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EXISTENCE RESULTS FOR NONAUTONOMOUS EVOLUTION EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT: We establish the existence of integral solutions to nonlocal Cauchy problems associated with time-dependent $m-$accretive operators in a general Banach space.

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1. INTRODUCTION

In this paper we first discuss the existence of solutions to the nonlocal Cauchy problem:

$$\begin{cases} u'(t) + A(t)u(t) \ni f(t, u(t)), & t \in [0, T], \\ u(0) = g(u), \end{cases} \tag{1.1}$$

in a real Banach space X. Here, $\{A(t) : t \in [0, T]\}$ are $m-$accretive operators in X, $g : C([0, T]; X) \to X$, and $f : [0, T] \times X \to X$. Subsequently, we study a nonautonomous evolution equation with a multivalued perturbation and a nonlocal initial condition, of the form:

$$\begin{cases} u'(t) + A(t)u(t) \ni F(t, u(t)), & t \in [0, T], \\ u(0) = g(u), \end{cases} \tag{1.2}$$

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where $\{A(t) : t \in [0,T]\}$ and g are as in (1.1), while $F : [0,T] \times X \to 2^X \setminus \{\phi\}$ is lower semicontinuous in its second argument.

The study of abstract nonlocal Cauchy problems was initiated by Byszewski [11], and has been developed by various authors, see, e.g.,Aizicovici and Gao [1], Aizicovici and Staicu [4], Benchohra et al [6], Benchohra et al [9], Xiao and Liang [22], and Xue [23]. In particular, existence results for nonlocal initial value problems associated with time-dependent fully nonlinear operators appear in Aizicovici and Gao [1], Aizicovici and McKibben [3], and Aizicovici and Staicu [4]. The present work may be viewed as an attempt to obtain nonautonomous versions of Theorem 3.1 of Aizicovici and Lee [2] and Theorem 3.8 in Aizicovici and McKibben [3] for equations (1.1) and (1.2), respectively, as well as to prove a counterpart of Theorem 8 in Aizicovici and Staicu [4] for equation (1.2), in the case when the multifunction F is nonconvex valued and lower semicontinuous in its second variable, as opposed to convex valued and upper semicontinuous. Our approach relies on the theory of evolution equations governed by time-dependent $m-$accretive operators, compactness methods and fixed point techniques. The plan of the paper is as follows. In Section 2 we review some background material on nonautonomous evolution equations and multifunctions. The main results are stated in Section 3, and the corresponding proofs are carried out in Section 4. Finally, Section 5 contains two examples to which our abstract theory applies.

2. PRELIMINARIES

For further background and details pertaining to this section, we refer the reader to Barbu [5], Deimling [12], Hu and Papageorgiou [13], Hu and Papageorgiou [14], Pavel [17], Vrabie [21], and Zeidler [24]. Throughout this paper, X denotes a real Banach space of norm $\|\cdot\|$ and dual $(X^*, \|\cdot\|_*)$. The duality mapping $J : X \to X^*$ is given by

$$J(x) = \{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|_*^2\}, \quad \forall x \in X,$$

while the so-called upper semi-inner product on X is defined by

$$< y, x >_+ = \sup\{x^*(y) : x^* \in J(x)\}.$$

Let A be a multivalued operator in X. The domain $D(A)$ and range $R(A)$ of A are defined by $D(A) = \{x \in A : Ax \neq \phi\}$ and $R(A) = \cup_{x \in D(A)} Ax$, respectively. The operator A is called accretive if $< y' - y, x' - x >_+ \geq 0$, for all $x, x' \in D(A)$, and all $y \in Ax$, $y' \in Ax$. If also, $R(I + \lambda A) = X$, for all $\lambda > 0$, where I is the identity on X, then A is said to be $m-$accretive.

Let $\{A(t) : t \in [0,T]\}$ be a family of (possibly multivalued) operators on X, of domains $D(A(t))$, with $\overline{D(A(t))} = D$ (independent of t), which satisfy the assumption:

$(H_{A(t)})$ (i) $R(I + \lambda A(t)) = X$, for all $\lambda > 0$ and $t \in [0,T]$,

(ii) there exist two continuous functions $m_1 : [0,T] \to X$ and $m_2 : \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+ := [0,\infty)$) such that

$$< y_1 - y_2, x_1 - x_2 >_+ \geq -\|m_1(t) - m_1(s)\| \|x_1 - x_2\| \cdot m_2(\max\{\|x_1\|, \|x_2\|\}),$$

for all $x_1 \in D(A(t)),\ y_1 \in A(t)x_1,\ x_2 \in D(A(s)),\ y_2 \in A(s)x_2$, and all $0 \leq s \leq t \leq T$.

In particular, for each $t \in [0,T]$, the operator $A(t)$ is $m-$accretive. If $(H_{A(t)})$ holds, then (see Pavel [17]), the family $\{A(t) : t \in [0,T]\}$ generates a so-called evolution operator U on D via the formula

$$U(t,s)x = \lim_{n\to\infty} \prod_{i=1}^{n} (I + \frac{t-s}{n} A(s + i\frac{t-s}{n}))^{-1} x, \tag{2.1}$$

for all $x \in D$ and all $0 \leq s \leq t \leq T$. Recall that $U(t,t) = I$ and $\|U(t,s)x - U(t,s)y\| \leq \|x - y\|$, for all $0 \leq s \leq t \leq T$ and $x, y \in D$. The evolution operator U is said to be compact if $U(t,s)$ maps bounded subsets of D into relatively compact subsets of D for all $0 \leq s < t \leq T$.

Consider the nonautonomous Cauchy problem:

$$\begin{cases} u'(t) + A(t)u(t) \ni f(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.2}$$

where $\{A(t)\}_{t\in[0,T]}$ satisfy $(H_{A(t)})$, $f \in L^1(0,T;X)$ and $u_0 \in D$.

Definition 2.1. An integral solution of (2.2) is a function $u \in C([0,T];D)$ satisfying $u(0) = u_0$ and the inequality

$$\|u(t)-x\|^2 - \|u(s)-x\|^2 \leq 2\int_s^t [< f(\tau)-y, u(\tau)-x >_+ + C\|u(\tau)-x\| \|m_1(\tau)-m_1(\theta)\|] d\tau,$$

for all $0 \leq s \leq t \leq T$, $\theta \in [0,T]$, $x \in D(A(\theta)),\ y \in A(\theta)x$, and
$C = m_2(\max\{\|x\|, \|u\|_{C([0,T];X)}\})$, with m_1, m_2 as in $(H_{A(t)})$ (ii).

It is well-known that problem (2.2) has a unique integral solution for each $u_0 \in D$ and $f \in L^1(0,T;X)$, provided that $(H_{A(t)})$ is satisfied. In particular, if $f \equiv 0$, then $U(t,0)u_0$ is the corresponding integral solution of (2.2). Moreover, the following result holds.

Proposition 2.2. *Let $(H_{A(t)})$ be satisfied, and let u and v be integral solutions of (2.2) corresponding to (u_0, f) and (u_0, g), respectively (with $u_0, v_0 \in D$ and $f, g \in L^1(0,T;X)$). Then*

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau, \tag{2.3}$$

for all $0 \leq s \leq t \leq T$.

The remainder of this section is devoted to a brief review of multifunctions. In what follows, the Banach space X will be assumed separable.

Let $\mathcal{P}_{cl}(X)$ denote the collection of all nonempty closed subsets of X. We also denote by $\mathcal{B}(X)$ the Borel $\sigma-$algebra on X, and by $\mathcal{L}$ the $\sigma-$algebra of Lebesgue measurable subsets on an interval $[0,T]$. Let (Ω,Σ) be a measurable space (we will particularly be interested in the case when $(\Omega,\Sigma)=([0,T]\times X,\mathcal{L}\otimes\mathcal{B}(X))$, where $\mathcal{L}\otimes\mathcal{B}(X)$ is the $\sigma-$algebra on $[0,T]\times X$ generated by sets of the form $A\times B$, with $A\in\mathcal{L}$ and $B\in\mathcal{B}(X)$). Let $\Phi:\Omega\to\mathcal{P}_{cl}(X)$. We say that Φ is measurable, if for all $x\in X$, the function

$$\omega\to d(x,\Phi(\omega))=\inf\{\|x-z\|:z\in\Phi(\omega)\}$$

is measurable.

By S^p_Φ $(1\le p<\infty)$, we denote the set of all measurable selections of Φ that belong to $L^p(\Omega;X)$, that is, $S^p_\Phi=\{\varphi\in L^p(\Omega;X):\varphi(\omega)\in\Phi(\omega),\text{ a.e. on }\Omega\}$. By the Kuratowski-Ryll Nardzewski Theorem (see, e.g., Hu and Papageorgiou [13], p. 175), it follows that for a measurable multifunction $\Phi:\Omega\to\mathcal{P}_{cl}(X)$, the set S^p_Φ is nonempty, iff $\inf\{\|x\|:x\in\Phi(\omega)\}\le h(\omega)$, a.e., for some $h\in L^p(\Omega;\mathbb{R}^+)$.

A set $K\subset L^p(0,T;X)(1\le p<\infty)$ is said to be decomposable, if for all $u,v\in K$ and all $A\in\mathcal{L}$, we have $u\mathcal{X}_A+v\mathcal{X}_{[0,T]\setminus A}\in K$, where $\mathcal{X}_A$ denotes the characteristic function of A. It is obvious that S^p_Φ has decomposable values.

Finally, let Y and Z be Hausdorff topological spaces, and let $\Psi:Y\to 2^Z$. We say that Ψ is lower semicontinuous (l.s.c., for short), if the set $\{y\in Y:\Psi(y)\subset A\}$ is closed in Y for each closed subset A of Z.

3. MAIN RESULTS

For fixed positive constants r,T, we set $B_r:=\{x\in X:\|x\|\le r\}$ and $K_r:=\{\phi\in C([0,T];X):\phi(t)\in B_r,\forall t\in[0,T]\}$.

We first consider problem (1.1) under the following conditions:

(H_1) $\{A(t)\}_{t\in[0,T]}$ satisfy $(H_{A(t)})$, and the corresponding evolution operator U (given by (2.1) with $D=\overline{D(A(t))}$, independent of t) is compact;

(H_2) $f:[0,T]\times B_r\to X$ is continuous in $t\in[0,T]$, and there exists a constant $L(r)>0$ such that $\|f(t,u)-f(t,v)\|\le L(r)\|u-v\|$, for all $t\in[0,T]$ and all $u,v\in B_r$;

(H_3) $g:C([0,T];X)\to D$ is a continuous mapping which maps K_r into a bounded set, and there is a $\delta=\delta(r)\in(0,T)$ such that $g(\phi)=g(\psi)$ for any $\phi,\psi\in K_r$ with $\phi(s)=\psi(s)$, $s\in[\delta,T]$;

(H_4) $T\sup\limits_{t\in[0,T],\,x\in B_r}\|f(t,x)\|+\sup\limits_{t\in[0,T],\phi\in K_r}\|U(t,0)g(\phi)\|\le r.$

Definition 3.1. A function $u \in C([0,T];D)$ is called an integral solution of problem (1.1), if u is an integral solution, in the sense of Definition 2.1, of (2.2) with $f(t,u(t))$ in place of $f(t)$ and $g(u)$ in place of u_0.

Our basic existence result is the following.

Theorem 3.2. *Let assumptions* $(H_1)-(H_4)$ *be satisfied. Then problem* (1.1) *has at least one integral solution.*

Remark 3.3.

(i) If $0 \in D(A(t))$, and $A(t)0 \ni 0,\ \ \forall t \in [0,T]$, then $U(t,0)0 = 0,\ \ \forall t \in [0,T]$ and $\|U(t,0)g(\phi)\| \leq \|g(\phi)\|,\ \ \forall \phi \in K_r,\ \ t \in [0,T]$. In this case, (H_4) holds if

$$T \sup_{t\in[0,T],\, x\in B_r} \|f(t,x)\| + \sup_{\phi\in K_r} \|g(\phi)\| \leq r. \tag{3.1}$$

(ii) Assume that $D = X,\ 0 \in D(A(t))$ and $A(t)0 \ni 0,\ \ \forall t \in [0,T]$.
Let $g : C([0,T];X) \to X$ be given by

$$g(u) = u_0 + \sum_{i=1}^{p} c_i u(t_i) \tag{3.2}$$

where $u_0 \in X$, p is a positive integer, $c_i (i = 1, \ldots, p)$ are given constants with $\sum\limits_{i=1}^{p} |c_i| < 1$, and $0 < t_1 < t_2 < \ldots < t_p \leq T$. Then (H_3) (with $\delta = t_1$) and (3.1) are satisfied, provided that

$$\|u_0\| + T \sup_{t\in[0,T],\, x\in B_r} \|f(t,x)\| \leq r(1 - \sum_{i=1}^{p} |c_i|).$$

We next study problem (1.2) in a real separable Banach space X, where $\{A(t)\}_{t\in[0,T]}$ satisfy (H_1), while g and F are subject to the following conditions:

(H_5) $g : C([0,T];D) \to D$ is such that

$$\|g(u) - g(v)\| \leq m\|u - v\|_{C([0,T];X)}$$

for all $u, v \in C([0,T];D)$ and some $0 < m < 1$.

(H_6) $F : [0,T] \times X \to \mathcal{P}_{cl}(X)$ satisfies

- (i) F is measurable,
- (ii) $x \to F(t,x)$ is l.s.c. for a.a. $t \in (0,T)$,
- (iii) there exists a function $\gamma : (0,T) \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(\cdot, r) \in L^1(0,T)$ for every $r \in \mathbb{R}^+$, $\gamma(t,\cdot)$ is continuous and nondecreasing for a.a. $t \in (0,T)$, with

$$\limsup_{r\to\infty} \frac{1}{r} \int_0^T \gamma(t,r)dt < 1 - m \tag{3.3}$$

where m is the same as in (H_5), and

$$|F(t,x)| := \sup\{\|w\| : w \in F(t,x)\} \leq \gamma(t, \|x\|), \tag{3.4}$$

for a.a. $t \in (0,T)$ and all $x \in D$.

Definition 3.4. A function $u \in C([0,T]; D)$ is said to be an integral solution of problem (1.2) if there exists $f \in L^1(0,T;X)$ with $f(t) \in F(t, u(t))$, a.e. on $(0,T)$, such that u is an integral solution, in the sense of Definition 2.1, of (2.2) where u_0 is replaced by $g(u)$.

The existence of integral solutions to problem (1.2) is established by the following result.

Theorem 3.5. *If X is separable and assumptions $(H_1), (H_5)$, and (H_6) are satisfied, then problem (1.2) has at least one integral solution.*

4. PROOFS

Proof of Theorem 3.2. Let $K_r(\delta) = \{\phi \in C([\delta,T]; X) : \phi(t) \in B_r, \forall t \in [\delta, T]\}$, where δ is as in (H_3). Clearly, $K_r(\delta)$ is a nonempty, closed, convex, bounded subset of $C([0,T]; X)$. For a fixed $v \in K_r(\delta)$, we define the mapping $\mathcal{F}_v : K_r \to C([0,T]; X)$ by $\mathcal{F}_v \phi = u_\phi$, where u_ϕ is the integral solution of

$$\begin{cases} u'_\phi(t) + A(t)u_\phi(t) \ni f(t, \phi(t)), \quad t \in [0,T], \\ u_\phi(0) = g(\tilde{v}), \end{cases} \tag{4.1}$$

with $\tilde{v} \in K_r$ given by

$$\tilde{v}(t) = \begin{cases} v(\delta) & \text{if } t \in [0,\delta], \\ v(t) & \text{if } t \in (\delta, T]. \end{cases}$$

We first remark that $\mathcal{F}_v$ maps K_r into itself. Indeed, from the definition of $\mathcal{F}_v$(cf. (4.1)) and (2.3), we obtain

$$\begin{aligned} \|(\mathcal{F}_v\phi)(t) - U(t,0)g(\tilde{v})(t)\| &\leq \textstyle\int_0^t \|f(s,\phi(s))\| ds \\ &\leq T \sup_{t\in[0,T],\, \phi\in K_r} \|f(t,\phi(t))\|, \quad \forall t \in [0,T],\ \phi \in K_r. \end{aligned}$$

This and (H_4) lead to

$$\begin{aligned} \|(\mathcal{F}_v\phi)(t)\| &\leq T \sup_{t\in[0,T],\, x\in B_r} \|f(t,x)\| + \sup_{t\in[0,T], \phi\in K_r} \|U(t,0)g(\phi)\| \\ &\leq r, \end{aligned}$$

for all $t \in [0,T]$ and $\phi \in K_r$. Hence $\mathcal{F}_v K_r \subset K_r$, as claimed.

Next, on account of (2.3) and (H_2), we deduce that for a positive integer n,

$$\|(\mathcal{F}_v^n \phi)(t) - (\mathcal{F}_v^n \psi)(t)\| \leq \frac{(tL(r))^n}{n!} \|\phi - \psi\|_{C([0,T];X)}, \ \forall t \in [0,T], \ \phi, \psi \in K_r.$$

Consequently, for n large enough, the mapping $\mathcal{F}_v^n$ is a strict contraction on K_r. Thus, by the Contraction Mapping Principle, $\mathcal{F}_v$ has a unique fixed point $\phi_v \in K_r$, which is the integral solution of

$$\begin{cases} \phi_v'(t) + A(t)\phi_v(t) \ni f(t, \phi_v(t)), & t \in [0,T], \\ \phi_v(0) = g(\tilde{v}). \end{cases} \tag{4.2}$$

We now define a map $\mathcal{G} : K_r(\delta) \to K_r(\delta)$ by $(\mathcal{G}v)(t) = \phi_v(t), t \in [\delta, T]$, where ϕ_v satisfies (4.2). From the definition of $\mathcal{G}$, (2.3) and (H_2), it follows that

$$\begin{aligned} &\|(\mathcal{G}v_1)(t) - (\mathcal{G}v_2)(t)\| = \|\phi_{v_1}(t) - \phi_{v_2}(t)\| \\ &\le \|\phi_{v_1}(0) - \phi_{v_2}(0)\| + \textstyle\int_0^t \|f(s, \phi_{v_1}(s)) - f(s, \phi_{v_2}(s))\| ds \\ &\le \|g(\tilde{v}_1) - g(\tilde{v}_2)\| + L(r) \textstyle\int_0^t \|\phi_{v_1}(s) - \phi_{v_2}(s)\| ds, \ \forall t \in [0,T]. \end{aligned}$$

Using Gronwall's inequality, we conclude that

$$\|\mathcal{G}v_1 - \mathcal{G}v_2\|_{C([0,T];X)} \le e^{TL(r)} \|g(\tilde{v}_1) - g(\tilde{v}_2)\|.$$

This, in conjunction with (H_3), implies the continuity of $\mathcal{G}$ on $K_r(\delta)$. We next adapt some of the arguments in Kartsatos and Shin [16] and Pavel [18].

Let $t \in [\delta, T]$ be fixed, and $0 < \varepsilon < t$. Define the function $v_\varepsilon : [t - \varepsilon, t] \to X$ by

$$v_\varepsilon(s) = U(s, t - \varepsilon)\phi_v(t - \varepsilon), \ \forall s \in [t - \varepsilon, T] \tag{4.3}$$

and note (cf., e.g., Pavel [18] and Pavel [19]) that v_ε is the integral solution of

$$u'(s) + A(s)u(s) \ni 0, \ \ t - \varepsilon \le s \le T; u(t - \varepsilon) = \phi_v(t - \varepsilon). \tag{4.4}$$

Then, by (2.3), (4.3) and (4.4), we derive

$$\|\mathcal{G}v(t) - v_\varepsilon(t)\| \le \int_{t-\varepsilon}^t \|f(\tau, \phi_v(\tau))\| d\tau \le M\varepsilon, \ \ \forall s \in [\delta, T], \tag{4.5}$$

where $M = \sup_{t \in [0,T],\, \phi \in K_r} \|f(t, \phi(t))\|$. Since $U(t, t - \varepsilon)$ is compact (cf. (H_1)), it follows that the set $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is relatively compact in X. Then (4.5) implies that the set $\{\mathcal{G}v(t) : v \in K_r(\delta)\}$ is relatively compact in X, as well.

Next, let us examine the equicontinuity of $\{\mathcal{G}v(t) : v \in K_r(\delta)\}$ at $t \in [\delta, T]$. On account of (H_1), we can invoke Theorem 1.1 in Pavel [19] to conclude that $\{v_\varepsilon(t) : v \in K_r(\delta)\}$ is equicontinuous at t, where v_ε is given by (4.3) for a fixed $t \in [\delta, T]$ and $\varepsilon \in (0, T)$. Therefore there exists $\gamma(t, \varepsilon) > 0$ such that

$$\|v_\varepsilon(s) - v_\varepsilon(t)\| \le M\varepsilon, \quad \forall v \in K_r(\delta) \tag{4.6}$$

for any $s \in [\delta, T]$ with $|s - t| \le \gamma(t, \varepsilon)$. Note that

$$\begin{aligned} \|\mathcal{G}v(s) - \mathcal{G}v(t)\| \le \ & \|\mathcal{G}v(s) - v_\varepsilon(s)\| + \|v_\varepsilon(s) - v_\varepsilon(t)\| \\ & + \|v_\varepsilon(t) - \mathcal{G}v(t)\|. \end{aligned} \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we obtain $\|\mathcal{G}v(s) - \mathcal{G}v(t)\| \le 3M\varepsilon, \ \ \forall v \in K_r(\delta)$, provided that $s \in [\delta, T]$, $|s - t| \le \gamma(t, \varepsilon)$. This proves the equicontinuity of $\{\mathcal{G}v(t) :$

$v \in K_r(\delta)\}$ at each $t \in [\delta, T]$. So, by Ascoli's Theorem, we infer that $\mathcal{G}(K_r(\delta))$ is relatively compact in $C([\delta, T]; X)$.

We can now apply Schauder's Fixed Point Theorem to conclude that $\mathcal{G}$ has at least one fixed point $v_* \in K_r(\delta)$. Let $u = \phi_{v_*}$ and remark that u is an integral solution, in the sense of Definition 3.1, of

$$\begin{cases} u'(t) + A(t)u(t) \ni f(t, u(t)), & t \in [0, T], \\ u(0) = g(\tilde{v}_*). \end{cases} \tag{4.8}$$

Inasmuch as $v_*(t) = (\mathcal{G}v_*)(t) = \phi_{v_*}(t) = u(t)$, for all $t \in [\delta, T]$, it follows by (H_3) that $g(\tilde{v}_*) = g(u)$. Hence (4.8) reduces to (1.1), so that u is an integral solution of problem (1.1), and the proof is complete. □

Proof of Theorem 3.5. Let $N : C([0, T]; X) \to 2^{L^1(0,T;X)}$ be defined by

$$N(u) = S^1_{F(\cdot, u(\cdot))}, \quad \forall u \in C([0, T]; X). \tag{4.9}$$

From (H_6) (i), (iii), it follows that N has nonempty, closed and decomposable values; cf. also Section 2. In addition, arguing as in Hu and Papageorgiou [13] p. 238, we see that (H_6) implies that N is l.s.c., as well. By the Bressan-Colombo Selection Theorem Bressan and Colombo [10], there exists a continuous function $f : C([0, T]; X) \to L^1(0, T; X)$ such that

$$f(u) \in N(u), \quad \forall u \in C([0, T]; X). \tag{4.10}$$

In other words (cf. (4.9), (4.10))

$$f(u)(t) \in F(t, u(t)), \text{ a.e. on } (0, T), \tag{4.11}$$

for all $u \in C([0, T]; X)$. In view of (4.11) and Definition 3.4, it is sufficient to prove the existence of an integral solution to the problem

$$\begin{cases} u'(t) + A(t)u(t) \ni f(u)(t), & t \in [0, T], \\ u(0) = g(u). \end{cases} \tag{4.12}$$

To accomplish this, we seek a fixed point of the map $\mathcal{F} : \mathcal{X} \to \mathcal{X}$, $\mathcal{X} = C([0, T]; X)$, defined by $\mathcal{F}v = u_v$, $\forall v \in \mathcal{X}$, where u_v is the unique integral solution of

$$\begin{cases} u'(t) + A(t)u(t) \ni f(v)(t), & t \in [0, T], \\ u(0) = g(u). \end{cases} \tag{4.13}$$

The existence and uniqueness of u_v follows as in the proof of Theorem 8 in Aizicovici and Staicu [4], on account of (H_1) and (H_5). We show that $\mathcal{F}$ is continuous and compact. Indeed, by (2.3) and (4.13), we have

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{X}} = \|u_{v_1} - u_{v_2}\|_{\mathcal{X}} \leq \|g(u_{v_1}) - g(u_{v_2})\| + \int_0^T \|f(v_1)(t) - f(v_2)(t)\| dt, \tag{4.14}$$

for all $v_1, v_2 \in \mathcal{X}$. Employing (H_5) in (4.14) yields

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\mathcal{X}} \leq \frac{1}{1-m}\|f(v_1) - f(v_2)\|_{L^1(0,T;X)},$$

which, by the continuity of f, implies that $\mathcal{F}$ is continuous on $\mathcal{X}$.

Next, let K be a bounded subset of $\mathcal{X}$. In view of (3.4) and (4.11), it follows that $\{f(v) : v \in K\}$ is a uniformly integrable subset of $L^1(0,T;X)$. In addition, by using this, (2.3), (4.13) and (H_5), it is easily seen that $\{g(u_v) : v \in K\}$ is bounded in $\mathcal{X}$. Consequently, invoking (H_1) and adapting the reasoning in the proof of Theorem 3 of Kartsatos and Shin [16], we conclude that $\{u_v : v \in K\}$ is relatively compact in $C([\varepsilon, T]; X)$, for any $0 < \varepsilon < T$. It actually follows that $\{u_v : v \in K\}$ is relatively compact in $\mathcal{X}$. This can be proved with the help of the operator $L : C([0,T]; D) \to \mathcal{X}_0$, given by

$$(Lw)(t) = w(t) - U(t,0)g(w), \forall t \in [0,T], w \in C([0,T]; D),$$

where $\mathcal{X}_0 = \{u \in \mathcal{X} : u(t) = w(t) - U(t,0)g(w), \forall t \in [0,T], \text{ for some } w \in \mathcal{X}\}$. Note (cf. Xue [23], Lemma 2.5) that L is one-to-one and onto, and L^{-1} is continuous on $\mathcal{X}_0$. Let $w_v(t) = u_v(t) - U(t,0)g(u_v)$, for all $t \in [0,T]$ and $v \in K$, so that $u_v(t) = L^{-1}(w_v)(t), t \in [0,T]$. In view of (H_1)(see, Pavel [19], Theorem 1.1), the set $\{U(t,0)g(u_v) : v \in K\}$ is relatively compact in $C([\varepsilon, T]; X), \forall 0 < \varepsilon < T$. Hence, $\{w_v : v \in K\}$ has the same property. Since $w_v(0) = 0, \forall v \in K$, the set $\{w_v(0) : v \in K\}$ is trivially compact in X. In addition, by (2.3), we have $\|w_v(t) - w_v(0)\| = \|u_v(t) - U(t,0)g(u_v)\| \leq \int_0^t \|f(v)(s)\|ds, \forall t \in [0,T]$. Recalling the uniform integrability of $\{f(v) : f \in K\}$, we conclude that $\{w_v(\cdot) : v \in K\}$ is equicontinuous at $t = 0$. So, finally, by Ascoli's Theorem, it follows that $\{w_v : v \in K\}$, and consequently $\{u_v : v \in K\}$ are relatively compact in $\mathcal{X}$. Therefore $\mathcal{F}$ is compact, as a map of $\mathcal{X}$ to $\mathcal{X}$, as claimed.

We can now apply the Leray-Schauder alternative (see Schaefer [20]) to establish that $\mathcal{F}$ has a fixed point in $\mathcal{X}$. To this end, we consider the set $S := \{v \in \mathcal{X} : \mathcal{F}v = \lambda v, \text{ for some } \lambda \geq 1\}$ and show that it is bounded. If $v \in S$, then by the definition of $\mathcal{F}$, λv is an integral solution of

$$\begin{cases} (\lambda v)'(t) + A(t)(\lambda v(t)) \ni f(v)(t), \quad t \in [0,T], \\ (\lambda v)(0) = g(\lambda v), \end{cases} \tag{4.15}$$

for some $\lambda \geq 1$. Let $z(t) = U(t,0)g(\bar{x})$, for a fixed constant function $\bar{x} : [0,T] \to D, \bar{x}(t) = x \ (x \in D)$. On account of (2.3) and (4.15), we obtain

$$\|\lambda v(t) - z(t)\| \leq \|g(\lambda v) - g(\bar{x})\| + \int_0^T \|f(v)(t)\|dt, \quad t \in [0,T]. \tag{4.16}$$

Employing (H_5) in (4.16) yields

$$\lambda\|v(t)\| \leq \|z(t)\| + \lambda m\|v\|_{\mathcal{X}} + m\|x\| + \int_0^T \|f(v)(t)\|dt, \quad t \in [0,T]. \tag{4.17}$$

Since $m \in (0,1)$ and $\lambda \geq 1$, (4.17) implies

$$(1-m)\|v\|_{\mathcal{X}} \leq C + \int_0^T \|f(v)(t)\|dt, \tag{4.18}$$

for some constant $C > 0$. Using (4.11) and (3.4) in (4.18), we arrive at

$$(1-m)\|v\|_{\mathcal{X}} \leq C + \int_0^T \gamma(t, \|v\|_{\mathcal{X}})dt.$$

This, in conjunction with (3.3), implies the existence of a positive constant M (independent of $v \in S$) such that $\|v\|_{\mathcal{X}} \leq M$, as desired. Consequently, by Schaefer's Fixed Point Theorem, Schaefer [20], we conclude that $\mathcal{F}$ has a fixed point $u \in \mathcal{X}$, which is an integral solution to (4.12), hence of problem (1.2). The proof is complete. □

5. APPLICATIONS

Throughout this section, Ω denotes a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$.

Example 5.1. Let $\rho : \mathbb{R} \to \mathbb{R}$ and $\alpha : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(H_ρ) $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ is nondecreasing, with $\rho(0) = 0$, and such that $\rho'(r) \geq K|r|^{p_0 - 1}, \forall r \in \mathbb{R} \setminus \{0\}$, for some constants $K > 0,\ p_0 \geq 1$;

(H_α) (i) $\alpha : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, $m-$accretive with respect to its third variable, with $\alpha(t,x,0) = 0, \forall (t,x) \in [0,T] \times \Omega$, and such that

$$|\alpha(t,x,u)| \leq q_1(t,x) + q_2(t)|u|, \quad \forall (t,x,u) \in [0,T] \times \Omega \times \mathbb{R},$$

where $q_1 : [0,T] \times \Omega \to \mathbb{R}^+$ is in $L^1(\Omega)$ for each $t \in [0,T]$ and $q_2 : [0,T] \to \mathbb{R}^+$,

(ii) there exists a function $h : [0,T] \to \mathbb{R}$, continuous and of bounded variation, and a function $q_3 \in L^1(\Omega)$ such that for any $t, s \in [0,T]$ and any $u \in \mathbb{R}$ one has

$$|\alpha(t,x,u) - \alpha(s,x,u)| \leq |h(t) - h(s)|(1 + |q_3(x)| + |u|),$$

for a.a. $x \in \Omega$,

(iii) there exists a constant $C > 0$ such that

$$|\alpha(t,x,u) - \alpha(t,x,v)| \leq C|u - v|,$$

for all $(t,x,u,v) \in [0,T] \times \overline{\Omega} \times \mathbb{R}^2$.

Let $X = L^1(\Omega)$ and define the operators $A(t)$ in $L^1(\Omega)$, for $t \in [0,T]$ by

$$\begin{cases} A(t)u(x) = -\Delta\rho(u(x)) + \alpha(t,x,u(x)), \text{ a.e. on } \Omega, \\ D(A(t)) = \{u \in L^1(\Omega); \rho(u) \in W_0^{1,1}(\Omega), \Delta\rho(u) \in L^1(\Omega)\}. \end{cases} \tag{5.1}$$

Clearly, $D(A(t))$ is independent of t, with $\overline{D(A(t))} = X$. According to the theory developed in Kartsatos [15] (see also Pavel [17]), under assumptions (H_ρ) and (H_α), the operators $A(t)$ satisfy (H_1) with $D = X$. Next, let $f : [0,T] \times X \to X$ be given by

$$f(t,u)(x) = \sin(u(x)), \forall u \in X, \text{ a.e. on } \Omega. \tag{5.2}$$

It is obvious that f satisfies (H_2) for any $r > 0$, with $L(r) = 1$. Finally, let $g : C([0,T]; X) \to X$ be as in (3.2), so that (H_3) also holds.

We consider the nonlocal initial boundary value problem:

$$\begin{cases} u_t(t,x) - \Delta\rho(u(t,x)) + \alpha(t,x,u(t,x)) = \sin(u(t,x)), & \text{a.e. on } (0,T)\times\Omega, \\ \rho(u(t,x)) = 0, \text{ a.e. on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) + \sum\limits_{i=1}^{p} c_i u(t_i,x), & \text{a.e. on } \Omega, \end{cases} \tag{5.3}$$

where $0 < t_1 < t_2 < \cdots < t_p \le T$ and $c_i (i = 1,2,\ldots,p)$ are given constants, with $\sum\limits_{i=1}^{p} |c_i| < 1$, and $u_0 \in L^1(\Omega)$. In view of the above discussion, it follows that (5.3) can be rewritten in the abstract form (1.1) in the space $X = L^1(\Omega)$, with $A(t), f$ and g given by (5.1), (5.2) and (3.2), respectively. By Remark 3.3 (ii), it is easily seen that (H_4) is satisfied if r is chosen large enough, so that

$$\|u_0\|_{L^1(\Omega)} + T \text{ meas}(\Omega) \le r(1 - \sum_{i=1}^{p} |c_i|),$$

where meas(Ω) denotes the Lebesgue measure of Ω.

An application of Theorem 3.2 yields:

Corollary 5.1. *Assume (H_ρ) and (H_α). If also $\sum\limits_{i=1}^{p} |c_i| < 1$, then problem* (5.3) *has at least one integral solution $u \in C([0,T]; L^1(\Omega))$.*

Example 5.2. Again, let $X = L^1(\Omega)$ and $A(t)$ be given by (5.1), where ρ and α satisfy (H_ρ) and (H_α), respectively. Let $\beta : [0,T] \times \Omega \times \mathbb{R}^l \to \mathbb{R} (l \ge 1)$ and $V : \Omega \times \mathbb{R} \to P_k(\mathbb{R}^l)$ (where $P_k(\mathbb{R}^l)$ denotes the collection of all nonempty compact subsets of $\mathbb{R}^l$) satisfy respectively:

(H_β) (i) $(t,x) \to \beta(t,x,z)$ is measurable for each $z \in \mathbb{R}^l$,

(ii) $z \to \beta(t,x,z)$ is continuous for a.a. $(t,x) \in [0,T] \times \Omega$,

(iii) $|\beta(t,x,z)| \le a_1(t,x) + a_2(t)\|z\|_{\mathbb{R}^l}$, a.e. on $[0,T] \times \Omega$, $\forall z \in \mathbb{R}^l$, with $a_1 \in L^1((0,T)\times\Omega; \mathbb{R}^+)$, $a_2 \in L^1(0,T;\mathbb{R}^+)$;

(H_V) (i) $(x,r) \to V(x,r)$ is measurable,

(ii) $r \to V(x,r)$ is l.s.c. for a.a. $x \in \Omega$, (*iii*) $|V(x,r)| := \sup\{\|w\|_{\mathbb{R}^l} : w \in V(x,r)\} \leq b_1(x)+b_2|r|$, a.e. on Ω, $\forall r \in \mathbb{R}$, with $b_1 \in L^1(\Omega;\mathbb{R}^+)$, $b_2 \geq 0$.

Define the multifunction $F : [0,T] \times L^1(\Omega) \to 2^{L^1(\Omega)}$ by

$$F(t,u) = \{\beta(t,\cdot,v(\cdot)) : v(x) \in V(x,u(x)),\ \text{a.e. on } \Omega, v \in L^1(\Omega;\mathbb{R}^l)\}. \tag{5.4}$$

Adapting the arguments given in Hu and Papageorgiou [14], p.186, we conclude that F is closed-valued and satisfies $(H_6)(i)$, (ii). Also, by $(H_\beta)(iii)$, $(H_V)(iii)$ and (5.4), it follows that (3.4) holds with

$$\gamma(t,r) = \|a_1(t,\cdot)\|_{L^1(\Omega)} + a_2(t)\|b_1\|_{L^1(\Omega)} + a_2(t)b_2 r. \tag{5.5}$$

Next, let $g : C([0,T];X) \to X$ be given by

$$g(u)(x) = \int_0^T G(s,u(s,x))ds,\ \ \forall u \in C([0,T];X),\ \text{a.e. on } \Omega, \tag{5.6}$$

where $G : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies:

(H_G) (*i*) $G(\cdot,r)$ is measurable for each $r \in \mathbb{R}$ and $G(\cdot,0) \in L^1(\Omega)$,
(*ii*) $G(t,\cdot)$ is continuous for a.a. $t \in [0,T]$,
(*iii*) there exists $k \in L^1(0,T;\mathbb{R}^+)$, with $\|k\|_{L^1} < 1$, such that

$$|G(t,r) - G(t,\bar{r})| \leq k(t)|r-\bar{r}|,$$

for all $r, \bar{r} \in \mathbb{R}$, and a.a. $t \in [0,T]$.

Then it is easily verified that g, as given by (5.6), is well-defined and satisfies (H_5), with $m = \|k\|_{L^1(0,T)}$. Finally, in view of (5.5), it follows that (3.3) holds provided that

$$b_2\|a_2\|_{L^1(0,T)} + \|k\|_{L^1(0,T)} < 1. \tag{5.7}$$

We now consider the problem:

$$\begin{cases} u_t(t,x) - \Delta\rho(u(t,x)) + \alpha(t,x,u(t,x)) \in \beta(t,x,V(x,u(t,x))), & \text{a.e. on } (0,T)\times\Omega, \\ \rho(u(t,x)) = 0,\ \text{a.e. on } (0,T)\times\partial\Omega, \\ u(0,x) = \int_0^T G(s,u(s,x))ds,\ \ \text{a.e. on } \Omega, \end{cases} \tag{5.8}$$

and remark that it can be reduced to the abstract form (1.2) in $X = L^1(\Omega)$, with $A(t), F$ and g given by (5.1), (5.4) and (5.6), respectively.

Applying Theorem 3.5, we obtain:

Corollary 5.2. *Assume (H_ρ), (H_α), (H_β), (H_V) and (H_G). If also* (5.7) *holds, then problem* (5.8) *has at least one integral solution $u \in C([0,T];L^1(\Omega))$.*

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