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FOURIER SERIES AND PERIODIC SOLUTIONS OF PERIODIC LINEAR NON-HOMOGENEOUS IMPULSIVE EQUATIONS

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ABSTRACT: The purpose of the present paper is to investigate the possibility to apply Fourier series in finding the T-periodic solution $x = \widetilde{x}(t)$ of the T-periodic linear non-homogeneous impulsive equation

$$\begin{aligned} x'(t) &= \alpha x(t) + f(t)\,, && t \neq t_k\,, && t \in \mathbb{R}\,,\\ x(t_k^+) &= \beta_k x(t_k^-) + \gamma_k\,, && t = t_k\,, && k \in \mathbb{Z}\,. \end{aligned}$$

Various problems are solved in connection with the determining of the Fourier coefficients of the solution $\widetilde{x}(t)$ and with the representation of this solution by Fourier series having a good convergence in the whole interval $[0, T]$.

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1. INTRODUCTION

Consider the T-periodic linear non-homogeneous impulsive equation

$$\begin{aligned} x'(t) &= \alpha x(t) + f(t)\,, && t \neq t_k\,, && t \in \mathbb{R}\,,\\ x(t_k^+) &= \beta_k x(t_k^-) + \gamma_k\,, && t = t_k\,, && k \in \mathbb{Z}\,, \end{aligned} \tag{1}$$

where $\{t_k\}$ are the moments of impulse effect and $x(t_k^{\pm}) = \lim_{t \to t_k \pm 0} x(t)$.

Suppose that the following conditions hold:

(H) $\alpha \in \mathbb{R}^{m\times m}$, $\beta_k \in \mathbb{R}^{m\times m}$, $\gamma_k \in \mathbb{R}^m$, $t_k < t_{k+1}$, $k \in \mathbb{Z}$, $f \in PC(\mathbb{R}, \mathbb{R}^m)$, $f(t+T) = f(t)$, $t \in \mathbb{R}$ and there exists $r \in \mathbb{N}$ such that

$$\beta_{k+r} = \beta_k\,, \quad \gamma_{k+r} = \gamma_k\,, \quad t_{k+r} = t_k + T\,, \quad k \in \mathbb{Z}\,.$$

Here $\mathbb{Z}$ is the set of all integers, $\mathbb{N}$ – the set of all positive integers and $PC(\mathbb{R}, \mathbb{R}^m)$ is the set of all functions $\psi : \mathbb{R} \to \mathbb{R}^m$, which are continuous at $t \neq t_k$, $k \in \mathbb{Z}$ and have discontinuities of the first kind at the points $\{t_k\}$.

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We shall investigate the problem of numerical determining of the T-periodic solution $x = \widetilde{x}(t)$ of equation (1) in the noncritical case, when the corresponding homogeneous equation

$$\begin{aligned} &x'(t) = \alpha x(t)\,, && t \neq t_k\,, && t \in \mathbb{R}\,, \\ &x(t_k^+) = \beta_k x(t_k^-)\,, && t = t_k\,, && k \in \mathbb{Z}\,, \end{aligned} \tag{2}$$

has no T-periodic solution other than $x \equiv 0$, that is, when

$$\det(E - X(T)) \neq 0\,. \tag{3}$$

Here E is the unit $m \times m$ matrix and $X(T)$ is the monodromy matrix of equation (2), where

$$X(T) = e^{\alpha(T-t_r)}\beta_r \dots e^{\alpha(t_2-t_1)}\beta_1 e^{\alpha t_1}\,,$$

if $0 < t_1 < t_2 < \dots < t_r \leq T$.

In the scalar case $m = 1$ condition (3) is equivalent to the condition

$$e^{\alpha T}\beta_1 \dots \beta_r \neq 1\,. \tag{4}$$

We recall Bainov and Simeonov [1], Theorem 4.1, that in the noncritical case equation (1) has a unique T-periodic solution

$$\widetilde{x}(t) = W(t,0)x_0 + \int_0^t W(t,s)f(s)ds + \sum_{0 \leq t_k < t} W(t, t_k^+)\gamma_k\,, \tag{5}$$

where

$$x_0 = [E - X(T)]^{-1}\left[\int_0^T W(T,s)f(s)ds + \sum_{0 \leq t_k < T} W(T, t_k^+)\gamma_k\right],$$

$X(T) = W(T,0)$ and $W(t,s)$ is the Cauchy matrix for equation (2):

$$W(t,s) = \begin{cases} e^{\alpha(t-s)}\,, & \text{for} \quad t_{k-1} < s \leq t \leq t_k\,, \\ e^{\alpha(t-t_k)}\beta_k e^{\alpha(t_k-s)}\,, & \text{for} \quad t_{k-1} < s \leq t_k < t \leq t_{k+1}\,, \\ e^{\alpha(t-t_k)}\beta_k \dots e^{\alpha(t_{j+1}-t_j)}\beta_j e^{\alpha(t_j-s)}\,, & \text{for} \quad t_{j-1} < s \leq t_j < t_k < t \leq t_{k+1}\,. \end{cases}$$

The application of formula (5) to concrete equation is difficult because of the complicated way of computing of $W(t,s)$. The computations become more difficult, when the function $f(t)$ has discontinuities of the first kind at the points $\{t_k\}$ and has various analytical representations in the intervals (t_k, t_{k+1}), $k \in \mathbb{Z}$.

An effective way to determine the T-periodic solution $\widetilde{x}(t)$ is its representation by Fourier series, or more precisely, the representation by its N-th partial sum. Then for the approximation of $\widetilde{x}(t)$ we have to find a finite number $(2N+1)$ of Fourier coefficient c_n $(n = 0, \pm 1, \dots, \pm N)$ of this solution.

In the present paper we shall investigate the possibility to apply Fourier series in finding the T-periodic solution $\widetilde{x}(t)$ of equation (1) and we solve the problems, which arise in this connection.

2. MAIN RESULTS

2.1. **THE CASE $\boldsymbol{m = 1}$.** In order to clarify the problems connected with the application of the Fourier series in finding the T-periodic solution $\widetilde{x}(t)$ of equation (1) we shall consider first the scalar case $m = 1$. Without loss of generality we shall suppose further that equation (1) is 2π-periodic, that is, $T = \pi$ and $-\pi < t_1 < t_2 < \cdots < t_r \leq \pi$.

Let the Fourier series of the function $f(t)$ be

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{int}\,, \tag{6}$$

where

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\,, \qquad n \in \mathbb{Z}\,. \tag{7}$$

We seek the 2π-periodic solution $\widetilde{x}(t)$ of equation (1) in the form

$$\widetilde{x}(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \tag{8}$$

and its approximation – in the form

$$\widetilde{x}(t) \approx \widetilde{x}_N(t) = \sum_{n=-N}^{N} c_n e^{int}\,, \tag{9}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{x}(t)e^{-int}dt\,, \qquad n \in \mathbb{Z}\,. \tag{10}$$

Let the Fourier series of the derivative $\widetilde{x}'(t)$ be

$$\widetilde{x}'(t) = \sum_{n=-\infty}^{\infty} s_n e^{int}\,. \tag{11}$$

Then

$$s_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{x}'(t)e^{-int}dt = \frac{1}{2\pi}\sum_{k=1}^{r}(\widetilde{x}(t_k^-) - \widetilde{x}(t_k^+))e^{-int_k} + inc_n\,. \tag{12}$$

Substituting (6), (8) and (11) into (1) and taking into account (12) we obtain

$$\frac{1}{\pi}\sum_{k=1}^{r} e^{-int_k} d_k + (\alpha - in)c_n = -f_n\,, \qquad n \in \mathbb{Z}\,, \tag{13}$$

where

$$d_k = \frac{1}{2}(\widetilde{x}(t_k^+) - \widetilde{x}(t_k^-))\,, \qquad k = 1, \ldots, r\,. \tag{14}$$

Since

$$\widetilde{x}(t_k) = \sum_{n=-\infty}^{\infty} c_n e^{int_k} = \frac{1}{2}(\widetilde{x}(t_k^+) + \widetilde{x}(t_k^-)) \tag{15}$$

and

$$\widetilde{x}(t_k^+) = \widetilde{x}(t_k) + d_k\,, \qquad \widetilde{x}(t_k^-) = \widetilde{x}(t_k) - d_k\,, \tag{16}$$

then it follows from the jump condition $\widetilde{x}(t_k^+) = \beta_k \widetilde{x}(t_k^-) + \gamma_k$ that

$$(1-\beta_k) \sum_{n=-\infty}^{\infty} e^{int_k} c_n + (1+\beta_k) d_k = \gamma_k \,, \qquad k = 1, \ldots, r \,. \tag{17}$$

Thus, in order to determine the Fourier coefficients of $\widetilde{x}(t)$ we have to solve with respect to c_n and d_k the infinite system (13), (17). For the approximation of $\widetilde{x}(t)$ by formula (9) we can solve the "shortened" system

$$\begin{cases} \frac{1}{\pi} \sum_{k=1}^{r} e^{-int_k} d_k + (\alpha - in) c_n = -f_n \,, & n = 0, \pm 1, \ldots, \pm N \,, \\ (1-\beta_k) \displaystyle\sum_{n=-N}^{N} e^{int_k} c_n + (1+\beta_k) d_k = \gamma_k \,, & k = 1, \ldots, r \,. \end{cases} \tag{18}$$

with $2N+1+r$ unknowns.

In order to reach the given precision we have to choose N sufficiently large. But then the error accumulated in solving of system (18) increases. One way to obtain the coefficients c_n more precisely is to determine first the unknowns d_k, $k = 1, \ldots, r$ exactly and then to calculate c_n by the formula

$$c_n = (in - \alpha)^{-1} \left[f_n + \frac{1}{\pi} \sum_{k=1}^{r} e^{-int_k} d_k \right] , \qquad n \in \mathbb{Z} \,, \tag{19}$$

which follows from (13). We substitute (19) in (17) and obtain the system for determining of d_k:

$$(\beta_k - 1) \sum_{j=1}^{r} F(t_k - t_j) d_j + (\beta_k + 1) d_k = R_k \,, \qquad k = 1, \ldots, r \,, \tag{20}$$

where

$$R_k = \gamma_k + (\beta_k - 1) \sum_{n=-\infty}^{\infty} (in - \alpha)^{-1} f_n e^{int_k} \,, \qquad k = 1, \ldots, r \,, \tag{21}$$

$$F(x) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (\alpha - in)^{-1} e^{inx} = \frac{1}{\pi} \left[\frac{1}{\alpha} + 2 \sum_{n=1}^{\infty} \frac{\alpha \cos nx - n \sin nx}{n^2 + \alpha^2} \right] . \tag{22}$$

In order to prove that system (20) is solvable we need the following two lemmas.

Lemma 1. *If $\alpha \neq 0$, then*

$$F(x) = \begin{cases} \dfrac{2e^{\alpha x}}{e^{2\pi\alpha} - 1} \,, & x \in (0, 2\pi) \,, \\ \\ \dfrac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1} \,, & x = 0 \,, \\ \\ \dfrac{2e^{\alpha(x+2\pi)}}{e^{2\pi\alpha} - 1} \,, & x \in (-2\pi, 0) \,. \end{cases} \tag{23}$$

Proof. Formula (23) follows from the formula for the sum of the Fourier series of the 2π-periodic extension $\widetilde{s}(x)$ of the function $s(x) = e^{\alpha x}$, $x \in (0, 2\pi)$:

$$\widetilde{s}(x) = \frac{e^{2\pi\alpha} - 1}{2\pi}\left[\frac{1}{\alpha} + 2\sum_{n=1}^{\infty}\frac{\alpha\cos nx - n\sin nx}{n^2+\alpha^2}\right]. \qquad \square$$

Corollary 1. *If* $h(\alpha) = \sum_{n=1}^{\infty}\frac{1}{n^2+\alpha^2}$, *then*

$$h(\alpha) = \begin{cases} \dfrac{1}{2\alpha}\left[\pi\dfrac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1} - \dfrac{1}{\alpha}\right], & \alpha \neq 0\,, \\[2ex] \dfrac{\pi^2}{6}\,, & \alpha = 0\,. \end{cases} \tag{24}$$

Lemma 2. *Let D be the matrix of system* (20) *and $\alpha \neq 0$. Then*

$$\det D = \frac{2^r}{e^{2\pi\alpha}-1}(e^{2\pi\alpha}\beta_1\ldots\beta_r - 1)\,. \tag{25}$$

Proof. We have

$$\det D = \begin{vmatrix} (\beta_1-1)F_{11}+\beta_1+1 & (\beta_1-1)F_{12} & \ldots & (\beta_1-1)F_{1r} \\ (\beta_2-1)F_{21} & (\beta_2-1)F_{22}+\beta_2+1 & \ldots & (\beta_2-1)F_{2r} \\ \vdots & \vdots & & \vdots \\ (\beta_r-1)F_{r1} & (\beta_r-1)F_{r2} & \ldots & (\beta_r-1)F_{rr}+\beta_r+1 \end{vmatrix},$$

where $F_{kj} = F(t_k - t_j)$.

Assume that $\beta_k \neq 1$, $k = 1, \ldots, r$ and set $B_k = \frac{\beta_k+1}{\beta_k-1}$. Then

$$\det D = \prod_{k=1}^{r}(\beta_k - 1)\begin{vmatrix} F_{11}+B_1 & F_{12} & \ldots & F_{1r} \\ F_{21} & F_{22}+B_2 & \ldots & F_{2r} \\ \vdots & \vdots & & \vdots \\ F_{r1} & F_{r2} & \ldots & F_{rr}+B_r \end{vmatrix}.$$

From Lemma 1 we have

$$F_{kj} = F(t_k - t_j) = \begin{cases} \dfrac{2}{v-1}e^{\alpha t_k}e^{-\alpha t_j}\,, & k > j\,, \\[2ex] \dfrac{v+1}{v-1}e^{\alpha t_k}e^{-\alpha t_j}\,, & k = j\,, \\[2ex] \dfrac{2v}{v-1}e^{\alpha t_k}e^{-\alpha t_j}\,, & k < j\,, \end{cases}$$

where $v = e^{2\pi\alpha}$. Then

$$\det D = \prod_{k=1}^{r}(\beta_k - 1) \begin{vmatrix} \frac{v+1}{v-1} + B_1 & \frac{2v}{v-1} & \dots & \frac{2v}{v-1} \\ \frac{2}{v-1} & \frac{v+1}{v-1} + B_2 & \dots & \frac{2v}{v-1} \\ \vdots & \vdots & & \vdots \\ \frac{2}{v-1} & \frac{2}{v-1} & \dots & \frac{v+1}{v-1} + B_r \end{vmatrix} . \tag{26}$$

We have Faddeev and Sominsky [2], Problem 250

$$\begin{vmatrix} a_1 & u & \dots & u \\ w & a_2 & \dots & u \\ \vdots & \vdots & & \vdots \\ w & w & \dots & a_r \end{vmatrix} = \frac{ug(w) - wg(u)}{u - w} , \tag{27}$$

where $g(z) = \prod_{k=1}^{r}(a_k - z)$. Then (25) follows from (26) and (27) in the case $\beta \neq 1$, $k = 1, \dots, r$. In the general cases (25) follows by continuity. $\square$

As a consequence of Lemma 2 we obtain the following theorem.

Theorem 1. *Let $\alpha \neq 0$ and $e^{2\pi\alpha}\beta_1 \dots \beta_r \neq 1$. Then system (20) has a unique solution.*

In the case, when $\alpha = 0$ we solve first the following system with respect to c_0 and d_k, $k = 1, \dots, r$, which follows from (13) and (17):

$$\begin{cases} \pi\alpha c_0 + \sum_{k=1}^{r} d_k = -\pi f_0 , \\ (1 - \beta_k)c_0 + (\beta_k - 1)\sum_{j=1}^{r} G_\alpha(t_k - t_j)d_j + (\beta_k + 1)d_k = H_k , \end{cases} \tag{28}$$

where

$$H_k = \gamma_k + (\beta_k - 1)\sum_{n\neq 0}(in - \alpha)^{-1} f_n e^{int_k} , \qquad k = 1, \dots, r , \tag{29}$$

$$G_\alpha(x) = \frac{1}{\pi}\sum_{n\neq 0}(\alpha - in)^{-1}e^{inx} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\alpha\cos nx - n\sin nx}{n^2 + \alpha^2} . \tag{30}$$

From Lemma 1 it follows that for $\alpha \neq 0$

$$G_\alpha(x) = \begin{cases} \dfrac{2e^{\alpha x}}{e^{2\pi\alpha}-1} - \dfrac{1}{\pi\alpha}\,, & x \in (0, 2\pi)\,, \\[2ex] \dfrac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1} - \dfrac{1}{\pi\alpha}\,, & x = 0\,, \\[2ex] \dfrac{2e^{\alpha(x+2\pi)}}{e^{2\pi\alpha}-1} - \dfrac{1}{\pi\alpha}\,, & x \in (-2\pi, 0)\,, \end{cases} \tag{31}$$

and

$$G_0(x) = -\sum_{n\neq 0} \frac{e^{inx}}{i\pi n} = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} \dfrac{x-\pi}{\pi}\,, & x \in (0, 2\pi)\,, \\ 0\,, & x = 0\,, \\ \dfrac{x+\pi}{\pi}\,, & x \in (-2\pi, 0)\,. \end{cases} \tag{32}$$

Proceeding as in the proof of Lemma 2 one can prove the following lemma.

Lemma 3. *Let D_0 be the matrix of system* (28)*. Then*

$$\det D_0 = \begin{cases} \dfrac{2^r\pi\alpha}{e^{2\pi\alpha}-1}(e^{2\pi\alpha}\beta_1\ldots\beta_r - 1)\,, & \alpha \neq 0\,, \\[3ex] 2^{r-1}(\beta_1\ldots\beta_r - 1)\,, & \alpha = 0\,. \end{cases} \tag{33}$$

As a consequence of Lemma 3 we obtain the following result.

Theorem 2. *Let $e^{2\pi\alpha}\beta_1\ldots\beta_r \neq 1$. Then system* (28) *has a unique solution.*

The solution c_0, d_k, $k = 1, \ldots, r$ of system (28) is determined exactly if the free terms H_k are determined exactly by (29).

Let H_k be obtained approximately by the following formula

$$H_k \approx H_k(N) = \gamma_k + (\beta_k - 1) \sum_{0<|n|\leq N} (in-\alpha)^{-1} f_n e^{int_k}\,.$$

Solving system (28) and using (19) we can find the approximations $c_0(N)$, $d_k(N)$, $k = 1, \ldots, r$ and $c_n(N)$, $n = \pm 1, \ldots, \pm N$ of c_0, d_k and c_n, respectively. Then the approximation $\widetilde{x}_N(t)$ of the 2π-periodic solution $\widetilde{x}(t)$ of equation (1) is

$$\widetilde{x}(t) \approx \widetilde{x}_N(t) = \sum_{|n|\leq N} c_n(N)e^{int}\,.$$

Taking into account that $\det D_0 \neq 0$ we can prove by standard way the following theorem.

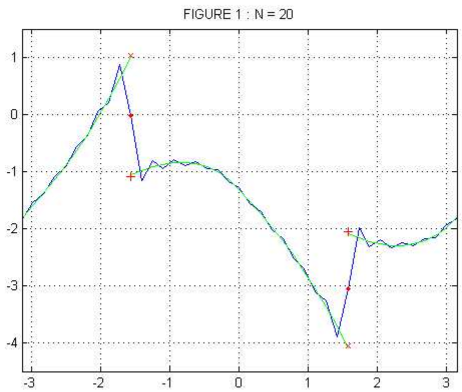

FIGURE 1. $N = 20$

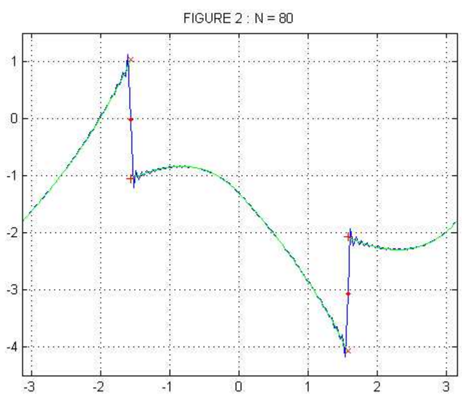

FIGURE 2. $N = 80$

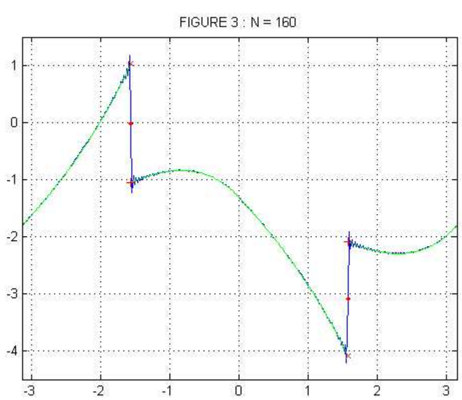

FIGURE 3. $N = 160$

Theorem 3. *Let conditions* (H) *and* (4) *hold. Then for each* $\varepsilon > 0$ *and* $\delta > 0$ *there exists* $N \in \mathbb{N}$ *such that*

$$|\widetilde{x}(t) - \widetilde{x}_N(t)| < \varepsilon \quad \textit{for} \quad t \in [0, T]\,, \quad |t - t_k| > \delta\,, \quad k = 1, \dots, r\,.$$

Remark 1. On Figures 1, 2 and 3 the graphs of the approximate solutions $\widetilde{x}_{20}(t)$, $\widetilde{x}_{80}(t)$ and $\widetilde{x}_{160}(t)$ are compared respectively with the graph of the 2π-periodic solution $\widetilde{x}(t)$ of the equation

$$\begin{aligned} &x'(t) = x(t) + T(t)\,, && t \neq t_k\,, \quad k \in \mathbb{Z}\,,\\ &x(t_k^+) = -2x(t_k^-) + 1\,, && t = t_k = -\tfrac{\pi}{2} + (k-1)\pi\,, \quad k \in \{\pm 1, \pm 3, \pm 5, \dots\}\,,\\ &x(t_k^+) = x(t_k^-) + 2\,, && t = t_k = \tfrac{\pi}{2} + (k-2)\pi\,, \quad k \in \{0, \pm 2, \pm 4, \dots\}\,, \end{aligned} \tag{34}$$

where $T(t)$ is the 2π-periodic extension of the function $y = |t|$, $t \in [-\pi, \pi]$.

We notice that when N increases the difference $|\widetilde{x}(t) - \widetilde{x}_N(t)|$ decreases at the points, which are "far" from the points $t_1 = -\frac{\pi}{2}$ and $t_2 = \frac{\pi}{2}$, where $\widetilde{x}(t)$ has discontinuities of the first kind, while in neighbourhoods of this points we observe a considerable deviation of $\widetilde{x}_N(t)$ from $\widetilde{x}(t)$. This defect of the convergence is characteristic of the partial sums of the Fourier series of any piecewise continuous function and is known as "Gibbs phenomenon", see Fikhtengolts [3], Point 700. The Gibbs phenomenon is observed in a neighbourhood of any point of discontinuity of such a function, even if at the rest points the function has continuous derivatives of higher order.

In order to improve the convergence of the Fourier series used for determining of $\widetilde{x}(t)$, we set $\widetilde{x}(t) = \varphi(t) + m(t)$, where the function $m(t)$ is 2π-periodic and piecewise continuous with points of discontinuity of the first kind $\{t_k\}$. Moreover, $m(t)$ is chosen so that the function $\varphi(t) = \widetilde{x}(t) - m(t)$ is continuous in $\mathbb{R}$. Then the Fourier series of $\varphi(t)$ has better convergence than this one for $\widetilde{x}(t)$ and the Gibbs phenomenon is not observed. This manner to improve the convergence of the Fourier series of piecewise continuous functions is known as "the method of isolation of the peculiarities" and is proposed by A.N. Krylov, see Fikhtengolts [3], Point 710.

The justification of the method to the our problem can be reduced to the following: We seek the 2π-periodic solution $\widetilde{x}(t)$ of equation (1) in the form

$$\widetilde{x}(t) = \varphi(t) + \sum_{k=1}^{r} \left(\sigma_k(t) - \frac{\tau(t)}{\pi}\right) d_k\,, \tag{35}$$

where $\sigma_k(t)$ and $\tau(t)$ are the 2π-periodic extensions of the functions

$$y = \operatorname{sign}(t - t_k)\,, \quad t \in (-\pi, \pi] \quad \text{and} \quad y = t\,, \quad t \in (-\pi, \pi]\,,$$

respectively.

It is easy to verify that the function $\varphi(t)$ defined by (35) is 2π-periodic and continuous in $\mathbb{R}$. Then the Fourier series of the function $\varphi(t)$ has a "good" convergence and for its N-th partial sum the Gibbs phenomenon is not observed.

In order to determine the Fourier coefficients φ_n of $\varphi(t)$ we use the relation which follows from (35)

$$c_n = \varphi_n + \sum_{k=1}^{r} \left(\sigma_{kn} - \frac{\tau_n}{\pi}\right) d_k\,, \tag{36}$$

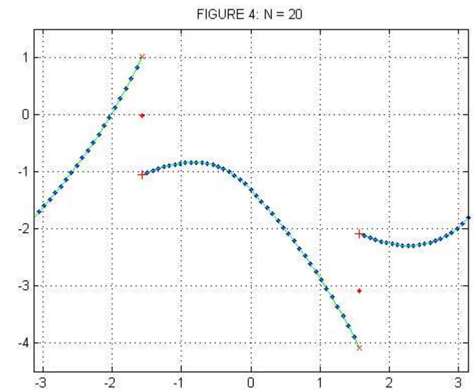

Figure 4. $N = 20$

where σ_{kn} and τ_n are the Fourier coefficients of $\sigma_k(t)$ and $\tau(t)$, respectively. Since

$$\sigma_{k0} = -\frac{t_k}{\pi}\,, \quad \tau_0 = 0\,, \quad \sigma_{kn} = \frac{i}{\pi n}((-1)^n - e^{-int_k})\,, \quad \tau_n = \frac{i(-1)^n}{n}\,, \quad n \neq 0\,,$$

we obtain from (36)

$$\begin{cases} \varphi_0 = c_0 + \dfrac{1}{\pi}\sum_{k=1}^{r} t_k d_k\,, \\ \\ \varphi_n = c_n + \dfrac{i}{\pi}\sum_{k=1}^{r} \dfrac{e^{-int_k}}{n} d_k\,, \qquad n \neq 0\,. \end{cases} \tag{37}$$

Thus, the 2π-periodic solution $\widetilde{x}(t)$ of equation (1) can be obtained approximately by formula (35), where

$$\varphi(t) \approx \sum_{|n|\leq N} \varphi_n e^{int} \tag{38}$$

and the coefficients φ_n are calculated by formula (37).

The approximate solution $\widetilde{x}_N(t)$ determined by this way is better than this one, obtained by formula (9).

Remark 2. The graph of the 2π-periodic solution $\widetilde{x}(t)$ of equation (34) is compared on Figure 4 with the graph of its approximation $\widetilde{x}_{20}(t)$, determined by formulas (35) and (38). Obviously, this approximation is better than the approximate solution from Figure 1, which is obtained by formula (9).

The procedure of "isolation of the peculiarities" can be repeated. Taking into account that the 2π-periodic function $\varphi'(t)$ is piecewise continuous with points of discontinuity $\{t_k\}$, where it has jumps

$$2\delta_k = \varphi'(t_k^+) - \varphi'(t_k^-) = x'(t_k^+) - x'(t_k^-) = 2\alpha d_k + f(t_k^+) - f(t_k^-)$$

we conclude that the function $\psi(t)$ defined by the equality

$$\varphi'(t) = \psi(t) + \sum_{k=1}^{r}\left(\sigma_k(t) - \frac{\tau(t)}{\pi}\right)\delta_k \tag{39}$$

is 2π-periodic and continuous in $\mathbb{R}$.

In accordance with (39) the Fourier coefficients ψ_n of $\psi(t)$ satisfy the relations

$$\psi_0 = \frac{1}{\pi}\sum_{k=1}^{r} t_k\delta_k\,, \tag{40}$$

$$\psi_n = in\varphi_n + \frac{i}{\pi}\sum_{k=1}^{r}\frac{e^{-int_k}}{n}\delta_k\,, \qquad n \neq 0\,, \tag{41}$$

where

$$\delta_k = \alpha d_k + \frac{1}{2}(f(t_k^+) - f(t_k^-))\,, \qquad k = 1, \dots, r\,. \tag{42}$$

Then integrating (39) and using (41) and the formula

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{1}{12}(3x^2 - 6\pi|x| + 2\pi^2)\,, \qquad x \in [-2\pi, 2\pi]\,,$$

we obtain for $t \in (-\pi, \pi]$

$$\varphi(t) = \varphi_0 + \psi_0 t + \sum_{n\neq 0}\frac{\psi_n}{in}e^{int} + \sum_{k=1}^{r}\left(|t - t_k| - \frac{t^2}{2\pi} - \frac{t_k^2}{2\pi} - \frac{\pi}{3}\right)\delta_k\,. \tag{43}$$

Now the 2π-periodic solution $\widetilde{x}(t)$ of equation (1) can be obtained using formulas (35), (43) and (40)-(42).

2.2. **THE CASE $\boldsymbol{m \geq 2}$.** In the case $m \geq 2$ the computations are analogous and the system for d_k, $k = 1, \dots, r$ has the form

$$(\beta_k - E)\sum_{j=1}^{r} F(t_k - t_j)d_k + (\beta_k + E)d_k = R_k\,, \qquad k = 1, \dots, r\,, \tag{44}$$

where

$$R_k = \gamma_k + (\beta_k - E)\sum_{n=-\infty}^{\infty}(inE - \alpha)^{-1}f_n e^{int_k}\,, \qquad k = 1, \dots, r\,, \tag{45}$$

$$F(x) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}(\alpha - inE)^{-1}e^{inx} = \begin{cases}(e^{2\pi\alpha} - E)^{-1}2e^{\alpha x}\,, & x \in (0, 2\pi)\,,\\ (e^{2\pi\alpha} - E)^{-1}(e^{2\pi\alpha} + E)\,, & x = 0\,,\\ (e^{2\pi\alpha} - E)^{-1}2e^{\alpha(x+2\pi)}\,, & x \in (-2\pi, 0)\,.\end{cases} \tag{46}$$

After defining of $d_k \in \mathbb{R}^m$ we obtain $c_n \in \mathbb{R}^m$ by the formula

$$c_n = (inE - \alpha)^{-1}\left[f_n + \frac{1}{\pi}\sum_{k=1}^{r} e^{-int_k}d_k\right]\,, \qquad n \in \mathbb{Z}\,. \tag{47}$$

Finally, we can find the approximate solution $\widetilde{x}_N(t)$ of the 2π-periodic solution $\widetilde{x}(t)$ of equation (1) applying one of the following way:

(i) by formula (9);
(ii) by formulas (35), (37) and (38);
(iii) by formulas (35), (37), (40)-(43).

Remark 3. Formulas (45)-(47) are applicable if the eigenvalues of the matrix α do not belong to the set $\mathbb{Z}_1 = \{0, \pm i, \pm 2i, \ldots, \pm ni, \ldots\}$. In this case the function

$$u(t) = \sum_{nj=-\infty}^{\infty} (inE - \alpha)^{-1} f_n e^{int} \tag{48}$$

is the unique 2π-periodic solution of the equation without impulses

$$u'(t) = \alpha u(t) + f(t)\,, \tag{49}$$

that is,

$$u(t) = e^{\alpha t}\left[u_0 + \int_0^t e^{-\alpha s} f(s) ds\right], \tag{50}$$

where

$$u_0 = [E - e^{2\pi\alpha}]^{-1} \int_0^{2\pi} e^{\alpha(2\pi - s)} f(s) ds\,. \tag{51}$$

Using (50) and (51) one can obtain the exact values of R_k in (45).

If some eigenvalues of the matrix α belong to the set $\mathbb{Z}_1$, say, $\pm in_1, \pm in_2, \ldots, \pm n_p$, then we have to define first the coefficients $c_{n_1}, c_{-n_1}, \ldots, c_{n_p}, c_{-n_p}$, $d_1 \ldots, d_r$ solving a system, which is obtained like system (28).

We shall consider all possible variants of such systems in the case $m = 2$.

2.3. **THE CASE $\boldsymbol{m = 2}$.** Let $m = 2$ and some eigenvalues of the real matrix α belong to the set $\mathbb{Z}_1$.

(i) Let $\det \alpha = 0$. Then α has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \mathrm{Tr}\alpha$ and we have to solve first the following system with respect to c_0 and d_k, $k = 1, \ldots, r$:

$$\begin{cases} \pi\alpha c_0 + \sum\limits_{k=1}^{r} d_k = -\pi f_0\,, \\ (E - \beta_k)c_0 + (\beta_k - E)\sum\limits_{=1}^{r} Q_\alpha^0(t_k - t_j)d_j + (\beta_k + E)d_k = H_k\,, \end{cases} \tag{52}$$

where

$$\begin{aligned} H_k &= \gamma_k + (\beta_k - E)\sum_{n\neq 0}(inE - \alpha)^{-1} f_n e^{int_k}\,, \qquad k = 1, \ldots, r\,, \\ Q_\alpha^0(x) &= \frac{1}{\pi}\sum_{n\neq 0}(\alpha - inE)^{-1} e^{inx}\,. \end{aligned} \tag{53}$$

Remark 4. The function

$$u(t) = \sum_{n\neq 0}(inE - \alpha)^{-1} f_n e^{int} \tag{54}$$

is the unique 2π-periodic solution of the equation

$$u'(t) = \alpha u(t) + f(t) - f_0\,, \tag{55}$$

such that

$$\int_{-\pi}^{\pi} u(t)dt = 0\,. \tag{56}$$

Summing the Fourier series (54) we obtain:

(i.1) If $\det\alpha = 0$ and $\mathrm{Tr}\alpha = \mu \neq 0$, then

$$Q^0_\alpha(x) = \begin{cases} \left(\dfrac{2e^{\mu x}}{e^{2\pi\mu}-1} - \dfrac{1}{\pi\mu} - \dfrac{x-\pi}{\pi}\right)\dfrac{\alpha}{\mu} + \dfrac{x-\pi}{\pi}E\,, & x \in (0, 2\pi)\,, \\ \left(\dfrac{e^{2\pi\mu}+1}{e^{2\pi\mu}-1} - \dfrac{1}{\pi\mu}\right)\dfrac{\alpha}{\mu}\,, & x = 0\,, \\ \left(\dfrac{2e^{\mu(x+2\pi)}}{e^{2\pi\mu}-1} - \dfrac{1}{\pi\mu} - \dfrac{x+\pi}{\pi}\right)\dfrac{\alpha}{\mu} + \dfrac{x+\pi}{\pi}E\,, & x \in (-2\pi, 0)\,. \end{cases} \tag{57}$$

(i.2) If $\det\alpha = 0$ and $\mathrm{Tr}\alpha = 0$, then

$$Q^0_\alpha(x) = \begin{cases} \left(\dfrac{(x-\pi)^2}{2\pi} - \dfrac{\pi}{6}\right)\alpha + \dfrac{x-\pi}{\pi}E\,, & x \in (0, 2\pi)\,, \\ \dfrac{\pi}{3}\alpha\,, & x = 0\,, \\ \left(\dfrac{(x+\pi)^2}{2\pi} - \dfrac{\pi}{6}\right)\alpha + \dfrac{x+\pi}{\pi}E\,, & x \in (-2\pi, 0)\,. \end{cases} \tag{58}$$

(ii) *Let* $\det\alpha \neq 0$. Then the eigenvalues λ_1, λ_2 of the real matrix α belong to the set $\mathbb{Z}_1$ if $\lambda_1 = in_0$, $\lambda_2 = -in_0$ for some $n_0 \in \mathbb{N}$. In this case we have to solve the following system with respect to c_{n_0}, c_{-n_0} and d_k, $k = 1, \ldots, r$:

$$\begin{cases} (\alpha - in_0E)c_{n_0} + \dfrac{1}{\pi}\sum_{k=1}^r e^{-in_0t_k}d_k = -f_{n_0}\,, \\ (\alpha + in_0E)c_{n_0} + \dfrac{1}{\pi}\sum_{k=1}^r e^{in_0t_k}d_k = -f_{-n_0}\,, \\ (E-\beta_k)e^{in_0t_k}c_{n_0} + (E-\beta_k)e^{-in_0t_k}c_{-n_0} \\ \qquad +(\beta_k - E)\displaystyle\sum_{j=1}^r Q^{n_0}_\alpha(t_k - t_j)d_j + (\beta_k+E)d_k = H_k(n_0)\,, \end{cases} \tag{59}$$

where

$$\begin{aligned} H_k(n_0) =& \gamma_k + (\beta_k - E)\sum_{|n|\neq n_0}(inE-\alpha)^{-1}f_n e^{int_k}\,, \qquad k = 1, \ldots, r\,, \\ Q^{n_0}_\alpha(x) =& \frac{1}{\pi}\sum_{|n|\neq n_0}(\alpha - inE)^{-1}e^{inx}\,. \end{aligned} \tag{60}$$

Remark 5. The function

$$u(t) = \sum_{|n| \neq n_0} (inE - \alpha)^{-1} f_n e^{int} \tag{61}$$

is the unique 2π-periodic solution of the equation

$$u'(t) = \alpha u(t) + f(t) - f_{n_0} e^{in_0 t} - f_{-n_0} e^{-in_0 t}\,, \tag{62}$$

such that

$$\int_{-\pi}^{\pi} u(t) e^{-in_0 t} dt = \int_{-\pi}^{\pi} u(t) e^{in_0 t} dt = 0\,. \tag{63}$$

Summing the Fourier series (62) we obtain

$$Q_\alpha^{n_0}(x) = \begin{cases} \left(\dfrac{x-\pi}{\pi n_0} \sin n_0 x + \dfrac{\cos n_0 x}{2\pi n_0^2}\right)\alpha \\ \qquad + \left(\dfrac{x-\pi}{\pi} \cos n_0 x + \dfrac{\sin n_0 x}{2\pi n_0}\right) E\,, & x \in (0, 2\pi)\,, \\ \dfrac{\alpha}{2\pi n_0^2}\,, & x = 0\,, \\ \left(\dfrac{x+\pi}{\pi n_0} \sin n_0 x + \dfrac{\cos n_0 x}{2\pi n_0^2}\right)\alpha \\ \qquad + \left(\dfrac{x+\pi}{\pi} \cos n_0 x + \dfrac{\sin n_0 x}{2\pi n_0}\right) E\,, & x \in (-2\pi, 0)\,. \end{cases} \tag{64}$$

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