such that* $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$, $n := [\nu]$ *and* $f_1, f_2 \in C^n(\bar{A})$ *and there exist* $\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^\nu}, \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^\nu} \in C(\bar{A})$, $A := B(0, R_2) - \overline{B(0, R_1)} \subseteq \mathbb{R}^N$, $N \geq 2$. *Furthermore assume* $\frac{\partial^j f_1}{\partial r^j} = \frac{\partial^j f_2}{\partial r^j} = 0$, *for* $j = 0, 1, \ldots, n-1$, *on* $\partial B(0, R_1)$.

Let $\lambda_\nu > 0$ *and* $\lambda_\alpha > 0$; $\lambda_\beta \geq 0$, $p := \lambda_\alpha + \lambda_\nu > 1$. *Set*

$$P_k(w) := \int_{R_1}^{w} (w-t)^{(\nu-\gamma_k-1)p/(p-1)} t^{\left(\frac{1-N}{p-1}\right)} dt, \tag{2.74}$$

$k = 1, 2, \quad R_1 \leq w \leq R_2$,

$$A(w) := \frac{w^{(N-1)\left(1-\frac{\lambda_\nu}{p}\right)} (P_1(w))^{\lambda_\alpha\left(\frac{p-1}{p}\right)} (P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)}}{(\Gamma(\nu-\gamma_1))^{\lambda_\alpha} (\Gamma(\nu-\gamma_2))^{\lambda_\beta}}, \tag{2.75}$$

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{\frac{p}{\lambda_\alpha}} dw\right)^{\frac{\lambda_\alpha}{p}}. \tag{2.76}$$

Take the case of $\lambda_\beta = 0$. *Then*

$$\int_A \left[\left|\frac{\partial_{R_1}^{\gamma_1}}{\partial r^{\gamma_1}} f_1(x)\right|^{\lambda_\alpha} \left|\left(\frac{\partial_{R_1}^{\nu} f_1}{\partial r^\nu}\right)(x)\right|^{\lambda_\nu} + \right.$$

$$\left.\left|\left(\frac{\partial_{R_1}^{\gamma_1} f_2}{\partial r^{\gamma_1}}\right)(x)\right|^{\lambda_\alpha} \left|\left(\frac{\partial_{R_1}^{\nu} f_2}{\partial r^\nu}\right)(x)\right|^{\lambda_\nu}\right] dx$$

$$\leq \left(A_0(R_2)|_{\lambda_\beta=0}\right) \left(\frac{\lambda_\nu}{p}\right)^{(\lambda_\nu/p)} \int_A \left[\left|\left(\frac{\partial_{R_1}^{\nu} f_1}{\partial r^\nu}\right)(x)\right|^p + \right.$$

$$\left.\left|\left(\frac{\partial_{R_1}^{\nu} f_2}{\partial r^\nu}\right)(x)\right|^p\right] dx. \tag{2.77}$$

We need

Theorem 2.13. (see Anastassiou [4]) *All here, as in Theorem 2.11. Denote*

$$\delta_3 := \begin{cases} 2^{\lambda_\beta/\lambda_\nu} - 1, & \text{if } \lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if } \lambda_\beta \leq \lambda_\nu. \end{cases}$$

If $\lambda_\alpha = 0$*, then, it holds*

$$\int_{x_0}^{x} q(w)\left[\left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\beta} \cdot \left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu} + \left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\beta} \cdot \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{\lambda_\nu}\right] dw$$

$$\leq \left(A_0(x)|_{\lambda_\alpha=0}\right) 2^{p-\lambda_\nu/p} \left(\frac{\lambda_\nu}{\lambda_\beta+\lambda_\nu}\right)^{\lambda_\nu/p} \delta_3^{\lambda_\nu/p} \cdot$$

$$\left(\int_{x_0}^{x} p(w)\left[\left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{p} + \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{p}\right] dw\right)^{((\lambda_\nu+\lambda_\beta)/p)}, \tag{2.78}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.78), we derive

Theorem 2.14. *All basic assumptions as in Theorem* 2.12. *Let* $\lambda_\nu > 0$, $\lambda_\alpha = 0$, $\lambda_\beta > 0$, $p := \lambda_\nu + \lambda_\beta > 1$, P_2 *defined by* (2.74)*. Now it is*

$$A(w) := \frac{w^{(N-1)\left(1-\frac{\lambda_\nu}{p}\right)} (P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)}}{(\Gamma(\nu-\gamma_2))^{\lambda_\beta}}, \tag{2.79}$$

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{\frac{p}{\lambda_\beta}} dw\right)^{\lambda_\beta/p}. \tag{2.80}$$

Denote

$$\delta_3 := \begin{cases} 2^{\lambda_\beta/\lambda_\nu} - 1, & \textit{if} \quad \lambda_\beta \geq \lambda_\nu, \\ 1, & \textit{if} \quad \lambda_\beta \leq \lambda_\nu. \end{cases} \tag{2.81}$$

Then

$$\int_A \left[\left|\frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta} \left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{\lambda_\nu} + \left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta} \left|\frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\right] dx$$

$$\leq A_0(R_2)\, 2^{\lambda_\beta/p} \left(\frac{\lambda_\nu}{p}\right)^{(\lambda_\nu/p)} \delta_3^{\lambda_\nu/p} \int_A \left(\left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{p} + \left|\frac{\partial_{R_2}^{\nu} f_2(x)}{\partial r^{\nu}}\right|^{p}\right) dx. \tag{2.82}$$

We need

Theorem 2.15. (see Anastassiou [4]) *All here, as in Theorem* 2.11 *(*$\lambda_\alpha,\ \lambda_\beta \neq 0$*). Denote*

$$\tilde{\gamma_1} := \begin{cases} 2^{((\lambda_\alpha+\lambda_\beta)/\lambda_\nu)-1}, & \textit{if} \quad \lambda_\alpha+\lambda_\beta \geq \lambda_\nu, \\ 1, & \textit{if} \quad \lambda_\alpha+\lambda_\beta \leq \lambda_\nu, \end{cases}$$

and

$$\tilde{\gamma_2} := \begin{cases} 1, & \textit{if} \quad \lambda_\alpha+\lambda_\beta+\lambda_\nu \geq p, \\ 2^{1-((\lambda_\alpha+\lambda_\beta+\lambda_\nu)/p)}, & \textit{if} \quad \lambda_\alpha+\lambda_\beta+\lambda_\nu \leq p. \end{cases} \tag{1}$$

Then, it holds

$$\int_{x_0}^{x} q(w)\left[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\beta} \cdot \left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu}\right.$$

$$+\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\beta} \cdot \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{\lambda_\nu}\Big] dw$$

$$\leq A_0(x)\left(\frac{\lambda_\nu}{(\lambda_\alpha+\lambda_\beta)(\lambda_\alpha+\lambda_\beta+\lambda_\nu)}\right)^{\lambda_\nu/p}\left[\lambda_\alpha^{\lambda_\nu/p}\,\tilde{\gamma}_2+2^{(p-\lambda_\nu)/p}(\,\tilde{\gamma}_1\lambda_\beta)^{\lambda_\nu/p}\right]\cdot$$

$$\left(\int_{x_0}^{x} p(w)\left(\left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^p+\left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^p\right) dw\right)^{((\lambda_\alpha+\lambda_\beta+\lambda_\nu)/p)}, \tag{2.85}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.85), we obtain

Theorem 2.16. *Let all basics as in Theorem* 2.12. *Here,* $\lambda_\nu, \lambda_\alpha, \lambda_\beta > 0$, $p := \lambda_\alpha+\lambda_\beta+\lambda_\nu > 1$. *Also* P_k, $k=1,2$ *as in* (2.74), *and* $A(w)$ *as in* (2.75). *Here it is*

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{\frac{p}{\lambda_\alpha+\lambda_\beta}} dw\right)^{\frac{\lambda_\alpha+\lambda_\beta}{p}}, \tag{2.86}$$

$$\tilde{\gamma_1} := \begin{cases} 2^{((\lambda_\alpha+\lambda_\beta)/\lambda_\nu)-1}, & \text{if } \lambda_\alpha+\lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if } \lambda_\alpha+\lambda_\beta \leq \lambda_\nu. \end{cases} \tag{2.87}$$

Then

$$\int_A \left[\left|\frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta}\left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}+\right.$$

$$\left.\left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta}\left|\frac{\partial_{R_1}^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\right] dx$$

$$\leq A_0(R_2)\left(\frac{\lambda_\nu}{(\lambda_\alpha+\lambda_\beta)p}\right)^{(\lambda_\nu/p)}\left[\lambda_\alpha^{\lambda_\nu/p}+2^{(\lambda_\alpha+\lambda_\beta)/p}(\tilde{\gamma_1}\lambda_\beta)^{\lambda_\nu/p}\right]$$

$$\int_A\left(\left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^p+\left|\frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}}\right|^p\right) dx. \tag{2.88}$$

We need

Theorem 2.17. (see Anastassiou [4]) Let $\nu \geq 3$ *and* $\gamma_1 \geq 1$, *such that* $\nu-\gamma_1 \geq 2$. *Let* $f_1, f_2 \in C_{x_0}^{\nu}([a,b])$ *with*

$$f_1^{(i)}(x_0)=f_2^{(i)}(x_0)=0, \quad i=0,1,\ldots,n-1,$$

$n := [\nu]$. *Here* $x, x_0 \in [a,b] : x \geq x_0$. *Consider also,* $p(t) > 0$, *and* $q(t) \geq 0$ *continuous functions on* $[x_0, b]$. *Let*

$$\lambda_\alpha \geq 0, \quad 0 < \lambda_{\alpha+1} < 1,$$

and $p > 1$. *Denote*

$$\theta_3 := \begin{cases} 2^{\lambda_\alpha/(\lambda_{\alpha+1})}-1, & \text{if } \lambda_\alpha \geq \lambda_{\alpha+1}, \\ 1, & \text{if } \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases}$$

$$L(x) := \left(2\int_{x_0}^{x} (q(w))^{(1/1-(\lambda_{\alpha+1}))} dw\right)^{(1-\lambda_{\alpha+1})}\left(\frac{\theta_3\lambda_{\alpha+1}}{\lambda_\alpha+\lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}, \tag{2.89}$$

and

$$P_1(x) := \int_{x_0}^{x} (x-t)^{(\nu-\gamma_1-1)p/(p-1)} (p(t))^{-1/(p-1)} \, dt, \tag{2.90}$$

$$T(x) := L(x) \cdot \left(\frac{P_1(x)^{((p-1)/p)}}{\Gamma(\nu-\gamma_1)} \right)^{(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{2.91}$$

and

$$\omega_1 := \begin{cases} 2^{1-(\lambda_\alpha+\lambda_{\alpha+1})/p,} & \text{if } \lambda_\alpha+\lambda_{\alpha+1} \le p, \\ 1, & \text{if } \lambda_\alpha+\lambda_{\alpha+1} \ge p, \end{cases} \tag{2.92}$$

$$\Phi(x) := T(x)\, \omega_1. \tag{2.93}$$

Then, it holds

$$\int_{x_0}^{x} q(w) \left[\left| \left(D_{x_0}^{\gamma_1} f_1\right)(w) \right|^{\lambda_\alpha} \cdot \left| \left(D_{x_0}^{\gamma_1+1} f_2\right)(w) \right|^{\lambda_{\alpha+1}} + \right.$$

$$\left. \left| \left(D_{x_0}^{\gamma_1} f_2\right)(w) \right|^{\lambda_\alpha} \cdot \left| \left(D_{x_0}^{\gamma_1+1} f_1\right)(w) \right|^{\lambda_{\alpha+1}} \right] dw$$

$$\le \Phi(x) \left[\int_{x_0}^{x} p(w) \cdot \left| \left(D_{x_0}^{\nu} f_1\right)(w) \right|^p + \left| \left(D_{x_0}^{\nu} f_2\right)(w) \right|^p dw \right]^{((\lambda_\alpha+\lambda_{\alpha+1})/p)}, \tag{2.94}$$

all $x_0 \le x \le b$.

Similarly, by (2. 94), we obtain

Theorem 2.18. *Let* $\nu \ge 3$, $\gamma_1 \ge 1$, *such that* $\nu - \gamma_1 \ge 2$, $n := [\nu]$. *Let* $f_1, f_2 \in C^n(\bar{A})$ *and there exist* $\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}, \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \in C(\bar{A})$, $A := B(0,R_2) - \overline{B(0,R_1)} \subseteq \mathbb{R}^N$, $N \ge 2$. *Furthermore assume* $\frac{\partial^j f_1}{\partial r^j} = \frac{\partial^j f_2}{\partial r^j} = 0$, $j = 0,1,\ldots,n-1$, *on* $\partial B(0,R_1)$. *Let* $\lambda_\alpha > 0$, $0 < \lambda_{\alpha+1} < 1$, *such that* $p := \lambda_\alpha + \lambda_{\alpha+1} > 1$.

Denote

$$\theta_3 := \begin{cases} 2^{(\lambda_\alpha/\lambda_{\alpha+1})} - 1 & \text{if } \lambda_\alpha \ge \lambda_{\alpha+1} \\ 1, & \text{if } \lambda_\alpha \le \lambda_{\alpha+1}, \end{cases} \tag{2.95}$$

$$L(R_2):$$

$$= \left[2 \frac{(1-\lambda_{\alpha+1})}{(N-\lambda_{\alpha+1})} \left(R_2^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} - R_1^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} \right) \right]^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \lambda_{\alpha+1}}{p} \right)^{\lambda_{\alpha+1}}, \tag{2.96}$$

and

$$P_1(R_2) := \int_{R_1}^{R_2} (R_2 - t)^{(\nu-\gamma_1-1)p/(p-1)} \, t^{\left(\frac{1-N}{p-1}\right)} \, dt, \tag{2.97}$$

$$\Phi(R_2) := L(R_2) \left(\frac{P_1(R_2)^{(p-1)}}{(\Gamma(\nu-\gamma_1))^p} \right). \tag{2.98}$$

Then

$$\int_A \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1+1} f_2(x)}{\partial r^{\gamma_1+1}} \right|^{\lambda_{\alpha+1}} + \left| \frac{\partial_{R_1}^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1+1} f_1(x)}{\partial r^{\gamma_1+1}} \right|^{\lambda_{\alpha+1}} \right] dx$$

$$\leq \Phi(R_2) \int_A \left(\left| \frac{\partial^{\nu}_{R_1} f_1(x)}{\partial r^{\nu}} \right|^p + \left| \frac{\partial^{\nu}_{R_1} f_2(x)}{\partial r^{\nu}} \right|^p \right) dx. \tag{2.99}$$

We need

Theorem 2.19. (see Anastassiou [4]) *All here, as in Theorem* 2.11. *Consider the special case* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. *Denote*

$$\tilde{T}(x) := A_0(x) \left(\frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu / p} 2^{(p - 2\lambda_\alpha - 3\lambda_\nu)/p}. \tag{2.100}$$

Then, it holds

$$\int_{x_0}^{x} q(w) \Big[\left| \left(D^{\gamma_1}_{x_0} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D^{\gamma_2}_{x_0} f_2 \right)(w) \right|^{\lambda_\alpha + \lambda_\nu} \left| \left(D^{\nu}_{x_0} f_1 \right)(w) \right|^{\lambda_\nu}$$

$$+ \left| \left(D^{\lambda_2}_{x_0} f_1 \right)(w) \right|^{\lambda_\alpha + \lambda_\nu} \left| \left(D^{\gamma_1}_{x_0} f_2 \right)(w) \right|^{\lambda_\alpha} \left| \left(D^{\nu}_{x_0} f_2 \right)(w) \right|^{\lambda_\nu} \Big] dw$$

$$\leq \tilde{T}(x) \left(\int_{x_0}^{x} p(w) \left(\left| \left(D^{\nu}_{x_0} f_1 \right)(w) \right|^p + \right. \right.$$

$$\left. \left. \left| \left(D^{\nu}_{x_0} f_2 \right)(w) \right|^p \right) dw \right)^{2((\lambda_\alpha + \lambda_\nu)/p)}, \tag{2.101}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.101), we get

Theorem 2.20. *Here all as in Theorem* 2.12. *Consider the case* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$; $\lambda_\alpha \geq 0$, $\lambda_\nu > 0$, $\lambda_\beta > \frac{1}{2}$, $p := 2\lambda_\beta$. *Here* P_k, $k = 1, 2$, *as in* (2.74) *and* $A(w)$ *as in* (2.75). *Set*

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{p/(2\lambda_\alpha + \lambda_\nu)} \right)^{\left(\frac{2\lambda_\alpha + \lambda_\nu}{p} \right)}. \tag{2.102}$$

Also put

$$\tilde{T}(R_2) := A_0(R_2) \left(\frac{\lambda_\nu}{\lambda_\beta} \right)^{\frac{\lambda_\nu}{p}} 2^{\left(- \frac{\lambda_\nu}{p} \right)}. \tag{2.103}$$

Then

$$\int_A \left[\left| \frac{\partial^{\gamma_1}_{R_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial^{\gamma_2}_{R_1} f_2(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial^{\nu}_{R_1} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \right.$$

$$\left. \left| \frac{\partial^{\gamma_2}_{R_1} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial^{\gamma_1}_{R_1} f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial^{\nu}_{R_1} f_2(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] dx$$

$$\leq \tilde{T}(R_2) \int_A \left(\left| \frac{\partial^{\nu}_{R_1} f_1(x)}{\partial r^{\nu}} \right|^p + \left| \frac{\partial^{\nu}_{R_1} f_2(x)}{\partial r^{\nu}} \right|^p \right) dx. \tag{2.104}$$

We need

Theorem 2.21. (see Anastassiou [4]) *Let* $\nu, \gamma_1, \gamma_2 \geq 1$, *such that* $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$ *and* $f_1, f_2 \in C^{\nu}_{x_0}([a, b])$ *with* $f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n :=$

$[\nu]$. *Here,* $x, x_0 \in [a,b] : x \geq x_0$. *Consider* $p(x) \geq 0$ *continuous functions on* $[x_0, b]$. *Let* $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. *Set*

$$\rho(x) := \frac{(x-x_0)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)} \|p(x)\|_\infty}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\beta}}. \tag{2.105}$$

Then, it holds

$$\int_{x_0}^{x} q(w) \Big[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\beta} \left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu} + \left|\left(D_{x_0}^{\lambda_2} f_1\right)(w)\right|^{\lambda_\beta} \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{\lambda_\nu} \Big] dw$$
$$\leq \frac{\rho(x)}{2} \Big[\left\|D_{x_0}^{\nu} f_1\right\|_\infty^{2(\lambda_\alpha+\lambda_\nu)} + \left\|D_{x_0}^{\nu} f_1\right\|_\infty^{2\lambda_\beta} + \left\|D_{x_0}^{\nu} f_2\right\|_\infty^{2\lambda_\beta} + \left\|D_{x_0}^{\nu} f_2\right\|_\infty^{2(\lambda_\alpha+\lambda_\nu)} \Big], \tag{2.106}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.106), we get

Theorem 2.22. *Same basic assumptions as in Theorem* 2.12. *Let* $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. *Set*

$$\rho(R_2) := \frac{R_2^{N-1}(R_2-R_1)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\beta}}. \tag{2.107}$$

Then

$$\int_A \Bigg[\left|\frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha} \left|\frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta} \left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{\lambda_\nu} + \left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta} \left|\frac{\partial_{R_1}^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha} \left|\frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}}\right|^{\lambda_\nu} \Bigg] dx$$
$$\leq \rho(R_2) \frac{\pi^{N/2}}{\Gamma(N/2)} \Bigg[\left\|\frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}}\right\|_\infty^{2(\lambda_\alpha+\lambda_\nu)} + \left\|\frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\beta} + \left\|\frac{\partial_{R_1}^{\nu} f_2}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\beta} \left\|\frac{\partial_{R_1}^{\nu} f_2}{\partial r^{\nu}}\right\|_\infty^{2(\lambda_\alpha+\lambda_\nu)} \Bigg]. \tag{2.108}$$

We need

Theorem 2.23. (see Anastassiou [4]) *(Assume, as in Theorem* 2.21, $\lambda_\beta = 0$.*) It holds*

$$\int_{x_0}^{x} p(w) \Big[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu} + \left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha} \cdot \left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{\lambda_\nu} \Big] dw$$
$$\leq \left(\frac{(x-x_0)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)} \|p(x)\|_\infty}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}} \right) \cdot \Big[\left\|D_{x_0}^{\nu} f_1\right\|_\infty^{\lambda_\alpha+\lambda_\nu} + \left\|D_{x_0}^{\nu} f_2\right\|_\infty^{\lambda_\alpha+\lambda_\nu} \Big], \tag{2.109}$$

all $x_0 \le x \le b$.

Similarly, by (2.109), we derive

Theorem 2.24. *All as in Theorem* 2.22. *Assume* $\lambda_\beta = 0$. *Then*

$$\int_A \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \left| \frac{\partial_{R_1}^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] dx$$

$$\le \frac{2\pi^{N/2}}{\Gamma(N/2)} \frac{R_2^{N-1}(R_2 - R_1)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \left(\left\| \frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}} \right\|_\infty^{\lambda_\alpha + \lambda_\nu} + \left\| \frac{\partial_{R_1}^{\nu} f_2}{\partial r^{\nu}} \right\|_\infty^{\lambda_\alpha + \lambda_\nu} \right). \tag{2.110}$$

We need

Theorem 2.25. (see Anastasiou [4]) *(In relationship to Theorem* 2.21, $\lambda_\beta = \lambda_\alpha + \lambda_\nu$.) *It holds*

$$\int_{x_0}^{x} p(w) \left[\left| \left(D_{x_0}^{\gamma_1} f_1\right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_2\right)(w) \right|^{\lambda_\alpha + \lambda_\nu} \left| \left(D_{x_0}^{\nu} f_1\right)(w) \right|^{\lambda_\nu} \right.$$

$$\left. + \left| \left(D_{x_0}^{\gamma_2} f_1\right)(w) \right|^{\lambda_\alpha + \lambda_\nu} \left| \left(D_{x_0}^{\gamma_1} f_2\right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\nu} f_2\right)(w) \right|^{\lambda_\nu} \right] dw$$

$$\le \left(\frac{(x - x_0)^{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)} \cdot \|p(x)\|_\infty}{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \right) \cdot$$

$$\frac{1}{(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\alpha + \lambda_\nu}} \cdot \left(\left\| D_{x_0}^{\nu} f_1 \right\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \left\| \left(D_{x_0}^{\nu} f_2\right) \right\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \right), \tag{2.111}$$

all $x_0 \le x \le b$.

Similarly, by (2.111), we derive

Theorem 2.26. *All as in Theorem* 2.22. *Assume* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. *Then*

$$\int_A \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha + \lambda_\nu} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \right.$$

$$\left. \left| \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha + \lambda_\nu} \left| \frac{\partial_{R_1}^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] dx$$

$$\le \left(\frac{2\pi^{N/2}}{\Gamma(N/2)} \right)$$

$$\left(\frac{R_2^{N-1}(R_2 - R_1)^{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)}}{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \right) \cdot$$

$$\frac{1}{(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\alpha + \lambda_\nu}} \left(\left\| \frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}} \right\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \left\| \frac{\partial_{R_1}^{\nu} f_2}{\partial r^{\nu}} \right\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \right). \tag{2.112}$$

We need

Theorem 2.27. (see Anastassiou [4]) *(In relationship to Theorem* 2.21*,* $\lambda_\nu = 0,\ \lambda_\alpha = \lambda_\beta$*.) It holds*

$$\int_{x_0}^{x} p(w)\Big[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha}\cdot\left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\alpha}+$$

$$\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\alpha}\cdot\left|\left(D_{x_0}^{\gamma_1} f_2\right)(w)\right|^{\lambda_\alpha}\Big]dw$$

$$\le \rho^*(x)\left[\left\|D_{x_0}^{\nu} f_1\right\|_\infty^{2\lambda_\alpha}+\left\|D_{x_0}^{\nu} f_2\right\|_\infty^{2\lambda_\alpha}\right], \tag{2.113}$$

all $x_0 \le x \le b$.

Here

$$\rho^*(x):$$

$$=\left(\frac{(x-x_0)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)}\cdot\|p(x)\|_\infty}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\alpha}}\right). \tag{2.114}$$

We get, by (2.113), the result

Theorem 2.28. *All as in Theorem* 2.22*. Assume* $\lambda_\nu = 0,\ \lambda_\alpha = \lambda_\beta$*. Then*

$$\int_A\left[\left|\frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha}+\left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\right]dx$$

$$\le\left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right)\rho^*(R_2)\left[\left\|\frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\alpha}+\left\|\frac{\partial_{R_1}^{\nu} f_2}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\alpha}\right], \tag{2.115}$$

where

$$\rho^*(R_2):$$

$$=\left(\frac{R_2^{N-1}(R_2-R_1)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)}}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\alpha}}\right). \tag{2.116}$$

We need

Theorem 2.29. (see Anastassiou [4]) *(In relationship to Theorem* 2.21*,* $\lambda_\alpha = 0,\ \lambda_\beta = \lambda_\nu$*.) It holds*

$$\int_{x_0}^{x} p(w)\Big[\left|\left(D_{x_0}^{\gamma_2} f_2\right)(w)\right|^{\lambda_\beta}\cdot\left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\beta}+$$

$$\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\beta}\cdot\left|\left(D_{x_0}^{\nu} f_2\right)(w)\right|^{\lambda_\beta}\Big]dw$$

$$\le\left(\frac{(x-x_0)^{(\nu\lambda_\beta-\gamma_2\lambda_\beta+1)}\cdot\|p(x)\|_\infty}{(\nu\lambda_\beta-\gamma_2\lambda_\beta+1)(\Gamma(\nu-\gamma_2+1))^{\lambda_\beta}}\right)\cdot$$

$$\left[\left\|D_{x_0}^{\nu} f_1\right\|_\infty^{2\lambda_\beta}+\left\|\left(D_{x_0}^{\nu} f_2\right)\right\|_\infty^{2\lambda_\beta}\right], \tag{2.117}$$

all $x_0 \le x \le b$.

We get, by (2.117), the next result.

Theorem 2.30. *All as in Theorem* 2.22*. Assume* $\lambda_\alpha = 0,\ \lambda_\beta = \lambda_\nu$*. Then*

$$\int_A\left[\left|\frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta}\left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{\lambda_\beta}+\left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\beta}\left|\frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}}\right|^{\lambda_\beta}\right]dx$$

$$\le \left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right)\left(\frac{R_2^{N-1}(R_2-R_1)^{(\nu\lambda_\beta-\gamma_2\lambda_\beta+1)}}{(\nu\lambda_\beta-\gamma_2\lambda_\beta+1)(\Gamma(\nu-\gamma_2+1))^{\lambda_\beta}}\right)\left[\left\|\frac{\partial_{R_1}^{\nu}f_1}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\beta}\right.$$

$$\left.+\left\|\frac{\partial_{R_1}^{\nu}f_2}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\beta}\right]. \tag{2.118}$$

We make

Assumption 2.1. Let $\nu \ge 1$, $n := [\nu]$, $f_j \in C^n(\bar{A})$, $j = 1, \dots, M \in \mathbb{N}$, and there exist $\frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}} \in C(\bar{A})$, $A := B(0, R_2) - \overline{B(0, R_1)} \subseteq \mathbb{R}^N$, $N \ge 2$. Furthermore assume that $\frac{\partial^i f_j}{\partial r^i} = 0$, $i = 0, 1, \dots, n-1$, on $\partial B(0, R_1)$, for all $j = 1, \dots, M$.

Next we present a set of multivariate fractional Opial type inequalities involving several functions over the shell.

We need

Theorem 2.31. (see Anastassiou [3]) *Let $\nu, \gamma_1, \gamma_2 \ge 1$, such that $\nu-\gamma_1 \ge 1$, $\nu-\gamma_2 \ge 1$ and $f_j \in C_{x_0}^{\nu}([a,b])$ with $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \dots, n-1$, $n := [\nu]$, $j = 1, \dots, M \in \mathbb{N}$. Here, $x, x_0 \in [a,b] : x \ge x_0$. Consider also $p(t) > 0$, and $q(t) \ge 0$ continuous functions on $[x_0, b]$. Let $\lambda_\nu > 0$ and $\lambda_\alpha, \lambda_\beta \ge 0$ such that $\lambda_\nu < p$, where $p > 1$. Set*

$$P_k(w) := \int_{x_0}^{w} (w-t)^{\frac{(\nu-\gamma_k-1)p}{p-1}} (p(t))^{-\frac{1}{p-1}}\, dt, \quad k = 1, 2,\ x_0 \le w \le b; \tag{2.119}$$

$$A(w) := \frac{q(w)\cdot (P_1(w))^{\lambda_\alpha\left(\frac{p-1}{p}\right)}\cdot (P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)}(p(w))^{-\frac{\lambda_\nu}{p}}}{(\Gamma(\nu-\gamma_1))^{\lambda_\alpha}\cdot(\Gamma(\nu-\gamma_2))^{\lambda_\beta}}; \tag{2.120}$$

$$A_0(x) := \left(\int_{x_0}^{x} A(w)^{\frac{p}{p-\lambda_\nu}}\, dw\right)^{\frac{p-\lambda_\nu}{p}}. \tag{2.121}$$

Call

$$\varphi_1(x) := \left(A_0(x)\,\big|_{\lambda_\beta=0}\right)\cdot\left(\frac{\lambda_\nu}{\lambda_\alpha+\lambda_\nu}\right)^{\frac{\lambda_\nu}{p}}, \tag{2.122}$$

$$\delta_1^* := \begin{cases} M^{1-\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha+\lambda_\nu \le p,\\ 2^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}-1, & \text{if } \lambda_\alpha+\lambda_\nu \ge p. \end{cases} \tag{2.123}$$

If $\lambda_\beta = 0$, we obtain that

$$\int_{x_0}^{x} q(w)\left(\sum_{j=1}^{M}\left|\left(D_{x_0}^{\gamma_1}f_j\right)(w)\right|^{\lambda_\alpha}\cdot\left|\left(D_{x_0}^{\nu}f_j\right)(w)\right|^{\lambda_\nu}\right)dw$$

$$\le \delta_1^*\cdot\varphi_1(x)\cdot\left[\int_{x_0}^{x} p(w)\left(\sum_{j=1}^{M}\left|\left(D_{x_0}^{\nu}f_j\right)(w)\right|^{p}\right)dw\right]^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)} \tag{2.124}$$

all $x_0 \le x \le b$.

Similarly, by (2.124), we derive

Theorem 2.32. *Let* $f_j, j = 1, \ldots, M,$ *as in Assumption* 2.1. *Let* $\gamma_1, \gamma_2 \geq 1,$ *such that* $\nu - \gamma_1 \geq 1,\ \nu - \gamma_2 \geq 1$ *Let* $\lambda_\nu > 0,$ *and* $\lambda_\alpha > 0;\ \lambda_\beta \geq 0,\ p := \lambda_\alpha + \lambda_\nu > 1.$ *Set*

$$P_k(w) := \int_{R_1}^{w} (w-t)^{(\nu-\gamma_k-1)\frac{p}{(p-1)}} t^{\left(\frac{1-N}{p-1}\right)} dt, \tag{2.125}$$

$k = 1, 2, \quad R_1 \leq w \leq R_2,$

$$A(w) := \frac{w^{(N-1)\left(1-\frac{\lambda_\nu}{p}\right)} (P_1(w))^{\lambda_\alpha\left(\frac{p-1}{p}\right)} (P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)}}{(\Gamma(\nu-\gamma_1))^{\lambda_\alpha} (\Gamma(\nu-\gamma_2))^{\lambda_\beta}}, \tag{2.126}$$

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{p/\lambda_\alpha} dw\right)^{\lambda_\alpha/p}. \tag{2.127}$$

Take the case of $\lambda_\beta = 0.$ *Then*

$$\sum_{j=1}^{M} \int_A \left|\frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha} \left|\frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}}\right|^{\lambda_\nu} dx$$

$$\leq \left(A_0(R_2)|_{\lambda_\beta=0}\right) \left(\frac{\lambda_\nu}{p}\right)^{\left(\frac{\lambda_\nu}{p}\right)} \left[\sum_{j=1}^{M} \left(\int_A \left|\frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}}\right|^p dx\right)\right]. \tag{2.128}$$

We need

Theorem 2.33. (see Anastassiou [3]) *All here as in Theorem* 2.31. *Denote*

$$\delta_3 := \begin{cases} 2^{\frac{\lambda_\beta}{\lambda_\nu}} - 1, & \text{if} \quad \lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if} \quad \lambda_\beta \leq \lambda_\nu, \end{cases} \tag{2.129}$$

$$\varepsilon_2 := \begin{cases} 1, & \text{if} \quad \lambda_\nu + \lambda_\beta \geq p, \\ M^{1-\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}, & \text{if} \quad \lambda_\nu + \lambda_\beta \leq p, \end{cases} \tag{2.130}$$

and

$$\varphi_2(x) := (A_0(x)|_{\lambda_\alpha=0}) 2^{\left(\frac{p-\lambda_\nu}{p}\right)} \left(\frac{\lambda_\nu}{\lambda_\beta+\lambda_\nu}\right)^{\lambda_\nu/p} \delta_3^{\lambda_\nu/p}. \tag{2.131}$$

If $\lambda_\alpha = 0,$ *then is holds*

$$\int_{x_0}^{x} q(w) \left\{\left\{\sum_{j=1}^{M-1} \left[\left|\left(D_{x_0}^{\gamma_2} f_{j+1}\right)(w)\right|^{\lambda_\beta} \left|\left(D_{x_0}^{\nu} f_j\right)(w)\right|^{\lambda_\nu}\right.\right.\right.$$

$$\left.\left. + \left|\left(D_{x_0}^{\gamma_2} f_j\right)(w)\right|^{\lambda_\beta} \left|\left(D_{x_0}^{\nu} f_{j+1}\right)(w)\right|^{\lambda_\nu}\right]\right\}$$

$$+ \left[\left|\left(D_{x_0}^{\gamma_2} f_M\right)(w)\right|^{\lambda_\beta} \left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu}\right.$$

$$\left.\left. + \left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\beta} \left|\left(D_{x_0}^{\nu} f_M\right)(w)\right|^{\lambda_\nu}\right]\right\} dw$$

$$\leq 2^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)} \varepsilon_2 \varphi_2(x) \cdot \left\{\int_{x_0}^{x} p(w) \cdot \left[\sum_{j=1}^{M} \left|\left(D_{x_0}^{\nu} f_j\right)(w)\right|^p\right] dw\right\}^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}, \tag{2.132}$$

$x \geq x_0.$

Similarly, by (2.132), we obtain

Theorem 2.34. *All basic assumptions as in Theorem* 2.32. *Let* $\lambda_\nu > 0$, $\lambda_\alpha = 0$*;* $\lambda_\beta > 0$, $p := \lambda_\nu + \lambda_\beta > 1$, P_2 *defined by* (2.125).

Now it is

$$A(w) := \frac{w^{(N-1)\left(1-\frac{\lambda_\nu}{p}\right)} (P_2(w))^{\lambda_\beta\left(\frac{p-1}{p}\right)}}{(\Gamma(\nu-\gamma_2))^{\lambda_\beta}}, \tag{2.133}$$

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{p/\lambda_\beta}\, dw\right)^{\lambda_\beta/p}. \tag{2.134}$$

Denote

$$\delta_3 := \begin{cases} 2^{\lambda_\beta/\lambda_\nu} - 1, & \text{if } \lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if } \lambda_\beta \leq \lambda_\nu. \end{cases} \tag{2.135}$$

Call

$$\varphi_2(R_2) := A_0(R_2) 2^{\lambda_\beta/p} \left(\frac{\lambda_\nu}{p}\right)^{\lambda_\nu/p} \delta_3^{\lambda_\nu/p}. \tag{2.136}$$

Then

$$\int_A \left\{ \left\{ \sum_{j=1}^{M-1} \left[\left| \frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \left| \frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_{j+1}(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] \right\} + \left[\left| \frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \left| \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_M(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] \right\} dx$$

$$\leq 2\varphi_2(R_2) \left[\sum_{j=1}^{M} \left(\int_A \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^p dx \right) \right]. \tag{2.137}$$

We need

Theorem 2.35. (see Anastassiou [3]) *All here as in Theorem* 2.31 $(\lambda_\alpha, \lambda_\beta \neq 0)$.

$$\tilde{\gamma}_1 := \begin{cases} 2^{\left(\frac{\lambda_\alpha+\lambda_\beta}{\lambda_\nu}\right)} - 1, & \text{if } \lambda_\alpha + \lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if } \lambda_\alpha + \lambda_\beta \leq \lambda_\nu, \end{cases} \tag{2.138}$$

and

$$\tilde{\gamma}_2 := \begin{cases} 1, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \geq p, \\ 2^{1-\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \leq p. \end{cases} \tag{2.139}$$

Set

$$\varphi_3(x) := A_0(x) \cdot \left(\frac{\lambda_\nu}{(\lambda_\alpha+\lambda_\beta)(\lambda_\alpha+\lambda_\beta+\lambda_\nu)}\right)^{\lambda_\nu/p} \tag{2.140}$$

and

$$\varepsilon_3 := \begin{cases} 1, & \text{if} \quad \lambda_\alpha + \lambda_\beta + \lambda_\nu \geq p, \\ M^{1-\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, & \text{if} \quad \lambda_\alpha + \lambda_\beta + \lambda_\nu \leq p, \end{cases} \tag{2.141}$$

Then it holds

$$\int_{x_0}^{x} q(w) \left[\sum_{j=1}^{M-1} \left[\left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_{j+1} \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_j \right)(w) \right|^{\lambda_\nu} \right.\right.$$

$$\left. + \left| \left(D_{x_0}^{\gamma_2} f_j \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\gamma_1} f_{j+1} \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\nu} f_{j+1} \right)(w) \right|^{\lambda_\nu} \right]$$

$$+ \left[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_M \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_1 \right)(w) \right|^{\lambda_\nu} \right.$$

$$\left.\left. + \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\gamma_1} f_M \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\nu} f_M \right)(w) \right|^{\lambda_\nu} \right] \right] dw$$

$$\leq 2^{\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)} \varepsilon_3 \varphi_3(x) \cdot \left\{ \int_{x_0}^{x} p(w) \left[\sum_{j=1}^{M} \left| \left(D_{x_0}^{\nu} f_j \right)(w) \right|^p \right] dw \right\}^{\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, \tag{2.142}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.142), we obtain

Theorem 2.36. *All basic assumptions as in Theorem* 2.32. *Here* $\lambda_\nu,\ \lambda_\alpha, \lambda_\beta > 0$, $p := \lambda_\alpha + \lambda_\beta + \lambda_\nu > 1$, P_k *as in* (2.125). A *as in* (2.126). *Here*

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{p/(\lambda_\alpha+\lambda_\beta)}, \right)^{\frac{\lambda_\alpha+\lambda_\beta}{p}}, \tag{2.143}$$

$$\tilde{\gamma_1} := \begin{cases} 2^{\left(\frac{\lambda_\alpha+\lambda_\beta}{\lambda_\nu}\right)} - 1, & \text{if} \quad \lambda_\alpha + \lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if} \quad \lambda_\alpha + \lambda_\beta \leq \lambda_\nu, \end{cases} \tag{2.144}$$

Put

$$\varphi_3(R_2) := A_0(R_2) \left(\frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)p} \right)^{(\lambda_\nu/p)} \left[\lambda_\alpha^{(\lambda_\nu/p)} + 2^{\left(\frac{\lambda_\alpha+\lambda_\beta}{p}\right)} (\tilde{\gamma_1}\lambda_\beta)^{\left(\frac{\lambda_\nu}{p}\right)} \right]. \tag{2.145}$$

Then

$$\int_A \left[\sum_{j=1}^{M-1} \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \right.\right.$$

$$\left. \left| \frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_{j+1}(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right]$$

$$+ \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \right.$$

$$\left.\left. \left| \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\gamma_1} f_M(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_M(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] \right] dx$$

$$\leq 2\varphi_3(R_2) \left[\sum_{j=1}^{M} \left(\int_A \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^p dx \right) \right]. \tag{2.146}$$

We need

Theorem 2.37. (see Anastassiou [3]) *Let* $\nu \geq 3$, *and* $\gamma_1 \geq 1$, *such that* $\nu - \gamma_1 \geq 2$. *Let*
$f_j \in C_{x_0}^{\nu}([a,b])$ *with* $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. *Here,* $x, x_0 \in [a,b] : x \geq x_0$. *Consider also* $p(t) > 0$, *and* $q(t) \geq 0$ *continuous functions on* $[x_0, b]$. *Let* $\lambda_\alpha \geq 0$, $0 < \lambda_{\alpha+1} < 1$ *and* $p > 1$. *Denote*

$$\theta_3 := \begin{cases} 2^{\left(\frac{\lambda_\alpha}{\lambda_{\alpha+1}}\right)} - 1, & if \quad \lambda_\alpha \geq \lambda_{\alpha+1}, \\ 1, & if \quad \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases} \tag{2.147}$$

$$L(x) := \left(2 \int_{x_0}^{x} (q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)} dw\right)^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}, \tag{2.148}$$

and

$$P_1(x) := \int_{x_0}^{x} (x-t)^{\left(\frac{(\nu-\gamma_1-1)p}{p-1}\right)} (p(t))^{-\frac{1}{p-1}} dt, \tag{2.149}$$

$$T(x) := L(x) \cdot \left(\frac{P_1(x)^{\left(\frac{p-1}{p}\right)}}{\Gamma(\nu - \gamma_1)}\right)^{(\lambda_\alpha + \lambda_{\alpha+1})}, \tag{2.150}$$

and

$$\omega_1 := \left\{ \begin{array}{ll} 2^{1-\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, & if \ \ \lambda_\alpha + \lambda_{\alpha+1} \leq p, \\ 1, & if \ \ \lambda_\alpha + \lambda_{\alpha+1} \geq p, \end{array} \right\}, \tag{2.151}$$

$$\Phi(x) := T(x)\, \omega_1.$$

Also put

$$\varepsilon_4 := \left\{ \begin{array}{ll} 1, & if \ \ \lambda_\alpha + \lambda_{\alpha+1} \geq p, \\ M^{1-\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, & if \ \ \lambda_\alpha + \lambda_{\alpha+1} \leq p \end{array} \right\}. \tag{2.152}$$

Then it holds

$$\begin{aligned} &\int_{x_0}^{x} q(w) \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[\left| \left(D_{x_0}^{\gamma_1} f_j\right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_1+1} f_{j+1}\right)(w) \right|^{\lambda_{\alpha+1}} \\ &\quad + \left| \left(D_{x_0}^{\gamma_1} f_{j+1}\right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_1+1} f_j\right)(w) \right|^{\lambda_{\alpha+1}} \Big] \Bigg\} \\ &\quad + \Big[\left| \left(D_{x_0}^{\gamma_1} f_1\right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_1+1} f_M\right)(w) \right|^{\lambda_{\alpha+1}} \\ &\quad + \left| \left(D_{x_0}^{\gamma_1} f_M\right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_1+1} f_1\right)(w) \right|^{\lambda_{\alpha+1}} \Big] \Bigg\} dw \\ &\leq 2^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)} \varepsilon_4 \Phi(x) \left[\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} \left| \left(D_{x_0}^{\nu} f_j\right)(w) \right|^p \right) dw \right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, \end{aligned} \tag{2.153}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.153), we get

Theorem 2.38. *Let all as in Assumptions 2.1. Here* $\nu \geq 3$, $\gamma_1 \geq 1$ *such that* $\nu - \gamma_1 \geq 2$. *Let* $\lambda_\alpha > 0$, $0 < \lambda_{\alpha+1} < 1$, *such that* $p := \lambda_\alpha + \lambda_{\alpha+1} > 1$. *Denote*

$$\theta_3 := \begin{cases} 2^{(\lambda_\alpha/\lambda_{\alpha+1})} - 1, & \text{if} \quad \lambda_\alpha \geq \lambda_{\alpha+1}, \\ 1, & \text{if} \quad \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases} \tag{2.154}$$

$$L(R_2) := \left[2 \left(\frac{1-\lambda_{\alpha+1}}{N-\lambda_{\alpha+1}} \right) \left(R_2^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} - R_1^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} \right) \right]^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}} \right)^{\lambda_{\alpha+1}}, \tag{2.155}$$

and

$$P(R_2) := \int_{R_1}^{R_2} (R_2 - t)^{(\nu-\gamma_1-1)\left(\frac{p}{p-1}\right)} t^{\left(\frac{1-N}{p-1}\right)} dt, \tag{2.156}$$

$$\Phi(R_2) := L(R_2) \left(\frac{P_1(R_2)^{(p-1)}}{(\Gamma(\nu-\gamma_1))^p} \right). \tag{2.157}$$

Then

$$\int_A \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Bigg[\left| \frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1+1} f_{j+1}(x)}{\partial r^{\gamma_1+1}} \right|^{\lambda_{\alpha+1}} + \left| \frac{\partial_{R_1}^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1+1} f_j(x)}{\partial r^{\gamma_1+1}} \right|^{\lambda_{\alpha+1}} \Bigg] \Bigg\}$$
$$+ \Bigg[\left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1+1} f_M(x)}{\partial r^{\gamma_1+1}} \right|^{\lambda_{\alpha+1}} + \left| \frac{\partial_{R_1}^{\gamma_1} f_M(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1+1} f_1(x)}{\partial r^{\gamma_1+1}} \right|^{\lambda_{\alpha+1}} \Bigg] \Bigg\} dx \leq$$
$$2\Phi(R_2) \left[\sum_{j=1}^{M} \left(\int_A \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^p dx \right) \right]. \tag{2.158}$$

We need

Theorem 2.39. (see Anastassiou [3]) *All here as in Theorem* 2.31. *Consider the special case* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. *Denote*

$$\tilde{T}(x) := A_0(x) \left(\frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\frac{\lambda_\nu}{p}} 2^{\left(\frac{p - 2\lambda_\alpha - 3\lambda_\nu}{p} \right)}, \tag{2.159}$$

$$\varepsilon_5 := \left\{ \begin{array}{ll} 1, & \text{if} \;\; 2(\lambda_\alpha + \lambda_\nu) \geq p, \\ M^{1-\left(\frac{2(\lambda_\alpha+\lambda_\nu)}{p}\right)}, & \text{if} \;\; 2(\lambda_\alpha + \lambda_\nu) \leq p \end{array} \right\}. \tag{2.160}$$

Then it holds

$$\int_{x_0}^{x} q(w) \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[\left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_{j+1} \right)(w) \right|^{\lambda_\alpha+\lambda_\nu} \left| \left(D_{x_0}^{\nu} f_j \right)(w) \right|^{\lambda_\nu}$$

$$+\left|\left(D_{x_0}^{\gamma_2} f_j\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\gamma_1} f_{j+1}\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\nu} f_{j+1}\right)(w)\right|^{\lambda_\nu}\Big]\Big\}$$

$$+\Big[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\gamma_2} f_M\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu}$$

$$+\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\gamma_1} f_M\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\nu} f_M\right)(w)\right|^{\lambda_\nu}\Big]\Big\}\,dw$$

$$\leq 2^{2\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}\varepsilon_5\,\tilde{T}(x)\left[\int_{x_0}^{x} p(w)\left(\sum_{j=1}^{M}\left|\left(D_{x_0}^{\nu} f_j\right)(w)\right|^p\right)dw\right]^{\left(2\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)\right)}, \tag{2.161}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.161), we have

Theorem 2.40. *Here all as in Theorem* 2.32. *Consider the case* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$; $\lambda_\alpha \geq 0,\ \lambda_\nu > 0,\ \lambda_\beta > 1/2,\ p := 2\lambda_\beta$. *Here* $P_k,\ k = 1, 2,$ *as in* (2.125) *and* A *as in* (2.126). *Set*

$$A_0(R_2) := \left(\int_{R_1}^{R_2} (A(w))^{p/(2\lambda_\alpha+\lambda_\nu)}dw\right)^{(2\lambda_\alpha+\lambda_\nu/p)}. \tag{2.162}$$

Also put

$$\tilde{T}(R_2) := A_0(R_2)\left(\frac{\lambda_\nu}{\lambda_\beta}\right)^{(\lambda_\nu/p)} 2^{-(\lambda_\nu/p)}. \tag{2.163}$$

Then

$$\int_A \Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Bigg[\left|\frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}$$

$$+\left|\frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\nu} f_{j+1}(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\Bigg]\Bigg\}$$

$$+\Bigg[\left|\frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}+$$

$$\left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\gamma_1} f_M(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\nu} f_M(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\Bigg]\Bigg\}\,dx$$

$$\leq 2\,\tilde{T}(R_2)\left[\sum_{j=1}^{M}\left(\int_A\left|\frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}}\right|^p dx\right)\right]. \tag{2.164}$$

We need

Theorem 2.41. (see Anastassiou [3]) *Let* $\nu, \gamma_1, \gamma_2 \geq 1,$ *such that* $\nu - \gamma_1 \geq 1,\ \nu - \gamma_2 \geq 1$ *and* $f_j \in C_{x_0}^{\nu}([a, b])$ *with* $f_j^{(i)}(x_0) = 0,\ i = 0, 1, \ldots, n-1,\ n := [\nu],\ j = 1, \ldots, M \in \mathbb{N}$. *Here,* $x, x_0 \in [a, b] : x \geq x_0$. *Consider* $p(x) \geq 0$ *continuous functions on* $[x_0, b]$. *Let* $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. *Set*

$$\rho(x) :$$

$$= \frac{(x-x_0)^{(\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\beta-\gamma_2\lambda_\beta+1)}\|p(x)\|_\infty}{(\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\beta-\gamma_2\lambda_\beta+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\beta}}. \tag{2.165}$$

Then it holds

$$\int_{x_0}^{x} p(w) \left\{ \left\{ \sum_{j=1}^{M-1} \left[\left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_{j+1} \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_j \right)(w) \right|^{\lambda_\nu} + \right.\right.\right.$$

$$\left. \left. \left| \left(D_{x_0}^{\gamma_2} f_j \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\gamma_1} f_{j+1} \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\nu} f_{j+1} \right)(w) \right|^{\lambda_\nu} \right] \right\}$$

$$+ \left[\left| \left(D_{x_0}^{\gamma_1} f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\gamma_2} f_M \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_1 \right)(w) \right|^{\lambda_\nu} + \right.$$

$$\left. \left. \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\gamma_1} f_M \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\nu} f_M \right)(w) \right|^{\lambda_\nu} \right] \right\} dw$$

$$\leq \rho(x) \left\{ \sum_{j=1}^{M} \left\{ \left\| \left(D_{x_0}^{\nu} f_j \right) \right\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \left\| \left(D_{x_0}^{\nu} f_j \right) \right\|_\infty^{2\lambda_\beta} \right\} \right\}, \tag{2.166}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.166), we have

Theorem 2.42. *All as in Assumption* 2.1. *Let* $\gamma_1, \gamma_2 \geq 1$, *such that* $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$; λ_α, λ_β, $\lambda_\nu \geq 0$. *Set*

$$\rho(R_2) := \frac{R_2^{N-1} (R_2 - R_1)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{2.167}$$

Then

$$\int_A \left\{ \left\{ \sum_{j=1}^{M-1} \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right.\right.\right.$$

$$\left.\left. + \left| \frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_{j+1}(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] \right\}$$

$$+ \left[\left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \right.$$

$$\left.\left. \left| \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\gamma_1} f_M(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\nu} f_M(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] \right\} dx$$

$$\leq \frac{2\pi^{N/2}}{\Gamma(N/2)} \rho(R_2) \left\{ \sum_{j=1}^{M} \left\{ \left\| \frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}} \right\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \left\| \frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}} \right\|_\infty^{2\lambda_\beta} \right\} \right\}. \tag{2.168}$$

We need

Theorem 2.43. (see Anastassiou [3]) *(As in Theorem* 2.41, $\lambda_\beta = 0$.*) It holds*

$$\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} \left| \left(D_{x_0}^{\gamma_1} f_j \right)(w) \right|^{\lambda_\alpha} \left| \left(D_{x_0}^{\nu} f_j \right)(w) \right|^{\lambda_\nu} \right) dw$$

$$\leq \left(\frac{(x - x_0)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)} \|p(x)\|_\infty}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \right) \cdot \left(\sum_{j=1}^{M} \left\| D_{x_0}^{\nu} f_j \right\|_\infty^{\lambda_\alpha + \lambda_\nu} \right), \tag{2.169}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.169), we obtain

Theorem 2.44. *Here all as in Theorem* 2.42. *Case of* $\lambda_\beta = 0$. *Then*

$$\sum_{j=1}^{M}\left(\int_A \left|\frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}}\right|^{\lambda_\nu} dx\right)$$

$$\leq \left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right)\left(\frac{R_2^{N-1}(R_2-R_1)^{(\nu\lambda_\alpha-\gamma_1\lambda_\alpha+1)}}{(\nu\lambda_\alpha-\gamma_1\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}}\right)\left(\sum_{j=1}^{M}\left\|\frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}}\right\|_\infty^{\lambda_\alpha+\lambda_\nu}\right). \quad (2.170)$$

We need

Theorem 2.45. (see Anastassiou [3]) *(As in Theorem* 2.41, $\lambda_\beta = \lambda_\alpha + \lambda_\nu$.*) It holds*

$$\int_{x_0}^{x} p(w)\left\{\left\{\sum_{j=1}^{M-1}\left[\left|\left(D_{x_0}^{\gamma_1} f_j\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\gamma_2} f_{j+1}\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\nu} f_j\right)(w)\right|^{\lambda_\nu}\right.\right.\right.$$

$$\left.\left.+\left|\left(D_{x_0}^{\gamma_2} f_j\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\gamma_1} f_{j+1}\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\nu} f_{j+1}\right)(w)\right|^{\lambda_\nu}\right]\right\}$$

$$+\left[\left|\left(D_{x_0}^{\gamma_1} f_1\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\gamma_2} f_M\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\nu} f_1\right)(w)\right|^{\lambda_\nu}+\right.$$

$$\left.\left.\left|\left(D_{x_0}^{\gamma_2} f_1\right)(w)\right|^{\lambda_\alpha+\lambda_\nu}\left|\left(D_{x_0}^{\gamma_1} f_M\right)(w)\right|^{\lambda_\alpha}\left|\left(D_{x_0}^{\nu} f_M\right)(w)\right|^{\lambda_\nu}\right]\right\} dw$$

$$\leq \left(\frac{2(x-x_0)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\nu-\gamma_2\lambda_\alpha-\gamma_2\lambda_\nu+1)}\|p(x)\|_\infty}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\nu-\gamma_2\lambda_\alpha-\gamma_2\lambda_\nu+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}}\right)$$

$$\frac{1}{(\Gamma(\nu-\gamma_2+1))^{(\lambda_\alpha+\lambda_\nu)}}\cdot\left(\sum_{j=1}^{M}\left\|D_{x_0}^{\nu} f_j\right\|_\infty^{2(\lambda_\alpha+\lambda_\nu)}\right), \quad (2.171)$$

all $x_0 \leq x \leq b$.

Similarly, by (2.171), we derive

Theorem 2.46. *Here all as in Theorem* 2.42. *Case of* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. *Then*

$$\int_A\left\{\left\{\sum_{j=1}^{M-1}\left[\left|\frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\right.\right.\right.$$

$$\left.\left.+\left|\frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\nu} f_{j+1}(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\right]\right\}$$

$$+\left[\left|\frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}+\right.$$

$$\left.\left.\left|\frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha+\lambda_\nu}\left|\frac{\partial_{R_1}^{\gamma_1} f_M(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial_{R_1}^{\nu} f_M(x)}{\partial r^{\nu}}\right|^{\lambda_\nu}\right]\right\} dx$$

$$\leq \frac{4\pi^{N/2}}{\Gamma(N/2)}\cdot$$

$$\left(\frac{R_2^{N-1}(R_2-R_1)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\nu-\gamma_2\lambda_\alpha-\gamma_2\lambda_\nu+1)}}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\nu-\gamma_2\lambda_\alpha-\gamma_2\lambda_\nu+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}}\right).$$

$$\frac{1}{(\Gamma(\nu-\gamma_2+1))^{(\lambda_\alpha+\lambda_\nu)}}\cdot\left(\sum_{j=1}^{M}\left\|\frac{\partial^{\nu}_{R_1}f_j}{\partial r^{\nu}}\right\|_\infty^{2(\lambda_\alpha+\lambda_\nu)}\right). \tag{2.172}$$

We need

Theorem 2.47. (see Anastassiou [3]) *(As in Theorem* 2.41, $\lambda_\nu=0,\ \lambda_\alpha=\lambda_\beta$*). It holds*

$$\int_{x_0}^{x}p(w)\left\{\left\{\sum_{j=1}^{M-1}\left[\left|\left(D^{\gamma_1}_{x_0}f_j\right)(w)\right|^{\lambda_\alpha}\left|\left(D^{\gamma_2}_{x_0}f_{j+1}\right)(w)\right|^{\lambda_\alpha}\right.\right.\right.$$
$$\left.\left.+\left|\left(D^{\gamma_2}_{x_0}f_j\right)(w)\right|^{\lambda_\alpha}\left|\left(D^{\gamma_1}_{x_0}f_{j+1}\right)(w)\right|^{\lambda_\alpha}\right]\right\}$$
$$+\left[\left|\left(D^{\gamma_1}_{x_0}f_1\right)(w)\right|^{\lambda_\alpha}\left|\left(D^{\gamma_2}_{x_0}f_M\right)(w)\right|^{\lambda_\alpha}\right.$$
$$\left.\left.+\left|\left(D^{\gamma_2}_{x_0}f_1\right)(w)\right|^{\lambda_\alpha}\left|\left(D^{\gamma_1}_{x_0}f_M\right)(w)\right|^{\lambda_\alpha}\right]\right\}dw$$
$$\le 2\,\rho^*(x)\left[\sum_{j=1}^{M}\left\|D^{\nu}_{x_0}f_j\right\|_\infty^{2\lambda_\alpha}\right], \tag{2.173}$$

all $x_0\le x\le b$. *Here we have*

$$\rho^*(x):=\left(\frac{(x-x_0)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)}\|p(x)\|_\infty}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\alpha}}\right). \tag{2.174}$$

Similarly, by (2.174), we derive

Theorem 2.48. *Here all as in Theorem* 2.42*. Case of* $\lambda_\nu=0,\ \lambda_\alpha=\lambda_\beta$.

Then

$$\int_A\left\{\left\{\sum_{j=1}^{M-1}\left[\left|\frac{\partial^{\gamma_1}_{R_1}f_j(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial^{\gamma_2}_{R_1}f_{j+1}(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha}\right.\right.\right.$$
$$\left.\left.+\left|\frac{\partial^{\gamma_2}_{R_1}f_j(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha}\left|\frac{\partial^{\gamma_1}_{R_1}f_{j+1}(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\right]\right\}$$
$$+\left[\left|\frac{\partial^{\gamma_1}_{R_1}f_1(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\left|\frac{\partial^{\gamma_2}_{R_1}f_M(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha}\right.$$
$$\left.\left.+\left|\frac{\partial^{\gamma_2}_{R_1}f_1(x)}{\partial r^{\gamma_2}}\right|^{\lambda_\alpha}\left|\frac{\partial^{\gamma_1}_{R_1}f_M(x)}{\partial r^{\gamma_1}}\right|^{\lambda_\alpha}\right]\right\}dx$$
$$\le\left(\frac{4\pi^{N/2}}{\Gamma(N/2)}\right)\rho^*(R_2)\left[\sum_{j=1}^{M}\left\|\frac{\partial^{\nu}_{R_1}f_j}{\partial r^{\nu}}\right\|_\infty^{2\lambda_\alpha}\right] \tag{2.175}$$

Here we have

$$\rho^*(R_2):=\frac{R_2^{N-1}(R_2-R_1)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)}}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\alpha}}. \tag{2.176}$$

We need

Theorem 2.49. (see Anastassiou [3]) *(As in Theorem 2.41,* $\lambda_\alpha = 0,\ \lambda_\beta = \lambda_\nu$*). It holds*

$$\int_{x_0}^{x} p(w) \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[\left| \left(D_{x_0}^{\gamma_2} f_{j+1} \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_j \right)(w) \right|^{\lambda_\beta}$$

$$+ \left| \left(D_{x_0}^{\gamma_2} f_j \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_{j+1} \right)(w) \right|^{\lambda_\beta} \Big] \Bigg\}$$

$$+ \Big[\left| \left(D_{x_0}^{\gamma_2} f_M \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_1 \right)(w) \right|^{\lambda_\beta}$$

$$+ \left| \left(D_{x_0}^{\gamma_2} f_1 \right)(w) \right|^{\lambda_\beta} \left| \left(D_{x_0}^{\nu} f_M \right)(w) \right|^{\lambda_\beta} \Big] \Bigg\} dw$$

$$\leq 2 \cdot \left(\frac{(x - x_0)^{(\nu\lambda_\beta - \gamma_2\lambda_\beta + 1)} \|p(x)\|_\infty}{(\nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}} \right) \left[\sum_{j=1}^{M} \left\| D_{x_0}^{\nu} f_j \right\|_\infty^{2\lambda_\beta} \right], \tag{2.177}$$

all $x_0 \leq x \leq b$.

Similarly, by (2.177), we give

Theorem 2.50. *Here all as in Theorem* 2.42*. Case of* $\lambda_\alpha = 0,\ \lambda_\beta = \lambda_\nu$*.*

Then

$$\int_A \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Bigg[\left| \frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial r^{\nu}} \right|^{\lambda_\beta}$$

$$+ \left| \frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_{j+1}(x)}{\partial r^{\nu}} \right|^{\lambda_\beta} \Bigg] \Bigg\}$$

$$+ \Bigg[\left| \frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\beta}$$

$$+ \left| \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_M(x)}{\partial r^{\nu}} \right|^{\lambda_\beta} \Bigg] \Bigg\} dx$$

$$\leq \left(\frac{4\pi^{N/2}}{\Gamma(N/2)} \right) \left(\frac{R_2^{N-1}(R_2 - R_1)^{(\nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}} \right) \left[\sum_{j=1}^{M} \left\| \frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}} \right\|_\infty^{2\lambda_\beta} \right]. \tag{2.178}$$

To extend the above research we give a motivational result regarding fractional integration by parts.

Proposition 2.1. *Let* $f \in C^1([0,1])$ *and* $g \in C^{\nu}([0,1])$, $\nu > 0,\ n := [\nu],\ \alpha := \nu - n$*.*

Then

$$\int_0^1 f(x) g^{(\nu)}(x) dx = f(1)(\mathcal{J}_{1-\alpha}\, g^{(n)})(1) -$$

$$\int_0^1 (\mathcal{J}_{1-\alpha}\, g^{(n)})(x) f'(x) dx. \tag{2.179}$$

Proof. Here $g^{(\nu)} = \frac{d\,(\mathcal{J}_{1-\alpha}\, g^{(n)})(x)}{dx}$, *i.e.* $d\,(\mathcal{J}_{1-\alpha}\, g^{(n)})(x) = g^{(\nu)}(x)\, dx$.

Hence, by ordinary integration by parts we have:

$$\int_0^1 f(x)g^{(\nu)}(x)\,dx = \int_0^1 f(x)\,d\,(\mathcal{J}_{1-\alpha}\,g^{(n)})(x)$$

$$= f(1)(\mathcal{J}_{1-\alpha}\,g^{(n)})(1) - \int_0^1 (\mathcal{J}_{1-\alpha}\,g^{(n)})(x)f'(x)\,dx,$$

by $(\mathcal{J}_{1-\alpha}\,g^{(n)})(0) = 0$. □

Now we are ready to give

Definition 2.3. Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $g : [0,1] \to \mathbb{R}$ such that there exists $g^{(n)}$ which is measurable. Assume that $(\mathcal{J}_{1-\alpha}\,g^{(n)}) \in L^1([0,1])$. We say that $g^{(\nu)} \in L^1([0,1])$ is a *weak fractional derivative of order* ν *for* g, iff

$$\int_0^1 u(x)\,g^{(\nu)}(x)\,dx = -\int_0^1 (\mathcal{J}_{1-\alpha}\,g^{(n)})(x)u'(x)\,dx, \tag{2.180}$$

$\forall\, u \in C^\infty([0,1]) :\ u(1) = 0$.

Based on the above Definition 2.3, we can extend the concept of weak fractional differentiation to anchor points $x_0 \neq 0$, and to the multivariate case, especially to the radial case. Then try to generalize the results of this article.

References

[1] R.P. Agarwal and P.Y.H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer, Dordrecht, London, 1995.

[2] G.A. Anastassiou, Multivariate fractional Taylor's formula, *Communications in Applied Analysis* **11** (2006), no. 2, To Appear.

[3] G.A. Anastassiou, Fractional Opial inequalities for several functions with applications, *J. Comput. Anal. Appl.*, **7** (2005), no. 3, 233-259.

[4] G.A. Anastassiou, Opial type inequalities involving fractional derivatives of two functions and applications, *Comput. Math. Appl.*, **48** (2004), no-s: 10-11, 1701-1731.

[5] G.A. Anastassiou, *Quantitative Approximations*, Chapman and Hall/CRC, Boca Raton, New York, 2001.

[6] G.A. Anastassiou, Opial type inequalities involving functions and their ordinary and fractional derivatives, *Commun. Appl. Anal.*, **4** (2000), no. 4, 547-560.

[7] G.A. Anastassiou, Opial type inequalities involving fractional derivatives of functions, *Nonlinear Stud.*, **6** (1999), no. 2, 207-230.

[8] G.A. Anastassiou, General fractional Opial type inequalities, *Acta Appl. Math.*, **54** (1998), no. 3, 303-317.

[9] G.A. Anastassiou and J.A. Goldstein, Fractional Opial type inequalities and fractional differential equations, *Results Math.*, **41** (2002), no-s. 3-4, 197-212.

[10] P.R. Beesack, On an integral inequality of Z. Opial, *Trans. Amer. Math. Soc.*, **104** (1962), 470-475.

[11] I.A. Canavati, The Riemann- Liouville Integral, *Nieuw Archief Voor Wiskunde*, **5** (1987), no. 1, 53-75.

[12] Z. Opial, Sur une inégalité, *Ann. Polon. Math.*, **8** (1960), 29-32.

[13] D. Willett, The existence-uniqueness theorem for an nth order linear ordinary differential equation, *Amer. Math. Monthly*, **75** (1968), 174-178.