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ENTROPY SOLUTION OF A QUASILINEAR ELLIPTIC PROBLEM WITH NONLINEAR BOUNDARY CONDITION

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ABSTRACT: In this paper, we consider the equation $u - \operatorname{div}\, a(u, Du) = f$ on a bounded domain with nonlinear boundary conditions of the form $-a(u, Du)\cdot\eta \in \beta(x, u)$. We introduce a notion of entropy solution for this problem and prove existence and uniqueness of this solution for general L^1-data.

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and $1 < p < N$. Consider the nonlinear stationary problem

$$(E)(f)\begin{cases} u - \operatorname{div}\, a(u, Du) = f & \text{in} \quad \Omega\,, \\ -\langle a(u, Du), \eta\rangle \in \beta(x, u) & \text{on} \quad \partial\Omega, \end{cases}$$

where η is the unit outward normal vector on $\partial\Omega$, $f \in L^1(\Omega)$, Du denotes the gradient of u and, for a.e. $x \in \partial\Omega$, $\beta(x, r) = \partial j(x, r)$ is the subdifferential of a function $j : \partial\Omega \times \mathbb{R} \to [0, \infty]$ which is convex, lower semicontinuous (l.s.c. for short) in $r \in \mathbb{R}$ for a.e. $x \in \partial\Omega$, measurable with respect to the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$ and such that $j(\cdot, 0) = 0$. The vector-valued function $a : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is continuous satisfying the following classical Leray-Lions-type conditions:

(H_1)- monotonicity in $\xi \in \mathbb{R}^N$:

$$(a(r, \xi) - a(r, \eta)) \cdot (\xi - \eta) \geq 0\,, \quad \forall r \in \mathbb{R},\ \forall \xi, \eta \in \mathbb{R}^N;$$

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(H_2)- coerciveness : there exist $\lambda_0 > 0$, $p > 1$ such that

$$(a(r,\xi) - a(r,0)) \cdot \xi \geq \lambda_0 |\xi|^p\,, \quad \forall r \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N;$$

(H_3)- growth restriction : there exists a function $\Lambda : \mathbb{R}^+ \to \mathbb{R}$ such that

$$|a(r,\xi)| \leq \Lambda(|r|)(1 + |\xi|^{p-1})\,, \quad \forall r \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N;$$

and, moreover,

(H_4)- there exists $C : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous such that

$$|a(r,\xi) - a(s,\xi)| \leq C(r,s)|r-s|(1 + |\xi|^{p-1})\,, \quad \forall r, s \in \mathbb{R}.$$

A typical example of a function a satisfying these hypotheses is $a(r,\xi) = |\xi|^{p-2}\xi + F(r)$, where $F : \mathbb{R} \to \mathbb{R}^N$ is a locally Lipschitz function. Note that the condition $-\langle a(u, Du), \eta\rangle \in \beta(x,u)$ on $\partial\Omega$ includes in particular mixed Dirichlet-Neumann conditions on the boundary and conditions of obstacle type. For many applications it is necessary to study such general type of boundary conditions.

Many results are known for elliptic problems in the variational setting for Dirichlet or Dirichlet Neumann problems (see Alt and Luckhaus [1], Bénilan and Wittbold [16], Carrillo [20], Prignet [27], Simondon [28]). In the L^1-setting, for elliptic and parabolic equations in divergence form, in last decade the new equivalent notions of entropy and renormalized solutions have been introduced and existence and uniqueness results for this new type of solution have been proved under various assumptions (see Ammar [2], Andreu et al [7], Bénilan et al [13], Boccardo et al [17]). In particular, in Andreu et al [7], a notion of entropy solution has been introduced for the nonlinear problem $(E)(f)$ with a being independent of u and the graph β being independent of the space variable. Under a regularity assumption on a and for particular graphs β, the authors prove existence and uniqueness of this entropy solution for arbitrary L^1-data.

Moreover, note that, in Ammar [2], a new notion of entropy solution was introduced for the problem

$$\begin{cases} Cf + f = \psi \ \text{ on } \ \partial\Omega\,, \\ \qquad \psi \in L^1(\partial\Omega), \end{cases}$$

where C is a capacity operator defined from $W^{\frac{1}{p'},p}(\partial\Omega)$ to his dual $W^{-\frac{1}{p'},p'}(\partial\Omega)$ by $\langle Cf, g\rangle = \int \langle a(x,Du), Dv\rangle$, $f, g \in W^{\frac{1}{p'},p}(\partial\Omega)$, where $u, v \in W^{1,p}(\partial\Omega), v_{|\partial\Omega} = g$ and - div $a(x,Du) = 0$, $u_{|\partial\Omega} = f$.. This approach allowed the author in the following to prove well-posedness of problems of type $(E)(f)$ and even of more general form. The disadvantage of this approach is that it is strictly restricted to the case, where the vector field a does not depend on the function u, but only on Du, and, possibly, the space variable.

Therefore, in the present paper, we use and extend the methods introduced in Andreu et al [7] to study the problem

$$\begin{cases} u - \operatorname{div} a(x, Du) = f & \text{in} \quad \Omega\,, \\ -a(x, Du) \cdot \eta \in \beta(u) & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$. We generalize their results for a divergence operator depending on u and for β depending also on the space variable x.

The present paper is organized as follows. In the next section we make precise the notations which will be used in the sequel and recall some facts on measures and capacities. In Section 3, we study the problem $(E)(f)$ by variational methods. We introduce an accretive operator A_δ related to some penalized version of problem $(E)(f)$ and show that $R(I + \alpha A_\delta) \supset L^\infty(\Omega)$ for all $\alpha > 0$. In Section 4, we introduce the notion of entropy solution of the original problem $(E)(f)$ and prove that the weak solutions of the penalized problem converge to the unique entropy solution of $(E)(f)$. By the way we characterize $\mathcal{A}$, the limit of the operator A_δ in $L^1(\Omega)$, which is associated to the limit equation. Finally, in Section 5 we discuss possible extensions of our results.

2. PRELIMINARIES

In this section, we introduce some notations and definitions used in this paper. We denote $|\cdot|$ and $d\sigma$ the N-dimensional Lebesgue measure in $\mathbb{R}^N$ and the $(N-1)$-dimensional Hausdorff measure of $\partial\Omega$, respectively. The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$, $1 \le p \le \infty$. $W^{1,p}(\Omega)$ denotes the classical Sobolev space endowed with the norm denoted $\|\cdot\|_{1,p}$. It is well-known (see Morrey [25], Nečas [26]) that if $u \in W^{1,p}(\Omega)$, it is possible to define the trace of u on $\partial\Omega$, where the continuous linear trace operator $\tau : W^{1,p}(\Omega) \to W^{\frac{1}{p'},p}(\partial\Omega)$ is surjective. In particular, as Ω is smooth, any function $v \in W^{\frac{1}{p'},p}(\partial\Omega)$ is the trace of a function $\widehat{v} \in W_0^{1,p}(G)$ such that $\widehat{v}_{|\partial\Omega} = v$, where G is an arbitrary fixed open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \subset G$.

For $k > 0$, we denote by T_k the truncation function at height $k \ge 0$, defined by

$$T_k(u) = \begin{cases} k \operatorname{sign}_0(u) & \text{if} \quad |u| > k\,, \\ u & \text{if} \quad |u| \le k, \end{cases}$$

where, $\operatorname{sign}_0(\cdot)$ denotes the single-valued function defined by $\operatorname{sign}_0(r) = -1$ if $r < 0$, $\operatorname{sign}_0(r) = 1$ if $r > 0$, $\operatorname{sign}_0(r) = 0$ if $r = 0$. In the sequel, C will denote a constant that may change from line to line. Throughout the paper, for the sake of simplicity, for any measurable function u defined on Ω and any $K \ge 0$, we denote by $\{|u| \le K\}$ the measurable subset $\{x \in \Omega; |u(x)| \le K\}$. We will write $\int_\Omega u = \int_\Omega u(x)dx$. We denote by $\overline{u}$ the average of u, i.e., $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx$. We denote by $\mathcal{P}$ the set of functions $\{S \in C^1(\mathbb{R})/ \ S(0) = 0,\ 0 \le S' \le \ 1,\ \operatorname{Supp}(S') \text{ is compact}\}$.

Let $\mathcal{A}$ be a multi-valued operator in $L^1(\Omega)$. Recall that $\mathcal{A}$ is said to be accretive in $L^1(\Omega)$ if $\|u-\tilde{u}\|_1 \leq \|u-\tilde{u}+\alpha(v-\tilde{v})\|_1$ for any $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{A};\ \alpha > 0$ i.e., for any $\alpha > 0$, the resolvent of $\mathcal{A}$, $(I+\alpha\mathcal{A})^{-1}$, is a single-valued operator and a contraction in L^1-norm. $\mathcal{A}$ is called T-accretive if $\|(u-\tilde{u})^+\|_1 \leq \|u-\tilde{u}+\alpha(v-\tilde{v})^+\|_1$ for any $(u,v),(\tilde{u},\tilde{v}) \in \mathcal{A}$ and for any $\alpha > 0$. Finally, $\mathcal{A}$ is called m-accretive (resp. m-T-accretive) in $L^1(\Omega)$ if $\mathcal{A}$ is accretive (resp. T-accretive) and moreover, $R(I+\alpha\mathcal{A}) = L^1(\Omega)$ for any $\alpha > 0$ (see Barbu [9], Bénilan [10], Bénilan et al [14] for the theory of accretive operators and nonlinear semigroups).

For a monotone graph β in $\mathbb{R}\times\mathbb{R}$ and $\lambda \in \mathbb{N}$ we denote by β_λ the Yosida approximation of β, given by $\beta_\lambda = \frac{1}{\lambda}(I-(I+\lambda\beta)^{-1})$. The function β_λ is monotone and Lipschitz. We recall the definition of the main section β^0 of β:

$$\beta^0(r) = \begin{cases} \inf\beta(r) & \text{if } r > 0 \\ 0 & \text{if } r = 0 \\ \sup\beta(r) & \text{if } r < 0, \end{cases}$$

with the usual convention $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$.

Now, let us introduce some notations about capacities and measures used throughout this paper (we refer the reader to Dal Maso et al [21], Dunfort and Schwartz [23]). Given $E \subseteq G$, $C_{1,p}(E)$ denotes the p-capacity of E with respect to the norm of $W^{1,p}(G)$ and it is defined in the following way: If $O \subset\subset G$ is open, then

$$C_{1,p}(O) = \inf\{\|\varphi\|_{1,p}; \varphi \in W_0^{1,p}(G), \varphi \geq \chi_O \text{ a.e. on } G\}.$$

The p-capacity of an arbitrary subset $E \subseteq G$ is defined by

$$C_{1,p}(E) = \inf\{C_{1,p}(O),\ O \text{ open}, E \subseteq O\}.$$

A function u defined on G is said to be cap-quasi continuous if for every $\varepsilon > 0$ there exists an open set $B \subseteq G$ with $C_{1,p}(B) < \varepsilon$ such that the restriction of u to $G\backslash B$ is continuous. It is well-known that every function in $W_0^{1,p}(G)$ has a cap-quasi continuous representative, i.e. a function $\tilde{u} : G \to \mathbb{R}$ such that $u = \tilde{u}$ a.e. on G and $\tilde{u}$ is cap-quasi-continuous. In particular, by the remarks above, any function $v \in W^{\frac{1}{p'},p}(\partial\Omega)$ has a cap-quasi-continuous representative $\tilde{v}$. Indeed, $\exists\widehat{v} \in W_0^{1,p}(G)$ such that $\tilde{\widehat{v}}$ is quasi-continuous represent of $\widehat{v}$ on G and $\tilde{\widehat{v}}_{|\partial\Omega} = v$ a.e. on $\partial\Omega$. As usual, a property will be said to hold cap-quasi everywhere (q.e. for short) if it holds everywhere except on a set of zero capacity.

Let $\mathcal{M}_b(\partial\Omega)$ be the space of all Radon measures on $\partial\Omega$ with bounded total variation. For $\mu \in \mathcal{M}_b(\partial\Omega)$ denote by μ^+, μ^- and $|\mu|$ the positive part, negative part and the total variation of the measure μ, respectively, and denote by $\mu = \mu_r d\sigma + \mu_s$ the Radon-Nikodym decomposition of μ relatively to the $(N-1)$-dimensional Hausdorff measure $d\sigma$.

We denote by $\mathcal{M}_b^p(\partial\Omega)$ the set of Radon measures μ which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \partial\Omega$ such that $C_{1,p}(B) = 0$, i.e., the Radon measures which do not charge sets of zero capacity.

We denote $\mathcal{J}_0(\partial\Omega) = \{j/j : \partial\Omega \times \mathbb{R} \to [0,\infty],\ j(\cdot, r)\ \sigma\text{-measurable } \forall r \in \mathbb{R},\ j(x,\cdot)$ convex, l.s.c. satisfying $j(x,0) = 0$ for a.e. $x \in \partial\Omega\}$. For a.e. x, we define $\beta(x,r) = \partial j(x,r)\ \forall j \in \mathcal{J}_0$. Given $j \in \mathcal{J}_0(\Omega)$, we define

$$\begin{aligned} \mathcal{J} : W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega) \quad &\longrightarrow \quad [0,\infty] \\ u \quad &\longmapsto \quad \int_{\partial\Omega} j(\cdot, u) d\sigma. \end{aligned}$$

Note that $\mathcal{J}$ naturally extends to a functional $\widehat{\mathcal{J}}$ on $W_0^{1,p}(G) \cap L^\infty(G)$ as follows: $\widehat{\mathcal{J}}(u) = \int_{\partial\Omega} j(\cdot, \tau(u)) d\sigma$ for any $u \in W_0^{1,p}(G)$. We recall that the closure of $D(\widehat{\mathcal{J}})$ in $W_0^{1,p}(G)$ is a convex bilateral set, so according to Attouch and Picard [8], there exist unique (in the sense q.e.) functions γ_+, γ_- which are cap-quasi-l.s.c. and cap-quasi-u.s.c. respectively, such that

$$\overline{\mathcal{D}(\mathcal{J})}^{\|\cdot\|_{\frac{1}{p'},p}} = \{u \in W^{\frac{1}{p'},p}(\partial\Omega);\ \gamma_-(x) \le \tilde{u}(x) \le \gamma_+(x) \ \text{ q.e. on } \ \partial\Omega\}.$$

Moreover, $\gamma_-(x) = \inf_n \tilde{u}_n(x) = \lim_n \inf_{1\le k\le n} \tilde{u}_k(x)$ q.e. $x \in \partial\Omega$. Analogous property holds for γ_+. For any $\|\cdot\|_{\frac{1}{p'},p}$ -dense sequence $(u_n)_n$ in $\mathcal{D}(\mathcal{J})$. Recall that the sub-differential operator $\partial\mathcal{J} \subseteq (W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)) \times (W^{-\frac{1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^*)$ is monotone and is given by

$$\begin{aligned} &M \in \partial\mathcal{J}(u) \Longleftrightarrow \\ &\begin{cases} u \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega);\ M \in W^{-\frac{1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^* \\ \text{and } \mathcal{J}(w) \ge \mathcal{J}(u) + \langle M, w-u\rangle \ \forall w \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega), \end{cases} \end{aligned}$$

where, here and in the sequel, if not explicitly stated otherwise, $\langle\cdot,\cdot\rangle$ denotes the duality between $W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and its dual.

3. VARIATIONAL APPROACH

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, $1 < p < N$, a a mapping $\Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the assumptions $(H_1) - (H_4)$ and β is such that $\beta(x,\cdot) = \partial j(x,\cdot)$ a.e. on $\partial\Omega$, where $j \in \mathcal{J}_0(\partial\Omega)$.

To apply the classical variational approach, we need an L^∞-estimate on u, which is not evident to obtain directly in our problem. The obstacle which we encounter is that we can not get rid of the term with $a(u,0)$. To overcome this difficulty, we first redefine and extend the function Λ, which appears in assumption (H_3), on an odd monotone function on $\mathbb{R}$ such that $|\frac{a(k,0)}{\Lambda(k)}| \to 0$ as $k \to \infty$. This will be possible by setting $\Lambda(r) := \sup_{-r\le z\le r} \{\Lambda(|z|), |z||a(z,0)|\}$ for $r \ge 0$. Secondly, we add a penalization term $\delta\Lambda(u)$ on the boundary for a fixed δ. This allows us to compensate the term with $a(u,0)$ by choosing k sufficient large such that $|\frac{a(k,0)}{\Lambda(k)}| < \delta$.

In the next section, we tend δ to zero and the penalization term disappears. Consequently we obtain the entropy solution of our initial problem $(E)(f)$.

Now, we define the operator A_δ as follows: $(u,f) \in A_\delta$ if and only if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$; $f \in L^1(\Omega)$ and there exists a measure $\mu \in \mathcal{M}_b^p(\partial\Omega)$ with $\mu_r(x) \in \partial j(x,u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x))$ a.e. $x \in \partial\Omega$ such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega a(u,Du)\cdot D(u-\phi) + \delta\int_{\partial\Omega}\Lambda(u)(u-\phi) \le \int_\Omega f(u-\phi) - \int_{\partial\Omega}(\tilde{u}-\tilde{\phi})d\mu,$$

$$\widetilde{u} = \gamma_+ \quad \mu_s^+ - \text{a.e.} \quad \text{on} \quad \partial\Omega, \qquad \widetilde{u} = \gamma_- \quad \mu_s^- - \text{a.e.} \quad \text{on} \quad \partial\Omega, \tag{3.1}$$

where for given interval $[a,b] \subset \mathbb{R}$, $I_{[a,b]}$ denotes the convex l.s.c. functional on $\mathbb{R}$ defined by 0 on $[a,b]$, $+\infty$ otherwise.

Remark 3.1. We will prove (see equation (3.14) below), that the measure $\mu \in \mathcal{M}_b(\partial\Omega) \cap (W^{-\frac{1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^*)$ and also $|\mu|$ does not charge sets of zero capacity. From $|\mu_s| \le |\mu|$, it follows that $|\mu_s|$ does not charge sets of 0-capacity. Consequently, the condition (3.1) is meaningful.

We can now state the first main result.

Theorem 3.1. *The operator A_δ satisfies the following properties:*

i) *A_δ is T-accretive in $L^1(\Omega)$,*
ii) *$L^\infty(\Omega) \subset R(I+\alpha A_\delta)$ for any $\alpha > 0$,*
iii) *$D(A_\delta)$ is dense in $L^1(\Omega)$.*

Proof. *i)* Let u, v such that

$$f \in u + A_\delta u, \quad g \in v + A_\delta v. \tag{3.2}$$

We must show that

$$\int_\Omega (u-v)^+ \le \int_\Omega (f-g)^+. \tag{3.3}$$

Taking $\phi_1 = u - \frac{1}{k}T_k(u-v)^+$ and $\phi_2 = v + \frac{1}{k}T_k(u-v)^+$ as test functions in (3.2) respectively, we get after adding inequalities

$$\begin{aligned}&\frac{1}{k}\int_{\{(u-v)^+<k\}}(a(u,Du)-a(v,Dv))\cdot D(u-v)^+\\&+\frac{1}{k}\delta\int_{\partial\Omega}(\Lambda(u)-\Lambda(v))T_k(u-v)^+\\&\le\frac{1}{k}\int_\Omega((f-u)-(g-v))T_k(u-v)^+ - \frac{1}{k}\Big(\int_{\partial\Omega}T_k(\tilde{u}-\tilde{v})^+d\mu_1\\&-\int_{\partial\Omega}T_k(\tilde{u}-\tilde{v})^+d\mu_2\Big).\end{aligned} \tag{3.4}$$

Denote by I_1 respectively I_2 the first, respectively the second integral in the left hand side of (3.4). Using assumptions (H_1) and (H_4), we have

$$\begin{aligned} I_1 &\geq \frac{1}{k}\int_{\{(u-v)^+<k\}} (a(u,Dv)-a(v,Dv))\cdot D(u-v)^+ \\ &\geq \frac{-Ck}{k}\int_{\{(u-v)^+<k\}} (1+|Dv|^{p-1})D(u-v)^+ \quad \to 0 \ \text{ as } \ k\to 0. \end{aligned}$$

Note that the properties of the measures μ_1 and μ_2 guarantee to us that the second term in the brackets in the right hand side of (3.4) is nonnegative. Indeed, these integrals can be written as $\int_{\partial\Omega} T_k(u-v)(\mu_{r,1}-\mu_{r,2})+\int_{\partial\Omega} T_k(\gamma_+-\tilde{v})d\mu_{s,1}^+ +\int_{\partial\Omega} -T_k(\tilde{u}-\gamma_+)d\mu_{s,2}^+ +\int_{\partial\Omega} -T_k(\gamma_- - \tilde{v})d\mu_{s,1}^- +\int_{\partial\Omega} T_k(\tilde{u}-\gamma_-)d\mu_{s,2}^-$, which are, clearly, nonnegative by properties of μ_1,μ_2 and $\gamma_{+/-}$.

Since $I_2 \geq 0$ (thanks to the monotonicity of Λ), we get after passing to the limit in (3.4) with $k\to 0$

$$\lim_{k\to 0}\frac{1}{k}\int_\Omega (u-v)T_k(u-v)^+ \leq \lim_{k\to 0}\frac{1}{k}\int_\Omega (f-g)T_k(u-v)^+ \leq \int_\Omega (f-g)^+.$$

Consequently (3.3) holds.

ii) It will be no restriction to assume that $\alpha=1$. In order to prove that $L^\infty(\Omega)\subset R(I+A_\delta)$, we approximate the problem

$$\begin{cases} u-\operatorname{div}\, a(u,Du)=f, & \text{in} \quad \Omega\,, \\ \\ -a(u,Du)\cdot\eta \in \beta(x,u)+\delta\Lambda(u) & \text{on} \quad \partial\Omega\,, \end{cases}$$

by problems of the form

$$\begin{cases} T_l(u_\lambda)-\operatorname{div}\, a(T_l(u_\lambda),Du_\lambda)=f & \text{in} \quad \Omega\,, \\ \\ -a(T_l(u_\lambda),Du_\lambda)\cdot\eta=\beta_\lambda(x,u_\lambda)+\delta T_l(\Lambda(u_\lambda)) & \text{on} \quad \partial\Omega, \end{cases}$$

where $l>\max\{k,\Lambda(k)\}, k>\|f\|_\infty+1$ and k satisfies $|\frac{a(k,0)}{\Lambda(k)}|<\delta$. Here for every $\lambda\in\mathbb{N}$, $\beta_\lambda(x,\cdot)$ is the Yosida approximation of $\beta(x,\cdot)$, i.e. $\beta_\lambda(x,\cdot)=1/\lambda(I-(I+\lambda\beta(x,\cdot))^{-1})$.

Consider the operator $A_{\delta,\lambda}: W^{1,p}(\Omega)\to (W^{1,p}(\Omega))^*$ defined by

$$\begin{aligned} &\langle A_{\delta,\lambda}u,\phi\rangle \\ &= \int_\Omega T_l(u)\phi + a(T_l(u),Du)\cdot D\phi + \int_{\partial\Omega}\beta_\lambda(\cdot,u)\phi+\delta T_l(\Lambda(u))\phi \end{aligned}$$

for all $\phi\in W^{1,p}(\Omega)$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$.

Lemma 3.1. *The operator $A_{\delta,\lambda}$ is bounded, coercive and verifies the (M)-property.*

The proof of this lemma is straightforward (see Lions [24]).

By Lemma 3.1 and the result of Browder (see Lions [24]), for all $f \in (W^{1,p}(\Omega))^*$ there exists $u_\lambda \in W^{1,p}(\Omega)$ such that for all $\phi \in W^{1,p}(\Omega)$

$$\langle A_{\delta,\lambda} u_\lambda - f, u_\lambda - \phi \rangle \leq 0. \tag{3.5}$$

In order to pass to the limit as $\lambda 0$ in inequality (3.5) we need a uniform $L^\infty-$estimates and the strong convergence of the solution u_λ. To this end, take $\phi = u_\lambda - p_\varepsilon(u_\lambda - k)$ as a test function in (3.5), where $p_\varepsilon(\cdot)$ is an approximation of $\text{sign}^+(\cdot)$ defined as follow

$$p_\varepsilon(r) = \begin{cases} 1 & \text{if} \quad r > \varepsilon \\ \frac{1}{\varepsilon} r & \text{if} \quad 0 < r < \varepsilon \\ 0 & \text{if} \quad r < 0, \end{cases}$$

using assumption (H_2), we have

$$\begin{aligned} & \int_\Omega p_\varepsilon(u_\lambda - k) T_l(u_\lambda) + \frac{1}{\varepsilon} \int_{\{k < u_\lambda < k+\varepsilon\}} a(T_l(u_\lambda), 0) \cdot Du_\lambda \\ & + \delta \int_{\partial\Omega} p_\varepsilon(u_\lambda - k) T_l(\Lambda(u_\lambda)) \\ \leq \; & \int_\Omega f p_\varepsilon(u_\lambda - k) - \int_{\partial\Omega} p_\varepsilon(u_\lambda - k) \beta_\lambda(\cdot, u_\lambda). \end{aligned} \tag{3.6}$$

Note that, since $l > k$

$$\begin{aligned} & \Big| \frac{1}{\varepsilon} \int_{\{k < u_\lambda < k+\varepsilon\}} a(T_l(u_\lambda), 0) \cdot Du_\lambda \Big| \\ & \leq \Big| \int_\Omega \text{div}\Big(\int_0^{\frac{(u_\lambda - k)^+}{\varepsilon} \wedge 1} a(T_l(\varepsilon r + k), 0) dr \Big) \Big| \\ & = \Big| \int_{\partial\Omega} \int_0^{\frac{(u_\lambda - k)^+}{\varepsilon} \wedge 1} a(T_l(\varepsilon r + k), 0) dr . \eta \; d\sigma \Big| \\ & \longrightarrow \Big| \int_{\partial\Omega} \text{sign}_0^+(u_\lambda - k) a(k, 0) \; d\sigma \Big| \quad \text{as } \varepsilon \to 0\,. \end{aligned} \tag{3.7}$$

Thus, we deduce that

$$\begin{aligned} & \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{k < u_\lambda < k+\varepsilon\}} a(T_l(u_\lambda), 0) \cdot Du_\lambda \\ & \geq -\frac{|a(k,0)|}{T_l(\Lambda(k))} \int_{\partial\Omega \cap \{u_\lambda > k\}} T_l(\Lambda(u_\lambda)) \\ & \geq -\delta \int_{\partial\Omega \cap \{u_\lambda > k\}} T_l(\Lambda(u_\lambda)). \end{aligned} \tag{3.8}$$

Passing to the limit in inequality (3.6) with $\varepsilon \to 0$, we get

$$\int_{\{u_\lambda > k\}} T_l(u_\lambda) - k \leq \int_{\{u_\lambda > k\}} f - k \leq \int_\Omega (f-k)^+.$$

Thus, since $k \geq \|f\|_\infty + 1$, we have

$$\int_{\{u_\lambda > k\}} (T_l(u_\lambda) - k)^+ \leq 0 \quad \forall l > k.$$

Then

$$T_l(u_\lambda) \leq k \;\text{ a.e. in } \quad \{u_\lambda > k\}.$$

We conclude that

$$u_\lambda \leq k \;\text{ a.e. in } \quad \Omega.$$

Similarly, we prove that $-k \leq u_\lambda$ a.e. in Ω, then

$$\|u_\lambda\|_\infty \leq C, \tag{3.9}$$

where C is a constant depending on $\|f\|_\infty$ and δ.

Taking $\phi = 0$ as a test function in (3.5), we get after using assumption (H_2), estimate (3.9) and Gauss-Green formula

$$\lambda_0 \int_\Omega |Du_\lambda|^p \leq \int_\Omega f u_\lambda + C. \tag{3.10}$$

From (3.9) and (3.10), it follows that $(u_\lambda)_\lambda$ is bounded in $W^{1,p}(\Omega)$. Hence there exists a subsequence, still denoted u_λ, such that $u_\lambda \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ as $\lambda \to 0$. By Rellich-Kondrachov Theorem, $u_\lambda \to u$ in $L^p(\Omega)$ and $\tau(u_\lambda) \to \tau(u)$ in $L^p(\partial\Omega)$ as $\lambda \to 0$. Then $T_l(\Lambda(u_\lambda)) \to \Lambda(u)$ on $\partial\Omega$. We may also assume that $u_\lambda \to u$ a.e. in Ω. Therefore, by (3.9), $\|u\|_\infty \leq C(\|f\|_\infty, \delta)$.
Taking $\phi = u_\lambda - \frac{1}{k} T_k(u_\lambda)$ as a test function in inequality (3.5), passing to the limit with $k \to 0$, we get

$$\int_{\partial\Omega} |\beta_\lambda(\cdot, u_\lambda)| + \delta \int_{\partial\Omega} |T_l(\Lambda(u_\lambda))| \leq \int_\Omega |f| < C. \tag{3.11}$$

Thus, passing to a subsequence if necessary, we have

$$\beta_\lambda(\cdot, u_\lambda) \rightharpoonup \mu \text{ in } \mathcal{M}_b(\partial\Omega) \text{ as } \lambda \to 0.$$

Note that for all $\nu > \lambda > 0$, we have $|\beta_\lambda(x,r)| \geq |\beta_\nu(x,r)|$ $\forall r \in \mathbb{R}$. Thus from (3.11), $\int_{\partial\Omega} |\beta_\nu(\cdot, u_\lambda)| \leq C$, passing to the limit with $\lambda \to 0$, we get $\int_{\partial\Omega} |\beta_\nu(\cdot, u)| \leq C$. As $\nu \to 0$, we obtain $\int_{\partial\Omega} |\beta^o(\cdot, u)| \leq C$.

Next, we need to pass to the limit in the nonlinearity $a(u_\lambda, Du_\lambda)$. Thanks to (3.9), (3.10) and assumption (H_3), we have $(a(u_\lambda, Du_\lambda))_\lambda$ is bounded in $\left(L^{p'}(\Omega)\right)^N$. After passing to a suitable subsequence, we can assume that $a(u_\lambda, Du_\lambda) \rightharpoonup \chi$ weakly in $\left(L^{p'}(\Omega)\right)^N$ as $\lambda \to 0$. The aim is to show, via the pseudo-monotonicity argument, that div $a(u, Du) =$ div χ. To this end, we must show that

$$\limsup_{\lambda \to 0} \int_\Omega a(u_\lambda, Du_\lambda) \cdot D(u_\lambda - u) = 0. \tag{3.12}$$

Taking $\phi = u_\lambda - (u_\lambda - u)^+$ as a test function in (3.5), we get

$$\begin{aligned}
&\int_\Omega a(u_\lambda, Du_\lambda)\cdot D(u_\lambda - u)^+ \\
\leq\ & \int_\Omega (f-u_\lambda)(u_\lambda-u)^+ - \int_{\partial\Omega}(\delta T_l(\Lambda(u_\lambda)) + \beta_\lambda(\cdot,u_\lambda))(u_\lambda-u)^+ \\
\leq\ & \int_\Omega (f-u_\lambda)(u_\lambda-u)^+ - \int_{\partial\Omega}(\delta T_l(\Lambda(u_\lambda)) + \beta_\lambda(\cdot,-u_\lambda^-))(u_\lambda-u)^+,
\end{aligned}$$

where we have used the fact that $\beta_\lambda(\cdot, u_\lambda^+)(u_\lambda - u)^+ \geq 0$. Having in mind that $(u_\lambda)_\lambda$ is uniformly bounded in $L^\infty(\partial\Omega)$, we have $\|(u_\lambda - u)^+\|_\infty \leq C$ and $(u_\lambda - u)^+ \to 0$ a.e., as $\lambda \to 0$. Next, observe that $\beta_\lambda(\cdot, -u_\lambda^-) \geq \beta_\lambda(\cdot, -u^-) \geq \beta^o(\cdot, -u^-)$ on $\{u_\lambda \geq u\}$. As $\beta^o(\cdot,-u^-) \in L^1(\partial\Omega)$, it follows that $\int_{\partial\Omega}\beta_\lambda(\cdot,-u_\lambda^-)(u_\lambda - u)^+ \to 0$. Consequently, $\limsup_{\lambda\to 0}\int_\Omega a(u_\lambda, Du_\lambda)\cdot D(u_\lambda - u)^+ \leq 0$, and $\limsup_{\lambda\to 0}\int_\Omega a(u_\lambda, Du_\lambda)\cdot D(-(u_\lambda - u)^-) \leq 0$ follows similarly. Hence $\limsup_{\lambda\to 0}\int_\Omega a(u_\lambda, Du_\lambda)\cdot D(u_\lambda - u) \leq 0$ and (3.12) follows from the monotonicity of a.

Up to now, we have shown that for all $\phi \in C_c^\infty(\mathbb{R}^N)$

$$\begin{aligned}
&\int_\Omega a(u,Du)\cdot D(u-\phi) + \delta\int_{\partial\Omega}\Lambda(u)(u-\phi) \\
&\leq \int_\Omega (f-u)(u-\phi) - \int_{\partial\Omega}(\tilde{u}-\phi)d\mu, \qquad (3.13)
\end{aligned}$$

which, by density, remains true for all $\phi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Then, we conclude that

$$\int_\Omega a(u,Du)\cdot D\phi + \delta\int_{\partial\Omega}\Lambda(u)\phi = \int_\Omega (f-u)\phi - \int_{\partial\Omega}\tilde{\phi}d\mu, \qquad (3.14)$$

for all $\phi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Finally, we must characterize the obtained measure μ. First, according to equation (3.14), $\mu \in \mathcal{M}_b(\partial\Omega)\cap(W^{-\frac{1}{p'},p'}(\partial\Omega) + (L^\infty(\partial\Omega))^*)$ and $|\mu|$ does not charge sets of zero capacity. Let us show now that $\mu \in \partial\mathcal{J}(u)$. For this, we proceed as in Bouchitté [18], Bouchitté [19]. Note that $\beta_\lambda = \partial j_\lambda$, where $j_\lambda \in \mathcal{J}_0(\partial\Omega)$, $j_\lambda(x,r) = \inf_{s\in\mathbb{R}}\{1/(2\lambda)|r-s|^2 + j(x,s)\}$. Recall that, for a.e. $x \in \partial\Omega$ and for all $r \in \mathbb{R}$, $j_\lambda(x,r) \uparrow j(x,r)$ as $\lambda \downarrow 0$. Thus, by definition of the subdifferential, for all $\nu > \lambda > 0$ and a.e. $x \in \partial\Omega$,

$$\begin{aligned}
j(x,r) &\geq j_\lambda(x,r) \\
&\geq j_\lambda(x,u_\lambda(x)) + \partial j_\lambda(x,u_\lambda(x))(r-u_\lambda(x)) \\
&\geq j_\nu(x,u_\lambda(x)) + \partial j_\lambda(x,u_\lambda(x))(r-u_\lambda(x));\ \forall r\in\mathbb{R}.
\end{aligned}$$

Therefore, for all $\xi \in W^{\frac{1}{p'},p}(\partial\Omega)\cap L^\infty(\partial\Omega)$

$$\int_{\partial\Omega} j(\cdot,\xi) \geq \int_{\partial\Omega} j_\nu(\cdot,u_\lambda) + \int_{\partial\Omega}\partial j_\lambda(\cdot,u_\lambda)(\xi-u_\lambda).$$

Having in mind that $u_\lambda \to u$ a.e. on Ω as $\lambda \to 0$ then, according to Fatou's Lemma and Monotone Convergence Theorem, passing first to the limit with $\lambda \to 0$ then with $\nu \to 0$, we get for all $\xi \in C(\partial\Omega)$ (the set of continuous functions on $\partial\Omega$)

$$\begin{aligned}\int_{\partial\Omega} j(\cdot,\xi) \geq \int_{\partial\Omega} j(\cdot,u) + \liminf_{\lambda\to 0}\int_{\partial\Omega} \beta_\lambda(\cdot,u_\lambda)(\xi-u_\lambda)\\ \geq \int_{\partial\Omega} j(\cdot,u) + \liminf_{\lambda\to 0}\int_{\partial\Omega} \beta_\lambda(\cdot,u_\lambda)(\xi-u)\\ + \liminf_{\lambda\to 0}\int_{\partial\Omega} \beta_\lambda(\cdot,u_\lambda)(u-u_\lambda).\end{aligned} \tag{3.15}$$

Now using (3.12), the monotonicity of Λ, the uniform L^∞-estimate on u_λ and the a.e. convergence of u_λ to u, we get from (3.5)

$$\begin{aligned}&\lim_{\lambda\to 0}\int_{\partial\Omega} \beta_\lambda(\cdot,u_\lambda)(u-u_\lambda)\\ \geq\;& \lim_{\lambda\to 0}\int_{\Omega}(f-u_\lambda)(u-u_\lambda) + \limsup_{\lambda\to 0}\int_\Omega a(u_\lambda,Du_\lambda)\cdot D(u_\lambda-u)\\ &+\delta\lim_{\lambda\to 0}\int_{\partial\Omega}(\Lambda(u_\lambda)-\Lambda(u))(u_\lambda-u) + \delta\lim_{\lambda\to 0}\int_{\partial\Omega}\Lambda(u)(u_\lambda-u)\\ \geq\;& 0.\end{aligned}$$

Consequently, we conclude from (3.15) that

$$\mathcal{J}(\xi) \geq \mathcal{J}(u) + \langle \mu, \xi-u\rangle \quad \forall \xi \in C(\partial\Omega).$$

Since $\mu \in \mathcal{M}_b^p(\partial\Omega)$, one can see that the last inequality holds for $\xi \in W^{\frac{1}{p'},p}(\partial\Omega) \cap L^\infty(\partial\Omega)$, and thus we deduce that $\mu \in \partial\mathcal{J}(u)$.

To conclude the proof of ii), we prove, using the same technics as Wittbold [29], Lemma 3.7 and Bouchitté [18], Proposition 20, that the elements in this subdifferential can be characterized as follows:

$$\begin{aligned}&\mu \in \partial\mathcal{J}(u)\\ \Longleftrightarrow\;&\\ &\begin{cases}\mu_r(x) \in \partial j(x,u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x)) \quad \text{a.e. } x \in \partial\Omega\\ \widetilde{u} = \gamma_- \quad \mu_s^- - \text{ a.e. on } \partial\Omega, \qquad \widetilde{u} = \gamma_+ \quad \mu_s^+ - \text{a.e. on } \partial\Omega.\end{cases}\end{aligned}$$

iii) We show that $D(A_\delta)$ is dense in $L^1(\Omega)$. To this end, it suffices to prove that $L^\infty(\Omega) \subset \overline{D(A_\delta)}^{\|\cdot\|_1}$. Let $\alpha > 0$. Given $f \in L^\infty(\Omega)$, if we set $u_\alpha := (I+\alpha A)^{-1} f$, then $(u_\alpha, \frac{1}{\alpha}(f-u_\alpha)) \in A_\delta$. So, taking $\phi = 0$ as a test function in the definition of the operator A_δ, we get

$$\begin{aligned}\int_\Omega a(u_\alpha, Du_\alpha)\cdot Du_\alpha &\le \frac{1}{\alpha}\int_\Omega (f-u_\alpha)u_\alpha - \int_{\partial\Omega}\delta\Lambda(u_\alpha)u_\alpha - \int_{\partial\Omega}\tilde{u}_\alpha d\mu_\alpha\\ &\le \frac{1}{\alpha}\int_\Omega (f-u_\alpha)u_\alpha\\ &\le \frac{1}{\alpha}C(\|f\|_1,\delta),\end{aligned}$$

where we have used the monotonicity of Λ, properties of μ and the L^∞-estimate on u_α. Now, using Hypotheses (H_2) and (H_3), it is easy to see that $\alpha\int_\Omega |a(u_\alpha, Du_\alpha)| \to 0$ as $\alpha \to 0$. On the other hand, if $\phi \in \mathcal{D}(\Omega)$, taking $u_\alpha+\phi$ and $u_\alpha-\phi$ as test functions in the definition of the operator A_δ we get

$$\lim_{\alpha\to 0}\int_\Omega u_\alpha\phi = \int_\Omega f\phi.$$

Since $\|u_\alpha\|_1 \le \|f\|_1$, we have $u_\alpha \to f$ in $L^1(\Omega)$. As a consequence $f \in \overline{D(A_\delta)}^{\|\cdot\|_1}$ and the proof is complete. □

4. ENTROPY SOLUTIONS

Before introducing the notion of entropy solutions for the problem $(E)(f)$, we define the following spaces similar to that introduced in Andreu et al, Bénilan [7, 11]. We note

$$\mathcal{T}^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable; } T_k(u) \in W^{1,p}(\Omega) \text{ for all } k>0\}.$$

In Bénilan [11], the authors prove that for $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $w : \Omega \to \mathbb{R}^N$ such that $DT_k(u) = w\chi_{\{|u|<k\}}$ $\forall k>0$. This function w will be denoted by Du.

Denote by $\mathcal{T}^{1,p}_{tr}(\Omega)$ the subset of $\mathcal{T}^{1,p}(\Omega)$ consisting of the functions that can be approximated by functions of $W^{1,p}(\Omega)$ in the following sense: a function $u \in \mathcal{T}^{1,p}(\Omega)$ belongs to $\mathcal{T}^{1,p}_{tr}(\Omega)$ if there exists a sequence $u_\delta \in W^{1,p}(\Omega)$ such that: - $u_\delta \to u$ a.e. in Ω,

- $DT_k(u_\delta) \rightharpoonup DT_k(u)$ weakly in $L^1(\Omega)$ for any $k>0$,

- there exists a measurable function $v : \partial\Omega \to \mathbb{R}$ such that $(\tau(u_\delta))_\delta$ converges a.e. in $\partial\Omega$ to v. The function v is called the trace of u, and denoted by $\tau(u)$.

We use notations $u, \tau(u)$ for the trace of $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ on $\partial\Omega$.

The concept of entropy solution for a problem with boundary conditions was introduced in Andreu et al [7] for the problem

$$\begin{cases} -\text{div } a(x,Du) = f & \text{in} \quad \Omega \\ \\ -a(x,Du)\cdot\eta \in \beta(u) & \text{on} \quad \partial\Omega. \end{cases}$$

Applying the same idea, we define an entropy solution for our problem $(E)(f)$.

Definition 1. A function $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ is an entropy solution for problem $(E)(f)$ if there exists a measure $\mu \in \mathcal{M}_b^p(\partial\Omega)$ with

$$\mu_r(x) \in \partial j(x, u(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u(x)) \quad \text{a.e. } x \in \partial\Omega \tag{4.1}$$

and

$$\tilde{u} = \gamma_+ \quad \mu_s^+ - a.e. \ \text{ on } \partial\Omega, \qquad \tilde{u} = \gamma_- \quad \mu_s^- - a.e. \ \text{ on } \partial\Omega, \tag{4.2}$$

such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega a(u, Du) \cdot DT_k(u - \phi) \le \int_\Omega (f - u)T_k(u - \phi) - \int_{\partial\Omega} T_k(\tilde{u} - \tilde{\phi})d\mu.$$

Remark 4.1. Note that each integral in the preceding definition is well defined. Indeed, the first term can be understood as $\int_\Omega a(T_l(u), DT_l(u)) \cdot DT_k(u - \phi)$ where $l \ge k + \|\phi\|_\infty$. Since $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have $u - \phi \in \mathcal{T}_{tr}^{1,p}(\Omega)$ (see Andreu et al [7], Theorem 3.1). Hence $T_k(u - \phi) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and admits a trace which has a quasi-continuous representative, according to the remarks made in Preliminaries. Thus the last integral in the above definition is well defined. Note also that condition (4.2) is meaningful.

We define an operator $\mathcal{A}$ by the rule: $(u, f - u) \in \mathcal{A}$ if and only if $u, f \in L^1(\Omega)$ and u is an entropy solution of Problem $(E)(f)$.

In the following, we use the notation $A_{m,n}$ (resp. $\Lambda_{m,n}$) instead of A_δ (resp. $\delta\Lambda$) where $\Lambda_{m,n}(u) = \frac{1}{m}\Lambda(u^+) - \frac{1}{n}\Lambda(u^-)$. This is done to be able to use the monotonicity of Λ.

Theorem 4.1. *The operator $\mathcal{A}$ is m-accretive with dense domain in $L^1(\Omega)$ and $\mathcal{A} = \liminf\limits_{m,n\to\infty} A_{m,n}$, where $\liminf\limits_{m,n\to\infty} A_{m,n}$ is the operator defined by $(x, y) \in \liminf\limits_{m,n\to\infty} A_{m,n}$, if for all $m > 0, n > 0$, there are $(x_{m,n}, y_{m,n}) \in A_{m,n}$, such that $(x, y) = \lim\limits_{m,n\to\infty}(x_{m,n}, y_{m,n})$ in $X \times X$.*

Proof. We divide the proof into six steps.

Step 1: A priori estimates.

Let $f \in L^1(\Omega)$. We approximate f by $f_{m,n} = (f \wedge m) \vee (-n)$ which is in $L^\infty(\Omega)$, non decreasing in m, non increasing in n and $\|f_{m,n}\|_1 \le \|f\|_1$. Then, by Theorem 3.1, $f_{m,n} \in R(I + A_{m,n})$ and there exist $u_{m,n} \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ and a measure $\mu_{m,n} \in \mathcal{M}_b^p(\partial\Omega)$ satisfying $(\mu_{m,n})_r(x) \in \partial j(x, u_{m,n}(x)) + \partial I_{[\gamma_-(x),\gamma_+(x)]}(u_{m,n}(x))$ a.e. $x \in \partial\Omega$, such that for all $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\begin{aligned}
&\int_\Omega a(u_{m,n}, Du_{m,n}) \cdot D(u_{m,n} - \phi) \\
&\quad + \frac{1}{m} \int_{\partial\Omega} \Lambda(u^+_{m,n})(u_{m,n} - \phi) - \frac{1}{n} \int_{\partial\Omega} \Lambda(u^-_{m,n})(u_{m,n} - \phi) \\
&\le \int_\Omega (f_{m,n} - u_{m,n})(u_{m,n} - \phi) - \int_{\partial\Omega} (\tilde{u}_{m,n} - \tilde{\phi}) d\mu_{m,n}.
\end{aligned} \tag{4.3}$$

In the following, let $k > 0$ be fixed. Using $u_{m,n} - T_k(u_{m,n})$ as a test function in inequality (4.3) and applying assumption (H_2) we obtain

$$\begin{aligned}
&\lambda_0 \int_\Omega |DT_k(u_{m,n})|^p \\
&\quad + \frac{1}{m} \int_{\partial\Omega} T_k(u_{m,n})\Lambda(u^+_{m,n}) - \frac{1}{n} \int_{\partial\Omega} T_k(u_{m,n})\Lambda(u^-_{m,n}) \\
\le\ & \int_\Omega T_k(u_{m,n})(f_{m,n} - u_{m,n}) - \int_{\partial\Omega} T_k(\tilde{u}_{m,n}) d\mu_{m,n} \\
&\quad - \int_\Omega a(u_{m,n}, 0) \cdot DT_k(u_{m,n}).
\end{aligned} \tag{4.4}$$

By Gauss-Green formula and assumption (H_3), we have

$$\begin{aligned}
\Big| \int_\Omega a(u_{m,n}, 0) \cdot DT_k(u_{m,n}) \Big| &\le \Big| \int_{\partial\Omega} \int_0^{T_k(u_{m,n})} a(r, 0) dr \cdot \eta d\sigma \Big| \\
&\le \int_{\partial\Omega} \int_0^{T_k(u_{m,n})} \Lambda(|r|) dr d\sigma \\
&\le C,
\end{aligned} \tag{4.5}$$

where C is a constant independent of m, n. Then from inequality (4.4), using the monotonicity of Λ, we conclude

$$\lambda_0 \int_\Omega |DT_k(u_{m,n})|^p \le Const(k, f, \Lambda(k)). \tag{4.6}$$

Thus $(T_k(u_{m,n}))_{m,n}$ is a bounded subset of $W^{1,p}(\Omega)$. Hence, after passing to a suitable subsequence if necessary, we have $(T_k(u_{m,n}))_{m,n}$ is weakly convergent in $W^{1,p}(\Omega)$. Then, $T_k(u_{m,n}) \to v_k$ in $L^p(\Omega)$ as $m, n \to \infty$. We may also suppose $DT_k(u_{m,n}) \rightharpoonup g_k$ weakly in $L^p(\Omega)$ as $m, n \to \infty$.

Now, we must prove the convergence almost everywhere of $u_{m,n}$. We recall that $\|u_{m,n}\|_1 \le \|f\|_1$. As $A_{m,n}$ is a T-accretive operator, using the monotonicity of $f_{m,n}$ and $\Lambda_{m,n}$, we have for all $m > m' > 0$ and for any $n > 0$, $u_{m,n} \ge u_{m',n}$ a.e. on Ω and, for any $n > n' > 0$, for all $m > 0, u_{m,n} \le u_{m,n'}$, a.e. on Ω. As a consequence, we have

$$u_{m,n} \uparrow_{m\uparrow\infty} u^n \downarrow_{n\downarrow\infty} u, \quad u_{m,n} \downarrow_{n\downarrow\infty} u_m \uparrow_{m\uparrow\infty} u \ \text{ in } L^1(\Omega). \tag{4.7}$$

Here, and in the sequel, we use the notation $\uparrow_n$, respectively $\downarrow_n$, to denote convergence of sequence which is monotone increasing, respectively decreasing in n.

Therefore from (4.7) we get the convergence in $L^1(\Omega)$ and also the convergence almost everywhere on Ω.

Obviously, we can conclude that $v_k = T_k(u)$ and $g_k = DT_k(u)$. Therefore, $T_k(u) \in W^{1,p}(\Omega)$ for every $k > 0$. Consequently, $u \in \mathcal{T}^{1,p}(\Omega)$.
Moreover, one can show exactly as in Andreu et al [7], that $(\tau(u_{m,n}))_{m,n}$ converges. Therefore, we have $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$.

Step 2: Existence of the measure.
It remains to show the existence of a measure $\mu \in \mathcal{M}^p_b(\partial\Omega)$ such that $\mu_{m,n} \to \mu$ strongly in $\mathcal{M}_b(\partial\Omega)$.

Let $u^\lambda_{m,n}$ be a solution of the following equation

$$\begin{aligned}&\int_\Omega a(u^\lambda_{m,n}, Du^\lambda_{m,n}) \cdot D\varphi + \frac{1}{m}\int_{\partial\Omega} \Lambda(u^{\lambda,+}_{m,n})\varphi - \frac{1}{n}\int_{\partial\Omega} \Lambda(u^{\lambda,-}_{m,n})\varphi \\ = &\int_\Omega (f_{m,n} - u^\lambda_{m,n})\varphi - \int_{\partial\Omega} \beta_\lambda(\cdot, u^\lambda_{m,n})\varphi, \end{aligned} \tag{4.8}$$

for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We know from Theorem 3.1 (Part *ii*) that $\|\beta_\lambda(\cdot, u^\lambda_{m,n})\|_1$ is uniformly bounded by a constant C independent of λ, thus $\beta_\lambda(\cdot, u^\lambda_{m,n}) \rightharpoonup \mu_{m,n}$ in $\mathcal{M}_b(\partial\Omega)$ as $\lambda \to 0$. Therefore

$$\|\mu_{m,n}\|_{\mathcal{M}_b(\partial\Omega)} \le \liminf_{\lambda\to 0} \|\beta_\lambda(\cdot, u^\lambda_{m,n})\|_{\mathcal{M}_b(\partial\Omega)} \le C$$

and we deduce, after extracting a subsequence if necessary, that $\mu_{m,n} \rightharpoonup \mu$ weakly in $\mathcal{M}_b(\partial\Omega)$ as $m, n \to \infty$.
In order to prove the strong convergence of $\mu_{m,n}$, we use the following comparison result:

Lemma 4.1. *Let $f_{m,n}, f_{\tilde{m},n} \in L^\infty(\Omega)$ and $u^\lambda_{m,n}, u^\lambda_{\tilde{m},n}$ be the weak solutions which verify (4.8). Assume that $f_{\tilde{m},n} \ge f_{m,n} > 0$ a.e. on Ω for $\tilde{m} > m > 0, n > 0$. Then*

$$u^\lambda_{m,n} \le u^\lambda_{\tilde{m},n} \quad a.e.\ on\ \ \Omega$$

and

$$\beta_\lambda(\cdot, u^\lambda_{m,n}) \le \beta_\lambda(\cdot, u^\lambda_{\tilde{m},n}) \quad a.e.\ on \quad \partial\Omega.$$

The proof of the comparison result is standard (see Ammar [2], Ammar and Wittbold [4]). Indeed, taking $\varphi = \frac{1}{k}T_k(u^\lambda_{m,n} - u^\lambda_{\tilde{m},n})^+$ as a test function in equation (4.8) and $\varphi = \frac{1}{k}T_k(u^\lambda_{\tilde{m},n} - u^\lambda_{m,n})^+$ in the equation corresponding to the solution $u^\lambda_{\tilde{m},n}$, passing to the limit in the sum of both equations with $k \to 0$, we get the result.

Note that the result of Lemma 4.1 remains true for the positive and negative parts, i.e. $\beta_\lambda(\cdot, u^{\lambda,+}_{m,n}) \le \beta_\lambda(\cdot, u^{\lambda,+}_{\tilde{m},n})$ and $\beta_\lambda(\cdot, u^{\lambda,-}_{m,n}) \le \beta_\lambda(\cdot, u^{\lambda,-}_{\tilde{m},n})$. Thus, by the previous results of convergence, we have $\mu^+_{m,n} \le \mu^+_{\tilde{m},n}$ and $\mu^-_{m,n} \le \mu^-_{\tilde{m},n}$, which is equivalent to say that the regular and the singular parts verify this comparison result. From this, we deduce that $\mu^+_{m,n} \uparrow \mu^+_n$ in $\mathcal{M}_b(\partial\Omega)$ as $m \to \infty$. Indeed, let $\mu^+_n : \mathcal{B}(\partial\Omega) \to [0,\infty]$ defined by $\mu^+_n(A) = \lim_{m\to\infty} \mu^+_{m,n}(A) < \infty$. Here $\mathcal{B}(\partial\Omega)$ denotes the set of Borel sets of $\partial\Omega$. Note that μ^+_n is a Radon measure. We have

$$\begin{aligned}
\|\mu^+_{m,n} - \mu^+_n\|_{\mathcal{M}_b(\partial\Omega)} &= \sup_{(E_i)_{i=1,n}\in\mathcal{B}(\partial\Omega)} \Big[\sum_{i=1}^{n}(\mu^+_{m,n} - \mu^+_n)(E_i)\Big] \\
&= \sum_{i=1}^{n} \big[\mu^+_{m,n}(E_i) - \mu^+_n(E_i)\big] \\
&= \mu^+_{m,n}(\partial\Omega) - \mu^+_n(\partial\Omega) \\
&\to 0 \ \text{ as } \ m \to \infty\,,
\end{aligned}$$

where $(E_i)_i$ denotes finite partition of $\partial\Omega$. We applied the same methods to show that $\mu^+_n \downarrow \mu^+$ as $n \to \infty$. Note that we get the same results for the negative parts, and this concludes the proof of Step 2.

Step 3: The pseudo-monotonicity argument.

Recall that $u_{m,n}$ satisfies, for all $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega a(u_{m,n}, Du_{m,n}) \cdot D\varphi + \frac{1}{m}\int_{\partial\Omega} \Lambda(u^+_{m,n})\varphi - \frac{1}{n}\int_{\partial\Omega} \Lambda(u^-_{m,n})\varphi \\
= &\int_\Omega (f_{m,n} - u_{m,n})\varphi - \int_{\partial\Omega} \tilde{\varphi}\, d\mu_{m,n}.
\end{aligned} \tag{4.9}$$

Since $T_k(u_{m,n})$ is bounded in $W^{1,p}(\Omega)$, then thanks to the growth assumption (H_3), there exists a vector fields $\chi_k \in \left(L^{p'}(\Omega)\right)^N$ such that $a(T_k(u_{m,n}), DT_k(u_{m,n})) \rightharpoonup \chi_k$ weakly in $\left(L^{p'}(\Omega)\right)^N$ as $m, n \to \infty$, for all $k \in \mathbb{N}^*$. The aim is to prove, via the pseudo-monotonicity argument, that $\operatorname{div} \chi_k = \operatorname{div} a(T_k(u), DT_k(u))$ in $\mathcal{D}'(\Omega)$. To this end, we define for $l < k$, the following integral

$$\begin{aligned}
I \ := \ &\int_\Omega \big[a(T_k(u_{m,n}), DT_k(u_{m,n})) - a(T_k(u_{m',n'}), DT_k(u_{m',n'}))\big] \\
&\qquad \cdot DT_l(T_k(u_{m,n}) - T_k(u_{m',n'})),
\end{aligned}$$

which, defining the sets $A_{1,k} = \{|u_{m,n}| < k, |u_{m',n'}| < k\}$, $A_{2,k} = \{|u_{m,n}| < k, |u_{m',n'}| \geq k\}$, $A_{3,k} = \{|u_{m,n}| \geq k, |u_{m',n'}| < k\}$ and $A_{4,k} = \{|u_{m,n}| \geq k, |u_{m',n'}| \geq k\}$, can be written as

$$\begin{aligned}
&\int_{A_{1,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big] \cdot DT_l(u_{m,n} - u_{m',n'}) \\
&+ \int_{A_{2,k}} \big[a(u_{m,n}, Du_{m,n}) - a(T_k(u_{m',n'}), 0)\big] \cdot DT_l(u_{m,n} - T_k(u_{m',n'})) \\
&+ \int_{A_{3,k}} \big[a(u_{m',n'}, Du_{m',n'}) - a(T_k(u_{m,n}), 0)\big] \cdot DT_l(u_{m',n'} - T_k(u_{m,n})) \\
=:\ & I_1 + I_2 + I_3.
\end{aligned}$$

We want to pass to the limit in I, in the following order, with $m', n' \to \infty, m, n \to \infty$ and then $l \to 0$. Note that the term I_1 can be written as

$$\begin{aligned}
&\int_\Omega \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big] \cdot DT_l(u_{m,n} - u_{m',n'}) \\
-& \int_{A_{2,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big] \cdot DT_l(u_{m,n} - u_{m',n'}) \\
-& \int_{A_{3,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big] \cdot DT_l(u_{m,n} - u_{m',n'}) \\
-& \int_{A_{4,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big] \cdot DT_l(u_{m,n} - u_{m',n'}) \\
=:& \ \ I_1^1 - I_1^2 - I_1^3 - I_1^4.
\end{aligned}$$

Choosing $T_l(u_{m,n} - u_{m',n'})$ as a test function in (4.9) and $T_l(u_{m',n'} - u_{m,n})$ in the equation corresponding to the solution $u_{m',n'}$, adding both equalities, using the fact that $u_{m,n}, u_{m',n'} \to u$ a.e. in Ω, $f_{m,n}, f_{m',n'} \to f$ in $L^1(\Omega)$, $\mu_{m,n}, \mu_{m',n'} \to \mu$ strongly in $\mathcal{M}_b(\partial\Omega)$ and $\int_{\partial\Omega} |\frac{1}{m}\Lambda(u^+_{m,n}) - \frac{1}{n}\Lambda(u^-_{m,n})|$ is bounded uniformly on m, n, we get

$$\lim_{l\to 0} \lim_{m,n\to\infty} \lim_{m',n'\to\infty} I_1^1 = 0.$$

By assumptions (H_1) and (H_4), Hölder's inequality and (4.6)

$$\begin{aligned}
I_1^2 \ &\geq \int_{A_{2,k}} \big[a(u_{m,n}, Du_{m',n'}) - a(u_{m',n'}, Du_{m',n'})\big] \cdot DT_l(u_{m,n} - u_{m',n'}) \\
&\geq -\int_{\mathcal{F}_1} \big|a(u_{m,n}, Du_{m',n'}) - a(u_{m',n'}, Du_{m',n'})\big| \big|D(u_{m,n} - u_{m',n'})\big| \\
&\geq -\Big[\int_{\mathcal{F}_1} 2^{p'} C(|u_{m,n}|, |u_{m',n'}|)^{p'} |u_{m,n} - u_{m',n'}|^{p'} (1 + |Du_{m',n'}|^p)\Big]^{1/p'} \\
&\qquad \times \Big[\int_{\mathcal{F}_1} |D(u_{m,n} - u_{m',n'})|^p\Big]^{1/p} \\
&\geq \ -Cl,
\end{aligned}$$

where $\mathcal{F}_1 := \{|u_{m,n}| < k, |u_{m',n'}| < 2k, |u_{m,n} - u_{m',n'}| < l\}$ and C is a constant depending on f, p and k. Clearly $\lim\limits_{l\to 0} \lim\limits_{m,n\to\infty} \lim\limits_{m',n'\to\infty} I_1^2 \geq 0$. By the same methods, $\lim\limits_{l\to 0} \lim\limits_{m,n\to\infty} \lim\limits_{m',n'\to\infty} I_1^3 \geq 0$. Now, let us show that $\lim\limits_{l\to 0} \lim\limits_{m,n\to\infty} \lim\limits_{m',n'\to\infty} I_1^4 = 0$.

Define the function h_k by

$$h_k(r) = \begin{cases} 0 & \text{if} \quad |r| < k \\ r - k\,\mathrm{sign}(r) & \text{if} \quad |r| \geq k. \end{cases}$$

Then I_1^4 is equal to

$$\begin{aligned}
&\int_\Omega \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big]\cdot DT_l(h_k(u_{m,n}) - h_k(u_{m',n'}))\\
&- \int_{A_{2,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big]\cdot DT_l(-h_k(u_{m',n'}))\\
&- \int_{A_{3,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})\big]\cdot DT_l(h_k(u_{m,n}))\\
&=: K_1 - K_2 - K_3. \qquad (4.10)
\end{aligned}$$

As in I_1^1, we prove that $\lim\limits_{l\to 0}\lim\limits_{m,n\to\infty}\lim\limits_{m',n'\to\infty} K_1 = 0$ by using $T_l(h_k(u_{m,n}) - h_k(u_{m',n'}))$ as a test function in the equations corresponding to the solutions $u_{m,n}$ and $u_{m',n'}$. Note that, by using $T_l(h_k(u_{m,n}))$ as a test function in (4.9) and a similar technics as in the proof of (4.5), it follows

$$\int_\Omega |DT_l(h_k(u_{m,n}))|^p \le lC, \qquad (4.11)$$

where C is a constant depending only on f and k.

Now, by Hölder's inequality

$$\begin{aligned}
&|K_2|\\
&\le \int_{\{|u_{m,n}|<k, |u_{m',n'}|\ge k, |h_k(u_{m',n'})|<l\}} |a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})||DT_l(h_k(u_{m',n'}))|\\
&\le \Big[\int_{\{|u_{m,n}|<k, |u_{m',n'}|<2k\}} |a(u_{m,n}, Du_{m,n}) - a(u_{m',n'}, Du_{m',n'})|^{p'}\Big]^{1/p'}\\
&\quad\times\Big[\int_\Omega |DT_l(h_k(u_{m',n'}))|^p\Big]^{1/p}.
\end{aligned}$$

Thus, clearly assumption (H_3), (4.6) and (4.11) yield

$$\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} K_2 = 0.$$

Similarly, $\lim\limits_{l\to 0}\lim\limits_{m,n\to\infty}\lim\limits_{m',n'\to\infty} K_3 = 0$.

Consequently, combining all limits in (4.10), we get

$$\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} I_1^4 = 0$$

and therefore

$$\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} I_1 \le 0.$$

Now, consider the term I_2. We remark that

$$\begin{aligned}
I_2 &= \int_{A_{2,k}} \big[a(u_{m,n}, Du_{m,n}) - a(u_{m,n}, 0)\big]\cdot DT_l(u_{m,n} - T_k(u_{m',n'}))\\
&\quad + \int_{A_{2,k}} \big[a(u_{m,n}, 0) - a(T_k(u_{m',n'}), 0)\big]\cdot DT_l(u_{m,n} - T_k(u_{m',n'}))\\
&=: I_2^1 + I_2^2.
\end{aligned}$$

Assumption (H_4), Hölder's inequality and (4.6) yield

$$\begin{aligned}|I_2^2| &\leq \int_{\mathcal{F}_2} C(|u_{m,n}|,|u_{m',n'}|)|T_k(u_{m,n})-T_k(u_{m',n'})||DT_k(u_{m,n})| \\ &\leq C\Big[\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|<l\}} |T_k(u_{m,n})-T_k(u_{m',n'})|^{p'}\Big]^{1/p'},\end{aligned}$$

where $\mathcal{F}_2 := \{|u_{m,n}|<k, |u_{m',n'}|<2k, |T_k(u_{m,n})-T_k(u_{m',n'})|<l\}$. Hence, obviously $\lim\limits_{l\to 0}\lim\limits_{m,n\to\infty}\lim\limits_{m',n'\to\infty} I_2^2 = 0$.

Assumption (H_2) ensures us that $I_2^1 \geq 0$. On the other hand

$$I_2^1 \leq \int_{\{k-l<|u_{m,n}|<k\}} \big[a(u_{m,n},Du_{m,n})-a(u_{m,n},0)\big]\cdot Du_{m,n}.$$

Now taking $T_k(u_{m,n})-T_{k-l}(u_{m,n})$ as a test function in (4.9), using the monotonicity of Λ, assumption (H_3) and the a.e. convergence $u_{m,n}\to u$ as $m,n\to\infty$, imply that the limit of the right hand side of the last inequality is non positive, thus $\lim\limits_{l\to 0}\lim\limits_{m,n\to\infty}\lim\limits_{m',n'\to\infty} I_2^1 = 0$.

An analogous decomposition and estimates can be applied to I_3. Thus combining all limits yields

$$\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} I \leq 0. \tag{4.12}$$

Now, thanks to this limit we are going to prove that div $a(T_k u, DT_k u)$ = div χ_k in $\mathcal{D}'(\Omega)$. Let $\varphi\in W^{1,p}(\Omega)$, using the limit (4.12), we have

$$\begin{aligned}&2\int_\Omega \chi_k\cdot D\varphi \\ \geq\ &\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} \\ &\Big[\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\leq l\}} a(T_k(u_{m,n}),DT_k(u_{m,n}))\cdot D(T_k(u_{m,n})-T_k(u_{m',n'})+\varphi) \\ +\ &\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|>l\}} a(T_k(u_{m,n}),DT_k(u_{m,n}))\cdot D\varphi \\ +\ &\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\leq l\}} a(T_k(u_{m',n'}),DT_k(u_{m',n'}))\cdot D(T_k(u_{m',n'})-T_k(u_{m,n})+\varphi) \\ +\ &\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|>l\}} a(T_k(u_{m',n'}),DT_k(u_{m',n'}))\cdot D\varphi\Big] \\ =:\ &J_1+J_2+J_3+J_4.\end{aligned} \tag{4.13}$$

We start with J_2. As $a(T_k(u_{m,n}),DT_k(u_{m,n}))$ is bounded in $\left(L^{p'}(\Omega)\right)^N$, Hölder's inequality applied to J_2 implies

$$|J_2| \leq C\Big[\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|>l\}} |D\varphi|^p\Big]^{1/p}.$$

Using Dominated Convergence Theorem and the fact that $T_k(u_{m',n'}) \to T_k(u)$ a.e. in Ω, we get $\lim\limits_{l\to 0}\lim\limits_{m,n\to\infty}\lim\limits_{m',n'\to\infty} J_2 = 0$. Analogously, we have also

$$\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} J_4 = 0.$$

Now, we treat the term J_1 by using hypotheses (H_1) and (H_3), the fact that $DT_k(u_{m,n}) \rightharpoonup DT_k(u)$ weakly in $L^p(\Omega)$ and $T_k(u_{m,n}) \to T_k(u)$ a.e. in Ω as $m,n\to\infty$. Indeed

$$\begin{aligned}
&\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} J_1 \\
&\geq \lim_{l\to 0}\lim_{m,n\to\infty}\int_{\{|T_k(u_{m,n})-T_k(u)|\leq l\}} a(T_k(u_{m,n}), D(T_k(u)-\varphi))\cdot D(T_k(u_{m,n})-T_k(u)+\varphi) \\
&\geq \int_\Omega a(T_k(u), D(T_k(u)-\varphi))\cdot D\varphi.
\end{aligned}$$

Now, we remark that the term J_3 can be written as

$$\begin{aligned}
&\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\leq l\}} a(T_k(u_{m',n'}), DT_k(u_{m',n'}))\cdot D(T_k(u_{m',n'})-T_k(u)+\varphi) \\
&+\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\leq l\}} a(T_k(u_{m',n'}), DT_k(u_{m',n'}))\cdot D(T_k(u)-T_k(u_{m,n})) \\
=:\ & J_3^1 + J_3^2.
\end{aligned}$$

By means of assumption (H_1) and Dominated Convergence Theorem we have

$$\begin{aligned}
&\lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty} J_3^1 \\
\geq\ & \lim_{l\to 0}\lim_{m,n\to\infty}\lim_{m',n'\to\infty}\Big[\\
&\int_{\{|T_k(u_{m,n})-T_k(u_{m',n'})|\leq l\}} a(T_k(u_{m',n'}), D(T_k(u)-\varphi))\cdot D(T_k(u_{m',n'})-T_k(u)+\varphi)\Big] \\
\geq\ & \int_\Omega a(T_k(u), D(T_k(u)-\varphi))\cdot D\varphi.
\end{aligned}$$

On the other hand, since

$$a(T_k(u_{m',n'}), DT_k(u_{m',n'})) \rightharpoonup \chi_k$$

weakly in $\left(L^{p'}(\Omega)\right)^N$ and $DT_k(u_{m,n}) \rightharpoonup DT_k(u)$ weakly in

$$L^p(\Omega) \quad \text{as} \quad m,n\to\infty,$$

$\lim\limits_{l\to 0}\lim\limits_{m,n\to\infty}\lim\limits_{m',n'\to\infty} J_3^2 = 0.$

Combining together all limits in (4.13), we obtain

$$2\int_\Omega \chi_k\cdot D\varphi \geq 2\int_\Omega a(T_k(u), D(T_k(u)-\varphi))\cdot D\varphi. \tag{4.14}$$

Taking $\varphi = t\zeta$ where $\zeta \in \mathcal{D}(\Omega)$ and $t \in \mathbb{R}$, dividing this inequality by $t > 0$, resp., $t < 0$, passing to the limit with $t \downarrow 0$, resp., $t \uparrow 0$, yields $\int_\Omega \chi_k \cdot D\zeta = \int_\Omega a(T_k(u), DT_k(u)) \cdot\ D\zeta$ for all $\zeta \in \mathcal{D}(\Omega)$ and the result follows.

Step 4: Passage to the limit in Equation (4.9).

Taking $\varphi = S(u_{m,n} - \phi)$ as a test function in (4.9), where $S \in \mathcal{P}$ and $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and define $l := \|\phi\|_\infty + \max\{|z|, z \in \mathrm{Supp}(S')\}$.

Let us pass to the limit with m, n in each term. Consider the first integral, using the monotonicity assumption on a we get

$$\begin{aligned}
&\int_\Omega a(u_{m,n}, D(u_{m,n})) \cdot DS(u_{m,n} - \phi) \\
= &\int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})) \cdot DS(u_{m,n} - \phi) \\
= &\int_\Omega \Big(a(T_l(u_{m,n}), DT_l(u_{m,n})) - a(T_l(u_{m,n}), DT_l(u))\Big) \\
&\quad \cdot D(T_l(u_{m,n}) - T_k(u))S'(u_{m,n} - \phi) \\
&\quad + \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})) \cdot DT_l(u)S'(u_{m,n} - \phi) \\
&\quad + \int a(T_l(u_{m,n}), DT_l(u)) \cdot D(T_l(u_{m,n}) - T_l(u))S'(u_{m,n} - \phi) \\
&\quad - \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})) \cdot D\phi S'(u_{m,n} - \phi) \\
\geq &\int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})) \cdot DT_l(u)S'(u_{m,n} - \phi) \\
&\quad + \int_\Omega a(T_l(u_{m,n}), DT_l(u)) \cdot D(T_l(u_{m,n}) - T_l(u))S'(u_{m,n} - \phi) \\
&\quad - \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})) \cdot D\phi S'(u_{m,n} - \phi).
\end{aligned} \tag{4.15}$$

As $S'(u_{m,n} - \phi) \to S'(u - \phi)$ a.e. in Ω, $DT_l(u_{m,n}) \rightharpoonup DT_l(u)$ weakly in $W^{1,p}(\Omega)$, $T_l(u_{m,n}) \to T_l(u)$ a.e. in Ω and $a(T_l(u_{m,n}), DT_l(u_{m,n})) \rightharpoonup \chi_l$ as $m, n \to \infty$, we get after passing to the limit in (4.15) with $m, n \to \infty$

$$\begin{aligned}
&\lim_{m,n\to\infty} \int_\Omega a(T_l(u_{m,n}), DT_l(u_{m,n})) \cdot DS(u_{m,n} - \phi) \\
\geq &\int_\Omega \chi_l \cdot DT_l(u)S'(u - \phi) - \int_\Omega \chi_l \cdot D\phi S'(u - \phi) \\
= &\int_\Omega \chi_l \cdot DS(u - \phi).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\lim_{m,n\to\infty} \int_\Omega a(u_{m,n}, Du_{m,n}) \cdot DS(u_{m,n} - \phi) \\
\geq &\int_\Omega a(u, Du) \cdot DS(u - \phi).
\end{aligned} \tag{4.16}$$

By Dominated Convergence Theorem, we get

$$\lim_{m,n\to\infty}\int_\Omega (f_{m,n}-u_{m,n})S(u_{m,n}-\phi)=\int_\Omega (f-u)S(u-\phi). \tag{4.17}$$

Now note that

$$\begin{aligned}
&\frac{1}{m}\int_{\partial\Omega}\Lambda(u^+_{m,n})S(u^+_{m,n}-\phi)-\frac{1}{n}\int_{\partial\Omega}\Lambda(u^-_{m,n})S(u^-_{m,n}-\phi)\\
&=\frac{1}{m}\int_{\partial\Omega}(\Lambda(u^+_{m,n})-\Lambda(\phi))S(u^+_{m,n}-\phi)\\
&-\frac{1}{n}\int_{\partial\Omega}(\Lambda(u^-_{m,n})-\Lambda(\phi))S(u^-_{m,n}-\phi)\\
&\ +\frac{1}{m}\int_{\partial\Omega}\Lambda(\phi)S(u^+_{m,n}-\phi)-\frac{1}{n}\int_{\partial\Omega}\Lambda(\phi)S(u^+_{m,n}-\phi).
\end{aligned} \tag{4.18}$$

The two first integrals of (4.18) are nonnegative while the two last converge to zero as $m,n\to\infty$.

To complete the proof, it remains to show that μ verifies (4.1), (4.2) and

$$\lim_{m,n\to\infty}\int_{\partial\Omega}S(\tilde{u}_{m,n}-\tilde{\phi})d\mu_{m,n}=\int_{\partial\Omega}S(\tilde{u}-\tilde{\phi})d\mu. \tag{4.19}$$

We know from the proof of Theorem 3.1 (part ii) that $\mu_{m,n}\in\partial\mathcal{J}(u_{m,n})$, thus

$$(\mu_{m,n})_r\in\partial j(\cdot,u_{m,n})+\partial I_{[\gamma_-,\gamma_+]}(u_{m,n}).$$

As $u_{m,n}\to u$ a.e. on Ω and $\|(\mu_{m,n})_r-\mu_r\|_{L^1(\partial\Omega)}\le\|\mu_{m,n}-\mu\|_{\mathcal{M}_b(\partial\Omega)}\longrightarrow 0$ as $m,n\to\infty$, then

$$\mu_r\in\partial j(\cdot,u)+\partial I_{[\gamma_-,\gamma_+]}(u).$$

On the other hand, we have

$$\int_{\partial\Omega}(\gamma_+-\tilde{u}_{m,n})d(\mu_{m,n})^+_s=0\ \text{ and }\ \int_{\partial\Omega}(\gamma_--\tilde{u}_{m,n})d(\mu_{m,n})^-_s=0,$$

which is equivalent to say that

$$\tilde{u}_{m,n}=\gamma_{+/-}\qquad(\mu_{m,n})^{+/-}_s-\text{a.e. on }\partial\Omega.$$

Thus, again as u is finite in the sense q.e. on $\partial\Omega$ and, moreover, $(\mu_{m,n})_s\to\mu_s$ in $\mathcal{M}_b(\partial\Omega)$ as $m,n\to\infty$, we get

$$\int_{\partial\Omega}(\gamma_+-\tilde{u})d\mu^+_s=0,\qquad\int_{\partial\Omega}(\gamma_--\tilde{u})d\mu^-_s=0,$$

which is equivalent to

$$\tilde{u}=\gamma_{+/-}\qquad\mu^{+/-}_s-\text{a.e. on }\partial\Omega.$$

As $u_{m,n}\to u$ a.e. on Ω and $\mu_{m,n}\to\mu$ strongly in $\mathcal{M}_b(\partial\Omega)$, it is easy to see that (4.19) holds.

Finally, collecting together all the limits (4.16)-(4.19), we conclude that

$$\int_\Omega a(u, Du)\cdot DS(u-\phi) + \int_{\partial\Omega} S(\tilde{u}-\tilde{\phi})d\mu \le \int_\Omega (f-u)S(u-\phi).$$

Taking S as an approximation of T_k, we get the desired entropy inequality . Therefore, we have shown that, for all $f \in L^\infty(\Omega)$, $(I+A_{m,n})^{-1}f$ converges in $L^1(\Omega)$ to an entropy solution of the problem $(E)(f)$, hence $\liminf\limits_{m,n\to\infty} A_{m,n} \subset \mathcal{A}$. For the inverse inclusion, we refer to the step below.

Step 5: The accretivity of $\mathcal{A}$.

To prove the accretivity of $\mathcal{A}$, we must show that

$$\int_\Omega |w-v| \le \int_\Omega |f-g| \tag{4.20}$$

where $f \in w+\mathcal{A}w, g \in v+\mathcal{A}v$.

Observe that $w = \lim\limits_{m,n\to\infty} w_{m,n}$ and $v = \lim\limits_{m,n\to\infty} v_{m,n}$ in $L^1(\Omega)$, where $w_{m,n} = (I+A_{m,n})^{-1}f$ and $v_{m,n} = (I+A_{m,n})^{-1}g$. Indeed, taking $\phi_1 = w_{m,n}$ and $\phi_2 = \frac{1}{h}T_h(w_{m,n}-T_l(w))$, where $l \ge \|w_{m,n}\|_\infty + h + 1$, as test functions in the inequalities corresponding to the solutions w and $w_{m,n}$ respectively, adding both inequalities, passing to the limit first with $h \to 0$ and $l \to \infty$, then with $m,n \to \infty$, we get the result.

We have shown in Theorem 3.1 that the operator $A_{m,n}$ is accretive, i.e. $\int_\Omega |w_{m,n}- v_{m,n}| \le \int_\Omega |f-g|$. Since $\int_\Omega |w-v| \le \int_\Omega |w-w_{m,n}| + \int_\Omega |w_{m,n}-v_{m,n}| + \int_\Omega |v_{m,n}-v|$, (4.20) follows.

Step 6: $D(\mathcal{A})$ is dense in $L^1(\Omega)$.

For this, we show that $L^\infty(\Omega) \subset \overline{D(\mathcal{A})}^{\|\cdot\|_1}$. Let $u \in L^\infty(\Omega)$. Consider $u^\alpha_{m,n}$ and $u_\alpha, \alpha > 0$ such that

$$u^\alpha_{m,n} + \alpha A_{m,n} u^\alpha_{m,n} \ni u, \qquad u_\alpha + \alpha \mathcal{A} u_\alpha \ni u. \tag{4.21}$$

We know from Theorem 3.1 that $D(A_{m,n})$ is dense in $L^1(\Omega)$, then for all $m > 0$ and $n > 0$ we have $u^\alpha_{m,n} \to u$ in $L^1(\Omega)$ as $\alpha \to 0$. We show now that $u^\alpha_{m,n} \to u_\alpha$ in $L^1(\Omega)$ as $m,n \to \infty$. To this end, taking $\frac{1}{l}T_l(u^\alpha_{m,n}-u_\alpha)$, respectively $u^\alpha_{m,n}$, as a test function in the entropy formulation of the problems defined in (4.21), adding both inequalities, we get for all $l > 0$

$$\begin{aligned}
&\frac{1}{l}\int_\Omega (a(u^\alpha_{m,n}, Du^\alpha_{m,n}) - a(u_\alpha, Du_\alpha)) \cdot DT_l(u^\alpha_{m,n} - u_\alpha) \\
&+\frac{1}{lm}\int_{\partial\Omega} \Lambda(u^{\alpha,+}_{m,n})T_l(u^\alpha_{m,n} - u_\alpha) - \frac{1}{ln}\int_{\partial\Omega} \Lambda(u^{\alpha,-}_{m,n})T_l(u^\alpha_{m,n} - u_\alpha) \\
\leq\ & -\int_\Omega (u^\alpha_{m,n} - u_\alpha)\frac{1}{l}T_l(u^\alpha_{m,n} - u_\alpha) - \frac{1}{l}\int_{\partial\Omega} T_l(\tilde{u}^\alpha_{m,n} - \tilde{u}_\alpha)d\mu^\alpha_{m,n} \\
&\quad -\int_{\partial\Omega} \frac{1}{l}T_l(\tilde{u}_\alpha - \tilde{u}^\alpha_{m,n})d\mu_\alpha. \qquad (4.22)
\end{aligned}$$

Using assumptions (H_1) and (H_4) we have

$$\begin{aligned}
&\frac{1}{l}\int_\Omega (a(u^\alpha_{m,n}, Du^\alpha_{m,n}) - a(u_\alpha, Du_\alpha)) \cdot DT_l(u^\alpha_{m,n} - u_\alpha) \\
\geq\ & \frac{1}{l}\int_\Omega (a(u^\alpha_{m,n}, Du^\alpha_{m,n}) - a(u_\alpha, Du^\alpha_{m,n}))DT_l(u^\alpha_{m,n} - u_\alpha) \\
\geq\ & -\frac{1}{l}\int_F C(\|u^\lambda_{m,n}\|_\infty, l)|u^\alpha_{m,n} - u_\alpha|(1 + |Du^\alpha_{m,n}|^{p-1})D(u^\alpha_{m,n} - u_\alpha) \\
\longrightarrow\ & 0 \ \text{ as } l \to 0,
\end{aligned}$$

where $F := \{|u_\alpha| \leq \|u^\alpha_{m,n}\|_\infty + l\} \cap \{|u^\alpha_{m,n} - u_\alpha| < l\}$. Noticing that the two last integrals in the right hand side of inequality (4.22) are nonnegative. Indeed these integrals can be written as $\int_{\partial\Omega} T_l(\tilde{u}^\alpha_{m,n} - \tilde{u}_\alpha)((\mu^\alpha_{m,n})_r - (\mu_\alpha)_r) + \int_{\partial\Omega} T_l(\gamma_+ - \tilde{u}_\alpha)(\mu^\alpha_{m,n})^+_s + \int_{\partial\Omega} -T_l(\gamma_- - \tilde{u}_\alpha)(\mu^\alpha_{m,n})^-_s + \int_{\partial\Omega} -T_l(\tilde{u}^\alpha_{m,n} - \gamma_+)(\mu_\alpha)^+_s + \int_{\partial\Omega} T_l(\tilde{u}^\alpha_{m,n} - \gamma_-)(\mu_\alpha)^-_s$, which are, clearly, nonnegative by properties of the measures and $\gamma_{+/-}$. Thus, after passing to the limit in (4.22) with $l \to 0$, we get

$$\int_\Omega |u^\alpha_{m,n} - u_\alpha| \leq \frac{1}{m}\int_{\partial\Omega} |\Lambda(u^{\alpha,+}_{m,n})| + \frac{1}{n}\int_{\partial\Omega} |\Lambda(u^{\alpha,-}_{m,n})|.$$

Therefore, $\|u^\alpha_{m,n} - u_\alpha\|_1 \to 0$ as $m, n \to \infty$. Since $\|u_\alpha - u\|_1 \leq \|u_\alpha - u^\alpha_{m,n}\|_1 + \|u^\alpha_{m,n} - u\|_1 \to 0$ as $\alpha \to 0$ and $m, n \to \infty$ we deduce that $u \in \overline{D(\mathcal{A})}^{\|\cdot\|_1}$. □

Corollary 4.1. *Under the assumptions of Theorem 4.1, we have existence and uniqueness of entropy solution for the problem $(E)(f)$.*

Remark 4.2. By the Nonlinear Semigroup Theory, it is possible to solve in the mild sense the evolution problem

$$\frac{du}{dt} + \mathcal{A}u = f, \qquad u(0) = u_0$$

for all $u_0 \in L^1(\Omega)$, $f \in L^1(0, T; L^1(\Omega))$, which transcribes the following problem

$$\begin{cases}
u_t - \operatorname{div}\, a(u, Du) = f & \text{in} \quad \Omega \times (0, T) \\
-\langle a(u, Du), \eta\rangle \in \beta(x, u) & \text{on} \quad \partial\Omega \times (0, T) \\
u(0) = u_0 & \text{in} \quad \Omega.
\end{cases}$$

In a forthcoming paper, the existence and uniqueness of entropy solutions of the above problem will be considered.

Remark 4.3. It is an open problem how to prove directly a comparison principle for entropy solutions without using the approximation method.

5. CONCLUSION

Note that assumption (H_4) is used to prove uniqueness of entropy solutions. The condition is not optimal. In fact, it is sufficient to assume that a satisfies some Hölder type continuity and a certain growth restriction in r instead (see Andreianov and Bouhssis [5]). However, it is not the purpose of this paper to present the weakness assumptions possible on a. In this paper we focus on the boundary condition. Our aim was to show which solution concept is suited for a nonlinear elliptic problem with general nonlinear boundary conditions in $(E)(f)$.

An interesting problem is to study the same problem with u replaced by $\gamma(u)$ and with the same nonlinear boundary conditions. The case where γ is continuous non-decreasing corresponds to the stationary problem associated with the elliptic-parabolic evolution problem arising as a model of fluid flow through porous media. In this type of problems, from the view point of applications, it is essential to study general nonlinear boundary conditions. The more general case where γ is a multi-valued monotone graph corresponds to a Stefan problem arising in applications in presence of phase transitions. The linear case with $a(u, Du) = Du$ and the boundary condition $u_\nu + \beta(u) \ni 0$ on $\partial\Omega$, β a maximal monotone graph, has already been studied in Bénilan et al [15]. In this case, some extra compatibility conditions on β and γ are necessary in order to get existence of a solution. In the linear case considered in Bénilan et al [15] these compatibility conditions include in particular the assumptions $D(\gamma) \cap D(\beta) \neq \emptyset$ and $D(\gamma) \cap \beta^{-1}(0) \neq \emptyset$.

Recently, Andreu et al [6], have studied the problem

$$\begin{cases} \gamma(u) - \operatorname{div} a(x, Du) \ni \phi & \text{in} \quad \Omega\,, \\ -a(x, Du) \cdot \eta + \beta(u) \ni \psi & \text{on} \quad \partial\Omega, \end{cases}$$

where γ, β are maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$, and $\phi \in L^1(\Omega), \psi \in L^1(\partial\Omega)$.

They prove existence and uniqueness of weak and entropy solutions for this problem. As in Bénilan et al [15], a range condition relating the average of ϕ and ψ to the range of β and γ are necessary for existence of weak and entropy solutions.

In a forthcoming paper, we will generalize their works to the case where the operator a depends on u and the graphs β and γ depend on the space variable x.

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