Communications in Applied Analysis **11** (2007), no. 2, 327–337

KERNELS OF AN INTEGRO-DIFFERENTIAL EQUATION FROM AN INITIAL PULSE

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Communicated by E.W. Cheney

ABSTRACT: We consider an integro-differential equation, proposed in the literature as a model of neuronal activity. We establish conditions under which an initial activity function exhibiting localized pattern formation completely characterizes the system. We also investigate how such an initial activity determines more complicated pattern formations.

AMS (MOS) Subject Classification: 45J05

1. INTRODUCTION

Nonlinear systems of partial integro-differential equations play an important role in the modelling of neuronal activity. A complete characterization of the solutions of such systems is very hard to determine. However, the search for solutions through numerical simulations allows us to reduce the initial system to a simpler one. Analysis of this new system often brings significant insight to the qualitative behavior of the original one. This technique has been useful in the study of nonlinear partial differential equations that arise in the modelling of neuronal systems.

Experimental research has shown that neuronal activity exhibits localized spatial regions of high excitability, designated “bumps” or “pulses”. Such brain activity is an indicator of selective and persistent neuronal response to stimuli. Experiments have suggested a close link between localized patterns of brain activity and the encoding of information such as visual cues or head direction, see Coombes et al [5], Laing and Troy [10], Stringer et al [19], Funahashi et al [20]. Realistic models must include

Received February 25, 2007

groups of spatially interconnected neurons exhibiting localized regions of high activity, see Murdock et al [9], Laing and Troy [10].

In 1977, Amari considered the following integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} \omega(x-y)\, H(u(y,t)-h)\, dy, \tag{1}$$

to model brain activity on a single layer of interconnected neurons (see Wilson and Cowan [21]).

The average membrane potential, at the location x and time t, is denoted by $u(x,t)$ and the average synaptic strength among neurons is represented by the coupling function ω. The Heaviside function H is used to represent the firing rate and the integral term, in equation (1), combines the activity emitted by the network and presented at the location x. The activity occurring at position y is filtered via the action of H and then changed by a multiplicative factor representing the neuronal synaptic strength and denoted by ω. Throughout this paper we consider ω to approach zero rapidly as x grows in absolute value. This assumption represents that activity occurring at a certain location decreases its effect with distance. The synaptic impact depends uniquely on the spatial distance, hence the system is said to be homogeneous, cf. Amari [2].

Several researchers have investigated, both analytically and numerically, multi-bump formation and respective linear stability (see Laing et al [11], Guo and Chow [6]). The numerical existence of single pulse solutions might be relatively easy to determine (see Amari [1], Laing and Troy [10], Guo and Chow [6]), but to establish their existence analytically is far more difficult, relying on careful estimates, cf. Botelho et al [4]. Moreover, the existence of multi-pulse solutions is extremely complicated to determine analytically, see Murdock et al [9]. A natural approach is to express multi-pulse solutions as a superposition of many simple basic structures, cf. Elphick et al [12]. Inspired by this idea, we investigate in this paper an inverse problem. We establish conditions that assure the existence of a coupling function ω, for which the system (1) has solution u. Under these conditions, we construct ω in terms of u and assert its uniqueness. In addition, we show that multi-bump solutions can be generated from an initial pulse. We also investigate rigidity questions and the system dependence on the initial pulse.

2. ONE-BUMP SOLUTIONS

In this section we discuss how a one-bump function, solution of an integro-differential equation (1), determines the coupling function ω. Coupling functions are continuously differentiable, integrable over $\mathbb{R}$, and $\lim_{x\to\pm\infty} \omega(x) = 0$, e.g. $(1-x^2)e^{-x^2/2}$. We derive a decomposition of ω in terms of an initial pulse. We start with the definition of an N-bump function or N-pulse.

Definition 2.1. A differentiable real-valued function f with integrable derivative is called an (N, h)-bump function or pulse if the positive support of $f - h$ (i.e. $\{x : f(x) > h\}$) is the union of N bounded and disjoint intervals.

The following theorem states necessary and sufficient conditions for an initial one-pulse to be a solution of the equation (1). We denote by γ a real number greater than 1.

Theorem 2.1. *If u is an integrable $(1, h)$-bump function with positive support $(0, a)$, such that $|u'(x)|\gamma^{|x|}$ is bounded, then there exists a unique ω for which u is a stationary solution of*

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} \omega(x-y)\, H(u(y,t)-h)\, dy$$

if and only if $\sum_{j=-\infty}^{\infty} u'(x - ja) = 0$, for every x.

Proof. First we show that $\sum_{j\in Z} u'(x - ja) \equiv 0$ is a sufficient condition for the existence of ω so that $u = \int_{-\infty}^{\infty} \omega(x-y)\, H(u(y)-h)\, dy$. The assumption on u implies the existence of positive constants M and λ so that, for $|x| > M$, $|u'(x)| \le \lambda\gamma^{-|x|}$. Therefore the series $\sum_{j=0}^{\infty} u'(x - ja)$ and $\sum_{j=0}^{\infty} u'(x + ja)$ are pointwise convergent. Furthermore, there exists a positive integer j_0, so that for every $x \in [-M, M]$ we have $x - j_0 a < -M$. We denote by L the maximum of $|u'(x)|$, for $x \in [-M, M]$. We have the following inequalities:

$$\begin{array}{ll} |\sum_{j=0}^{\infty} u'(x-ja)| \le \lambda\gamma^{x} \sum_{j=0}^{\infty} \gamma^{-ja} & \text{for } x \le -M\,, \\ |\sum_{j=0}^{\infty} u'(x-ja)| \le j_0\, L + \lambda\gamma^{x} \sum_{j=0}^{\infty} \gamma^{-ja} & \text{for } x \in [-M,\, M]\,, \\ |\sum_{j=1}^{\infty} u'(x+ja)| \le \lambda\gamma^{-x} \sum_{j=1}^{\infty} \gamma^{-ja} & \text{for } x > M. \end{array}$$

Therefore, the sequence of partial sums

$$s_n(x) = \begin{cases} \sum_{j=0}^{n} u'(x-ja) & \text{if } x \le M\,, \\ -\sum_{j=1}^{n} u'(x+ja) & \text{if } x > M \end{cases}$$

is pointwise convergent and we set $\omega(x) = \lim_n s_n(x)$. Since

$$-\sum_{j=1}^{\infty} u'(x+ja) = \sum_{j=0}^{\infty} u'(x-ja),$$

we have $\omega = \sum_{j=0}^{\infty} u'(x - ja)$. In order to show that u is a stationary solution of the equation (1) we consider the integrable function

$$f(x) = \begin{cases} A\gamma^{x} & \text{for } x \le -M\,, \\ j_0 L + A\gamma^{x} & \text{for } -M < x < M\,, \\ A\gamma^{-x} & \text{for } x \ge M, \end{cases}$$

where $A = \sum_{j=0}^{\infty} \gamma^{-ja}$. The Lebesgue Convergence Theorem implies that ω is integrable and $\lim_n \int_0^a s_n(x)\,dx = \int_0^a \omega(x)\,dx$. Therefore we have

$$\int_{-\infty}^{\infty} \omega(x-y)H(u(y)-h)\,dy = \lim_n \sum_{j=0}^{n} \int_0^a u'(x-y-ja)\,dy = \lim_n (u(x)-u(x-na)).$$

Since $-\lambda e^{-x} \leq u'(x) \leq \lambda e^{-x}$ $(x>0)$, given $x_0 > 0$, we have that

$$-\lambda \int_{x_0}^{x} e^{-y}\,dy \leq \int_{x_0}^{x} u'(y)\,dy \leq \lambda \int_{x_0}^{x} e^{-y}\,dy$$

or equivalently

$$\lambda(e^{-x} - e^{-x_0}) \leq u(x) - u(x_0) \leq \lambda(-e^{-x} + e^{-x_0}).$$

Consequently we have

$$-\lambda e^{-x_0} \leq \overline{\lim}_{x\to\infty} u(x) - u(x_0) \leq \lambda e^{-x_0}$$

and $\overline{\lim}_{x\to\infty} u(x) = \underline{\lim}_{x\to\infty} u(x)$. Similarly it can be shown that

$$\overline{\lim}_{x\to-\infty} u(x) = \underline{\lim}_{x\to-\infty} u(x).$$

Since u is integrable then $\lim_{x\to\pm\infty} u(x) = 0$. Therefore $\lim_n u(x \pm na) = 0$. This concludes the proof that u is a solution of equation (1). The uniqueness follows from the construction of ω. Conversely, if u is an one-bump function over $(0, a)$ such that $u = \int_0^a \omega(x-y)\,dy$ then $u(x) = \int_0^x \omega(y)\,dy - \int_0^x \omega(y-a)\,dy$. Therefore, $u'(x) = \omega(x) - \omega(x-a)$ and $\omega(x) = \sum_{j=0}^{\infty} u'(x-ja)$. □

Remarks.

(1) The statement above is still true under weaker conditions on u'. It is sufficient to assume that $|u'(x)P(x)|$ is bounded, with P a nonlinear polynomial function.

(2) We also observe that Theorem 2.1 can be restated in an "almost everywhere" setting. In fact, the same proof will also assert that, under the weaker assumption $u'(x)\gamma^{|x|} \in L_\infty(R)$ (or $\|u'(x)\gamma^{|x|}\|_\infty < +\infty$), there exists ω uniquely defined almost everywhere so that $\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} \omega(x-y)\,H(u(y,t) - h)\,dy$ if and only if $\sum_{j=-\infty}^{\infty} u'(x-ja) = 0$, for almost every x.

We describe a class $\mathcal{A}$ of initial one-pulses that satisfies the conditions of the theorem. A function $u_m^n \in \mathcal{A}$ is defined to be:

$$u_m^n(x) = \begin{cases} xe^{-\frac{x^{2n}}{2}} - (x-1)e^{-\frac{(x-1)^{2n}}{2}} & x > 1\,, \\ xe^{-\frac{x^{2n}}{2}} - (x-1)e^{-\frac{(x-1)^{2m}}{2}} & 0 \leq x \leq 1\,, \quad m, n \in Z^+. \\ xe^{-\frac{x^{2m}}{2}} - (x-1)e^{-\frac{(x-1)^{2m}}{2}} & x < 0\,, \end{cases}$$

Proposition 2.1. *For every $u_m^n \in \mathcal{A}$ there exists a unique ω_m^n so that u_m^n is a $(1, e^{-\frac{1}{2}})$-bump solution of the equation*

$$u_m^n(x,t) = \int_{-\infty}^{\infty} \omega_m^n(x-y)\, H(u_m^n(y,t) - h)\, dy.$$

Proof. First, we observe that $u_m^n(0) = u_m^n(1) = e^{-\frac{1}{2}}$, ${u'}_m^n(0) = 1 - (1-m)e^{-\frac{1}{2}}$ and ${u'}_m^n(1) = (1-n)e^{-\frac{1}{2}} - 1 < 0$. We set $z_m = -\left(\frac{1}{m}\right)^{\frac{1}{2m}}$ and $z_n = \left(\frac{1}{n}\right)^{\frac{1}{2n}}$. Moreover, $(1 - nx^{2n})e^{-x^{2n}/2}$ is positive implying that $xe^{-x^{2n}/2}$ is strictly increasing on the open interval $(0,\ z_n)$; $(1-m(x-1)^{2m})e^{-(x-1)^{2m}/2}$ is negative, thus $(x-1)e^{-(x-1)^{2m}/2}$ is strictly decreasing over the open interval $(1+z_m,\ 1)$; and the product $(1-nx^{2n})e^{-x^{2n}/2}(1-m(x-1)^{2m})e^{-(x-1)^{2m}/2}$ is negative over the union $(0,\ \max\{z_n,\ -z_m\}) \cup (1+\max\{-z_n,\ z_m\},\ 1)$. Therefore ${u'}_m^n(x)$ is equal to zero at some point $x \in (z_n,\ 1+z_m)$, which is the unique critical point of u in $(0,\ 1)$. Next, we establish that $u_m^n(x) < e^{-\frac{1}{2}}$ on the complement of the closed interval $[0,\ 1]$. We observe that $z_n e^{-z_n^{2n}/2} \le e^{-1/2}$, therefore $u_m^n(x) - e^{-\frac{1}{2}} \le 0$, for $x > 1$. Similarly, it can be shown for $x < 0$. A telescoping argument implies that $\lim_{N\to\infty} \sum_{j=-N}^{N} {u'}_m^n(x-j) = 0$, therefore Theorem 2.1 assures the existence of a unique ω_m^n so that u_m^n is a stationary solution of the equation

$$u_m^n(x,t) = \int_{-\infty}^{\infty} \omega_m^n(x-y)\, H(u_m^n(y,t) - h)\, dy.$$

Furthermore, a similar telescoping argument also implies that

$$\omega_m^n(x) = \sum_{j=0}^{\infty} {u'}_m^n(x-j) = \begin{cases} (1-nx^{2n})\, e^{-\frac{x^{2n}}{2}} & x \ge 0 \\ (1-mx^{2m})\, e^{-\frac{x^{2m}}{2}} & x < 0. \end{cases}$$

Remark. Every function u in class $\mathcal{A}$, determines a unique coupling function ω for which that function is an (1,h)-bump solution of $u(x) = \int_{-\infty}^{\infty} \omega(x-y)\, H(u(y)-h)\, dy$. This class provides examples of asymmetric pulses and coupling functions. As a consequence of Theorem 2.1 and Proposition 2.1, we have provided an analytical proof that the connection given above supports stationary solutions of equation (1). Moreover, for $m \ne n$, ω_m^n is not symmetric hence Amari's Theorem on existence of one-bump solutions does not imply our result. In a later section we will show that this stationary solution is linearly stable.

The following two corollaries of theorem 2.1 investigate the inter-dependence between initial pulse and coupling function. They are stated for a (1,h)-bump function u with positive support over the open interval $(0,\ a)$ and $|u'(x)|\gamma^{|x|}$ $(\gamma > 1)$ bounded.

Corollary 2.1. *u is symmetric relatively to $\frac{a}{2}$ (i.e. $u(\frac{a}{2}+x) = u(\frac{a}{2}-x)$) if and only if ω is symmetric (i.e. $\omega(x) = \omega(-x)$).*

Proof. This statement follows directly from the theorem, since

$$\begin{aligned}\omega(x) &= \textstyle\sum_{j=0}^{\infty} u'(x-ja) = -\sum_{j=0}^{\infty} u'(-x+(j+1)a)\\ &= -\textstyle\sum_{j=1}^{\infty} u'(-x+ja) = \sum_{j=-\infty}^{0} u'(-x+ja)\\ &= \textstyle\sum_{j=0}^{\infty} u'(-x-ja) = \omega(-x).\end{aligned}$$

Corollary 2.2. *If $u(x) = F(x) - F(x-a)$ for some continuously differentiable function $F(x)$ so that $\lim_{x\to\pm\infty} F'(x) = 0$, then $\sum_{j=-\infty}^{\infty} u'(x-ja) = 0$, $\omega(x) = F(x) = \sum_{j=0}^{\infty} u'(x-ja)$, and $u = \omega * H(u-h)$.*

Proof. We notice that

$$\sum_{j=-\infty}^{\infty} u'(x-ja) = \lim_{N\to\infty} [F'(x+Na) - F'(x-(N+1)a)] = 0,$$

therefore the statement in the corollary follows from the theorem 2.1. □

3. ILLUSTRATIVE EXAMPLES

In this section we investigate the applicability of the conditions stated in Theorem 2.1 to some particular classes of examples. Our first example considers the function $u(x) = (1-(1-x)^2)e^{-(x-1)^2/2}$ representing an initial pulse positive over the interval (0, 2). Though, a priori this function is candidate for a one-bump stationary solution to an equation of type (1), we show that in fact it does not satisfy the convolution equation $u(x) = \int_{-\infty}^{\infty} \omega(x-y)\, H(u(y)-h)\, dy$, for every coupling function ω that is continuously differentiable, integrable over $\mathbb{R}$, and $\lim_{x\to\pm\infty} \omega(x) = 0$. This follows from the fact that u does not satisfy the condition stated in Theorem 2.1, $\sum_{j=-\infty}^{\infty} u'(x-2j) = 0$. We show that

$$\sum_{j=-\infty}^{\infty} u'(x-2j) = \sum_{j=-\infty}^{\infty} [-3(x-2j-1) + (x-2j-1)^3]e^{-(x-2j-1)^2/2},$$

defines a nontrivial period two function that is equal to zero at every even integer.

If $x = 1/2$ we have

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} [-3(-2j-\frac{1}{2}) + (-2j-\frac{1}{2})^3]e^{-(2j+\frac{1}{2})^2/2}\\ &= \left(\frac{11}{8}\right) e^{-\frac{1}{8}} + \sum_{j=1}^{\infty}[3(2j+\frac{1}{2}) - (2j+\frac{1}{2})^3]e^{-(2j+\frac{1}{2})^2/2}\\ &+ \qquad + \sum_{j=1}^{\infty}[3(-2j+\frac{1}{2}) - (-2j+\frac{1}{2})^3]e^{-(2j-\frac{1}{2})^2/2}.\end{aligned}$$

We estimate these two infinite sums as follows:

$$\begin{aligned}
&\textstyle\sum_{j=1}^{\infty} \left(3(2j+\frac{1}{2}) - (2j+\frac{1}{2})^3\right) e^{-(2j+\frac{1}{2})^2/2} \\
&= \left[7.5 - 2.5^3\right] e^{-\frac{2.5^2}{2}} + \sum_{j=2}^{\infty}[3(2j+\frac{1}{2}) - (2j+\frac{1}{2})^3]e^{-(2j+\frac{1}{2})^2/2} \\
&\geq -.4 + \int_{2.5}^{\infty}(3x - x^3)e^{-x^2/2}dx \geq -.7,
\end{aligned}$$

and

$$\begin{aligned}
&\textstyle\sum_{j=1}^{\infty} \left[3\left(-2j+\frac{1}{2}\right) - \left(-2j+\frac{1}{2}\right)^3\right] e^{-(2j-\frac{1}{2})^2/2} \\
&= (3(-1.5) + 1.5^3)e^{-1.5^2/2} + \sum_{j=2}^{\infty}(3(-2j+\frac{1}{2}) - (-2j+\frac{1}{2})^3)e^{-(2j-\frac{1}{2})^2/2} \\
&\geq -.4 + \textstyle\int_{-\infty}^{-3.5}(3x - x^3)e^{-x^2/2}dx = -.4 + (3.5^2 - 1)e^{-3.5^2/2} \geq -.4.
\end{aligned}$$

These inequalities imply that $\sum_{j=-\infty}^{\infty} u'(x-2j) \neq 0$, as asserted.

We describe next a natural construction to extend an observable pulse defined over a finite spatial interval in order to obtain a a function over $\mathbb{R}$ that satisfies the conditions stated in Theorem 2.1. For simplicity of exposition the initial pulse is differentiable but not continuously differentiable, however a careful perturbation will yield a continuously differentiable one. We outline this construction for a particular example. The activation function $u(x) = x(2-x),\ 0 \leq x \leq 2$ has derivative $2-2x$, hence we extend its derivative as follows:

$$u'(x) = \begin{cases} 2-2x & 0 \leq x < 2 \\ 2x-6 & 2 \leq x < 4 \\ 0 & \text{elsewhere.} \end{cases}$$

The proof of the theorem 2.1 implies that

$$\sum_{j=-\infty}^{\infty} u'(x-2j) = 0,$$

$$\omega(x) = \begin{cases} 0 & x < 0 \\ 2-2x & 0 \leq x < 2 \\ 0 & x \geq 2, \end{cases}$$

and u satisfies $u(x) = \int_{-\infty}^{\infty} \omega(x-y)H(u(y))\,dy$.

4. RELATING STATIONARY SOLUTIONS

In this section we establish necessary conditions for the co-existence of distinct one-bump stationary solutions for the same equation. We observe that if v is another one-pulse function obtained from u via a spatial translation, i.e. $v(x) = u(x-r)$, then v is also a solution of equation (1) and the positive support of v is the interval $(r,\, r+a)$ and

$$\omega(x)=\sum_{j=0}^{\infty} v(x+r-ja)=\sum_{j=0}^{\infty} u(x-ja).$$

Proposition 4.1. *If* u_1 *and* u_2 *are one-bump stationary solutions of*

$$\frac{\partial u(x,t)}{\partial t}=-u(x,t)+\int_{-\infty}^{\infty}\omega(x-y)\,H(u(y,t)-h)\,dy,$$

with positive supports $(0,\,a)$ *and* $(0,\,b)$ *respectively, then:*

(1) $a\int_{-\infty}^{\infty} u_2(x)\,dx = b\,\int_{-\infty}^{\infty} u_1(x)\,dx$,
(2) $u_1(a+b)=u_2(a+b)$,
(3) $-\int_0^a \omega(s)\,ds=\int_0^{-a}\omega(s)\,ds=-\int_0^b\omega(s)\,ds=\int_0^{-b}\omega(s)\,ds=h$,
(4) $u_2(x)=\sum_{j=0}^{\infty}\left(u_1(x-ja)-u_1(x-b-ja)\right)$.

Proof. If u_1 is a solution of the equation above then $u_1=\int_{-\infty}^{\infty}\omega(x-y)\,H(u_1(y)-h)\,dy$ with Fourier transform $\mathcal{F}(u_1)(\xi)=\mathcal{F}(\omega)(\xi)\frac{1-e^{-ia\xi}}{i\xi}$. Equivalently $i\,\xi\,\mathcal{F}(u_1)(\xi)=\mathcal{F}(\omega)(\xi)\,(1-e^{-ia\xi})$. Differentiating this last equation and setting $\xi=0$ we prove the statement in part (1). On the other hand, we also have that $(1-e^{-ib\xi})\mathcal{F}(u_1)(\xi)=(1-e^{-ia\xi})\mathcal{F}(u_2)(\xi)$. The inverse Fourier transform applied to the last equation yields $u_1(x)-u_1(x-b)=u_2(x)-u_2(x-a)$ and for $x=a+b$ we derive part (2). Part (3) follows from $u_1(0)=u_1(a)=u_2(0)=u_2(b)=h$. Moreover, we also have that $i\xi\,(1-e^{-ib\xi})\mathcal{F}(u_1)(\xi)=i\xi\,(1-e^{-ia\xi})\,\mathcal{F}(u_2)(\xi)$ and therefore u_2 can be written in terms of u_1 as stated in (4). This concludes the proof. □

Similarly an N-bump solution can be generated from a given one-bump function.

Proposition 4.2. *If* u *and* v *are stationary solutions of equation (1) and* u *is a one-pulse over* $(0,\,a)$ *and* v *a* N*-pulse over the intervals* $(b_1,\,c_1),\,(b_2,\,c_2),\,\ldots,(b_N,\,c_N)$ *then*

$$v(x)=\sum_{j=0}^{\infty}\sum_{k=1}^{N}[u(x-b_k-ja)-u(x-c_k-ja)].$$

Proof. We observe that $v'(x)=\sum_{j=1}^{N}(w(x-b_j)-w(x-c_j))$ and Fourier transformed

$$\mathcal{F}(v)(\xi)=\mathcal{F}(\omega)(\xi)\,\sum_{j=1}^{N}\frac{e^{-ib_j\xi}-e^{-ic_j\xi}}{i\xi}.$$

Since $\mathcal{F}(\omega)(\xi)=\frac{i\,\xi}{1-e^{-ia\xi}}\mathcal{F}(u)(\xi)$ we have that $\mathcal{F}(v)(\xi)=\mathcal{F}(u)(\xi)\,\sum_{j=1}^{N}(e^{-ib_j\xi}-e^{-ic_j\xi})$. The inverse of the Fourier transform applied to this last equation yields the statement in the proposition. □

5. STABILITY

We consider an initial pulse u_0 over the interval $(0, a)$ that determines a unique ω under which the system has u_0 as stationary solution. Now, we search for a solution the system of the form $u(x,t) = u_0 + v(x,t)$, we linearize around u_0 and derive a condition for the local linear stability of this initial pulse. If $u(x,t) = u_0 + v(x,t)$ is a solution of equation (1) then we have:

$$\begin{aligned} \tfrac{\partial v(x,t)}{\partial t} &= -v(x,t) + \textstyle\int_{-\infty}^{\infty} \left[\omega(x-y)\, H(u_0 + v(y,t) - h) - H(u_0(y) - h)\right] dy \\ &\approx -v(x,t) + \tfrac{\omega(x)}{u_0'(0)} v(0,t) - \tfrac{\omega(x-a)}{u_0'(a)} v(a,t). \end{aligned}$$

We solve the linear equation

$$\frac{\partial v(x,t)}{\partial t} = -v(x,t) + \frac{\omega(x)}{u_0'(0)} v(0,t) - \frac{\omega(x-a)}{u_0'(a)} v(a,t). \tag{2}$$

A solution to equation (2) must also satisfy, for $x = 0$ and $x = a$, the system

$$\begin{cases} \dot{v}(0,t) = & -v(0,t) + \frac{\sum_{j=0}^{\infty} u_0'(-ja)}{u_0'(0)} v(0,t) - \frac{\sum_{j=0}^{\infty} u_0'(-a-ja)}{u_0'(a)} v(a,t) \\ \dot{v}(a,t) = & -v(a,t) + \frac{\sum_{j=0}^{\infty} u_0'(a-ja)}{u_0'(0)} v(0,t) - \frac{\sum_{j=0}^{\infty} u_0'(-ja)}{u_0'(a)} v(a,t) \end{cases}$$

with the matrix of coefficients equal to

$$M = \begin{bmatrix} -1 + \frac{\sum_{j=0}^{\infty} u_0'(-ja)}{u_0'(0)} & -\frac{\sum_{j=0}^{\infty} u_0'(-a-ja)}{u_0'(a)} \\ \frac{\sum_{j=0}^{\infty} u_0'(a-ja)}{u_0'(0)} & -1 - \frac{\sum_{j=0}^{\infty} u_0'(-ja)}{u_0'(a)} \end{bmatrix}.$$

Definition 5.1. The solution $u_0(x)$ is said to be linearly stable along a vector space V if V is an invariant space of M and the point spectra of M restricted to V, $\sigma_p(L|_V)$ has negative real parts.

We set the quantities $A = \frac{\omega(0)}{u_0'(0)}$ and $B = \frac{u_0'(0)}{u_0'(a)}$ and

$$M = \begin{bmatrix} -1 + A & -AB + B \\ 1/B + A & -1 - AB \end{bmatrix},$$

with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = A - AB - 2$. We conclude that u_0 is a linearly stable stationary solution provided that $A - AB < 2$ and linearly unstable if $A - AB > 2$. We summarize these considerations in the next proposition.

Proposition 5.1. *If u_0 is a one pulse stationary solution of*

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} \omega(x-y)\, H(u(y,t) - h)\, dy, \tag{3}$$

so that $\sum_{j=-\infty}^{\infty} u_0'(x - ja) \equiv 0$ for all x, then $\sum_{j=0}^{\infty} u_0'(-ja) \left[\frac{1}{u_0'(0)} - \frac{1}{u_0'(a)}\right] < 2$ if and only if u_0 is linearly stable.

Remark. If u_0 is a one pulse function so that $\sum_{j=-\infty}^{\infty} u_0'(x-ja) \equiv 0$ for all x and $\omega(0)\omega(a) - 2\omega(-a)\omega(a) + \omega(-a)\omega(0) < 0$ then there exists a unique system of form (3) for which u_0 is a linearly stable. If $\omega(0) > 0$ and both $\omega(a) < 0$ and $\omega(-a) < 0$, then u_0 is a linearly stable stationary solution.

Example. We consider $u_m^n \in \mathcal{A}$, presented in Proposition 2.1. This function is one-bump stationary solution of equation (1) for

$$\omega(x) = \omega_m^n(x) = \sum_{j=0}^{\infty} {u'}_m^n(x-j) = \begin{cases} (1-nx^{2n})\, e^{-\frac{x^{2n}}{2}} & x \geq 0 \\ (1-mx^{2m})\, e^{-\frac{x^{2m}}{2}} & x < 0. \end{cases}$$

Therefore the quantity $\omega_m^n(0)\left[\frac{1}{u_0'(0)} - \frac{1}{u_0'(a)}\right] = e^{1/2}\left[\frac{1}{e^{1/2}-1+m} + \frac{1}{e^{1/2}-1+n}\right] < 2$, unless $m = n = 1$.

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