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PERSPECTIVES OF FUZZY INITIAL VALUE PROBLEMS

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ABSTRACT. Recently it is proposed that the behavior of the solutions of fuzzy differential equations (FDEs) could be tamed by a suitable forcing term. In this context a case has been made that FDEs need to be investigated as a separate discipline instead of treating them as fuzzy analogues of crisp counterparts. Here in this paper, we support this argument also by showing how different formulations of a fuzzy differential equation can lead to solutions with different behaviors, adding richness to the theory of FDEs. For this aim we use the notions of Hukuhara differential, generalized differentiability, differential inclusions and the interpretation of FDEs by using Zadeh's extension principle on the classical solution. We also point out several possible research directions in the study of FDEs.

Key Words. fuzzy differential equations, modeling under uncertainty

INTRODUCTION

In the modeling of real world phenomena, often some or most of the pertinent information may be uncertain. For example, the precise initial state may not be known or information about various parameters required as a part of the model may be imprecise. Many times, the nature of the uncertainty involved may not be statistical. In such situations involving uncertainties, Fuzzy differential equations (FDEs) are a natural way to model dynamical systems.

Here, we are interested in issues concerning Fuzzy Initial Value Problems (FIVP) of the form

$$u' = f(t, u(t)), \qquad u(0) = u_0, \tag{1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ and $u_0 \in \mathbb{R}_{\mathcal{F}}$. Here $\mathbb{R}_{\mathcal{F}}$ is the space of fuzzy real numbers.

There are several approaches to the study of fuzzy differential equations. The approach based on the Hukuhara derivative ([19]) has the disadvantage that any

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solution of a FDE has increasing length of its support ([8]). This shortcoming was resolved by interpreting a FDE as a family of differential inclusions (see e.g. [10]).

$$u'_\alpha(t) \in [f(t, u_\alpha(t))]^\alpha, \quad u_\alpha(0) \in [u_0]^\alpha, \quad \alpha \in [0,1]. \tag{2}$$

This approach has been successfully and widely adapted in several applications. (see [15] and the references therein). However, the differential inclusion formulation circumvents the situation and addresses a slightly different IVP. Namely, the f in (2) is $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_\mathcal{F}$.

On the other hand, motivated by the desire to avoid the above shortcomings, [2] introduced the notion of strongly (weakly) generalized differentiability. Using this notion, it is shown in [2] that if $c \in \mathbb{R}_\mathcal{F}$, $g : (a, b) \to \mathbb{R}$ is a differentiable function satisfying suitable conditions, then $f(x) = c \cdot g(x)$ is a strongly (weakly) generalized differentiable on (a, b) and that $f'(x) = c \cdot g'(x)$. Using such a result, [2] established some fundamental results on FIVP (1). However, while the solutions of the IVP may have decreasing length of support, the uniqueness is lost.

Recently, [9] analyzed the situation for FDE under the original formulation of the FDE's by Kaleva [11] and the Hullermeier's formulation [10] of fuzzy differential inclusions. The main argument of [9] is that dynamic systems involving fuzziness, randomness or uncertainties would naturally exhibit and reflect the effect of such an underlying phenomena. To expect a behavior similar to that of the usual dynamical systems is unnatural. One of the following three possible situations may arise in the study of such systems. The presence of uncertainties (fuzzy terms)

(i) has no effect on the dynamical system and the behavior of the solutions is exactly same as that of the corresponding crisp systems,

(ii) alters some of the essential features of the solutions but some properties of the solutions may be similar to that of the crisp counterparts, or

(iii) changes the behavior of the solutions of the system entirely and these solutions have nothing in common with the crisp counterparts.

Thus, the theory of such systems would be a lot richer than the theory of ODEs and demands an investigation as an independent discipline. This requires an effort not to fit all the results in the existing frameworks and being flexible enough to adapt new paradigms, even if such an approach seems to threaten the existing, established approaches. Better still, if one can evolve an inclusive approach, where the existing approaches are accommodated and enriched in the new paradigms.

The need for such a fresh approach was demonstrated in [9] via different inequivalent formulations of the FIVP (1). This work was further improved in [12]. Other works studying different interpretations and different formulations of FIVP, further supporting our conclusions (not always explicitely), are e.g. [3], [18], [16], [20], [6].

Continuing this spirit, in the present paper, we consider several (inequivalent) fuzzy differential equations arising from fuzzifications of crisp ODEs and also an FDE that arises naturally from a physical phenomena. We analyze in detail these FDEs with each of the above mentioned interpretations of fuzzy differential equations. We also study the influence of a forcing term on the behavior of the solutions of a FDE, under each of these different interpretations.

Our main hypothesis is that one may have to make a choice of the suitable formulation in any given application in order to obtain meaningful conclusions. The ideas proposed in this paper also have several implications to the study of Set Differential Equations (SDEs). Moreover, since Differential inclusions (see e.g. [1]) are particular cases of fuzzy differential inclusions, different formulations of the same crisp ODE lead to inequivalent differential inclusions. So, the results of the present paper bring new light and add richness to the theory of SDEs and differential inclusions. Surely, a thorough study of these issues is a subject of further research.

The organization of the paper is as follows. In Section 2, we present some mathematical preliminaries and a few relevant results. we consider several examples of fuzzy initial value problems (FIVP) in Section 3, under the notion of Hukuhara differentiability and study the effect of a forcing term on the behavior of the solutions. Section 4 deals with the FIVPs with (i) and (ii) differentiability notions. In Section 5, the Hüllermeier formulation of FIVPs in terms of IVPs involving differential inclusions are considered and in Section 6 the interpretation using Zadeh's extension principle is investigated. Throughout, we demonstrate, via several examples, the various possibilities that arise due to the different formulations.

1. MATHEMATICAL PRELIMINARIES

Let $\mathbb{R}_{\mathcal{F}}$ denote the space of fuzzy numbers. For $0 < r \leq 1$, and $u \in \mathbb{R}_{\mathcal{F}}$, denote $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. It is well-known that for any $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$, and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda \cdot u$ are defined by $[u+v]^r = [u]^r + [v]^r$, $[\lambda \cdot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$.

We denote a triangular number as $u = (a, b, c)$, where a, c are endpoints of the 0-level set (support) and $\{b\}$ is the 1-level set (core) of the fuzzy number.

Definition 1.1. (see e.g. [19]). Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$.

In this paper the "$\ominus$" sign stands always for H-difference and let us remark that $x \ominus y \neq x + (-1)y$. Usually we denote $x + (-1)y$ by $x - y$, while $x \ominus y$ stands for the H-difference. Here, the H-difference is consistent with constrained fuzzy arithmetic when y is known to be a "part" of x.

We now recall the definition of strongly generalized differentiability introduced in [2].

Definition 1.2. Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0 - h) \ominus f(x_0)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0) \ominus f(x_0 - h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

(h and $(-h)$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

Case (i) of the previous definition corresponds to the H-derivative (Hukuhara derivative) introduced in [19].

A function that is strongly generalized differentiable as in cases (i) and (ii) , will be referred as (i)-differentiable or as (ii) differentiable, respectively. As for cases (iii) and (iv), it is known that (see [2]), a function may be differentiable as in (iii) or (iv) only on a discrete set of points and these points are exactly those where differentiability switches between cases (i) and (ii)).

The following lemma is useful throughout the rest of the paper.

Lemma 1.3. *If $u(t) = (x(t), y(t), z(t))$ is triangular number valued function, then*

a) If u is (i)-differentiable (Hukuhara differentiable) then $u' = (x', y', z')$.

b) If u is (ii)-differentiable then $u' = (z', y', x')$.

Proof. The proof of b) is as follows. Let us suppose that the H-difference $u(t) \ominus u(t+h)$ exists. Then, by direct computation we get:

$$\begin{aligned}
&\lim_{h \searrow 0} \frac{u(t) \ominus u(t+h)}{-h} \\
&= \frac{(x(t) - x(t+h), y(t) - y(t+h), z(t) - z(t+h))}{-h} \\
&= \left(\frac{z(t) - z(t+h)}{-h}, \frac{y(t) - y(t+h)}{-h}, \frac{x(t) - x(t+h)}{-h} \right) \\
&= (z', y', x').
\end{aligned}$$

Similarly

$$\lim_{h \searrow 0} \frac{u(t-h) \ominus u(t)}{-h} = (z', y', x')$$

and the required conclusion follows. □

Remark 1.4. If the derivatives in a FDE are in the sense of (i) or (ii) of Definition 1.2, then one can see as a consequence of the lemma that the FDE can be translated into a system of ODEs.

Remark 1.5. The converse of the Lemma is not necessarily true. The solutions of the system of ODEs above, may result in triplets of functions (x, y, z) that may not form a triangular-number valued function. Even if they result in a triangular-number valued function, it is not in necessarily differentiable in the notion of differentiability under consideration. Therefore, employing the above Lemma and obtaining solutions for the ODEs requires that one has to verify if they form a triangular-number valued function and if it is a solution of the equation under study with the corresponding notion of differentiability.

Remark 1.6. It is shown in [2] that a fuzzy differential equation $u' = f(t, u(t))$ considered under strongly generalized differentiability concept has locally two solutions. As a result, one can obtain infinitely many functions that are solutions of a FDE almost everywhere, by pasting together the local solutions of IVPs (each subsequent IVP having as initial values the terminal value of the solution of the preceding IVP). Requiring that the solution should be a solution of the FDE on the whole domain leads to a switch-point between (i) and (ii) differentiability by (iii) or (iv) differentiability. According to a result in [2] this leads to the fact that u' is real at the switch-points. That is, differentiable switch is possible only for t and u such that $f(t, u) \in \mathbb{R}$. This condition restricts the set of solutions to a finite set.

Here, we consider only those solutions for which there is no switch between the different differentiability types. Thus, we find solutions of a FDE that are (i) or (ii) differentiable on an interval (t_0, t_1). We intend to pursue in a future work, the study of the solutions that are allowed to switch between (i) and (ii) differentiability.

2. EXISTENCE OF SEVERAL SOLUTIONS UNDER HUKUHARA DIFFERENTIABILITY

In [9] the following equivalent crisp differential equations are considered:

$$u' = -u \text{ and } u' + u = 0,\ u(0) = u_0.$$

When these equations are fuzzified we get two different fuzzy differential equations. Clearly, the second one does not have any fuzzy non-real solution. Therefore, in [9] a forcing term is added, so that it becomes

$$u' + u = \sigma(t),\ u(0) = u_0.$$

It is shown that the forcing term affects the behavior of solutions. The fuzzy IVP

$$u' + u = e^{-t}(-1, 0, 1),\ u(0) = u_0$$

with $u_0 = (-1, 0, 1)$ was studied and it was claimed that $u(t) = (-1, 0, 1)e^{-t}(1 + t)$ is the solution of the problem. In [12], it was pointed out that the above u is not a solution of the FDE but the choice of a modified $\sigma(t) = 2e^{-t}(-1, 0, 1)$ corrects the situation, and gives a fuzzy solution $u(t) = (-1, 0, 1)e^{-t}(1 + 2t)$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$, supporting the main claim of [9] that the nature of the fuzzy solution strongly depends on the choice of the forcing term.

We demonstrate below that while the crisp IVPs

$$u' = -u + \sigma(t), \quad u' - \sigma(t) = -u \quad \text{and} \quad u' + u = \sigma(t)$$
$$\text{with the initial condition } u(0) = u_0$$

are equivalent, the solutions of the corresponding FIVP's

$$u' = -u + \sigma(t), \qquad u(0) = u_0, \tag{3}$$

$$u' - \sigma(t) = -u, \qquad u(0) = u_0, \tag{4}$$

and

$$u' + u = \sigma(t), \qquad u(0) = u_0, \tag{5}$$

for $u_0 \in \mathbb{R}_{\mathcal{F}}$ and $\sigma : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, exhibit very different behaviors.

In this section, we begin with the inequivalent homogeneous FIVPs, under the notion of Hukuhara differentiability and then contrast their behavior with the behavior of the solutions of the corresponding nonhomogeneous FDEs.

Example 2.1. Consider the well studied FIVP

$$u' = -u,\ u(0) = (-1, 0, 1). \tag{6}$$

The solution of this problem is $u(t) = (-e^t, 0, e^t)$. Its graphical representation can be seen in Fig. 1 (in all the figures which appear in the present paper we have represented

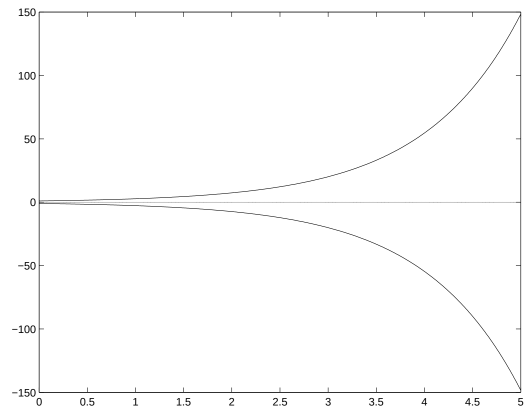

FIGURE 1. Solution of the fuzzy IVP (6)

the core and the endpoints of the support of the triangular number valued functions).

Now, if we consider the corresponding nonhomogeneous FIVP, (see [9] and [12]) one obtains a solution with a very different behavior.

In fact [12], considered the following problem.

$$u' + u = 2e^{-t}(-1,0,1), \qquad u(0) = (-1,0,1). \tag{7}$$

The solution is given by

$$u(t) = (-1,0,1)e^{-t}(1+2t) \tag{8}$$

$$= ((-2t-1)e^{-t}, 0, (2t+1)e^{-t}). \tag{9}$$

In this case,

$$u(t+h) = (-1,0,1)e^{-t-h}(1+2t+2h)$$

and so the H-difference

$$u(t+h) \ominus u(t) = [2he^{-h}e^{-t} + (2t+1)e^{-t}(e^{-h}-1)](-1,0,1)$$

exists (similarly $u(t) \ominus u(t-h)$ exists). Hence, u is a solution of (7) on $\left[0, \frac{1}{2}\right]$ (we consider it on this interval only, however it exists on $\left[-\frac{1}{2}, \frac{1}{2}\right]$).

We now demonstrate the behavior of the solution when we consider the FIVP in a different formulation.

$$u' = -u + 2e^{-t}(-1,0,1), \; u(0) = (-1,0,1). \tag{10}$$

We wish to find a triangular solution of the form $u = (x, y, z)$. (10) translates by Lemma 1.3 to

$$\begin{cases} x' = -z - 2e^{-t} \\ y' = -y \\ z' = -x + 2e^{-t} \end{cases}$$

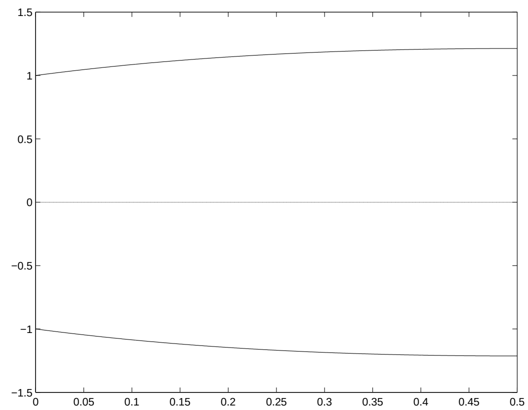

FIGURE 2. Solution (8) of the equation (7).

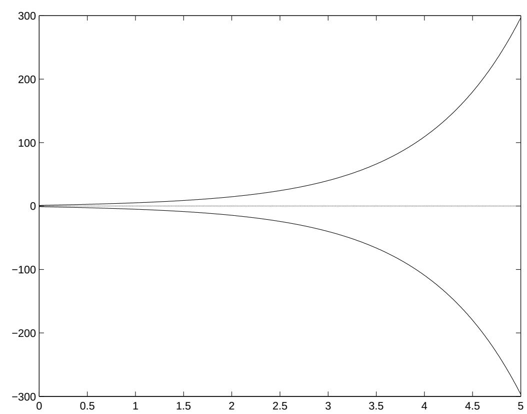

FIGURE 3. Solution (11) of the equation (10).

and we get the solution

$$u(t) = (e^{-t} - 2e^{t}, 0, 2e^{t} - e^{-t}), \qquad t \in (0, \infty). \tag{11}$$

Let us consider the last equation on the list of inequivalent FDEs obtained from equivalent crisp ODEs. i.e.,

$$u' - 2e^{-t}(-1, 0, 1) = -u, \ u(0) = (-1, 0, 1). \tag{12}$$

In this case, the equation translates to the system

$$\begin{cases} x' - 2e^{-t} = -z \\ y' = -y \\ z' + 2e^{-t} = -x \end{cases}$$

and we obtain $u(t) = (-e^{-t}, 0, e^{-t})$. But u is not H-differentiable since the H-differences $u(t+h) \ominus u(t)$ and $u(t) \ominus u(t-h)$ do not exist.

We observe that the solutions (8) and (11) of the equations (7) and (10) behave in quite different ways, as shown in Figures 2 and 3.

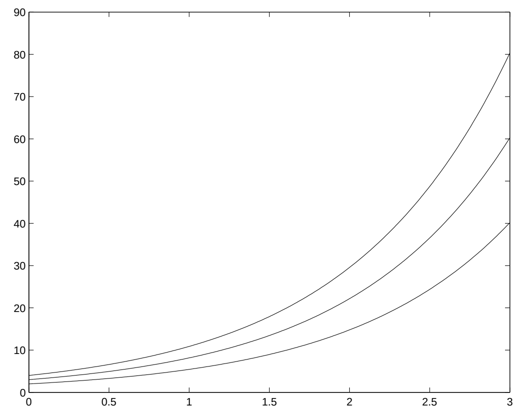

FIGURE 4. Solution of the homogeneous FDE (13)

We present another example, [3], in two different formulations and illustrate the situation.

Example 2.2. Consider the homogeneous FDE

$$u' = u,\ u(0) = (2,3,4). \tag{13}$$

It is easy to check that $u(t) = e^t(2,3,4)$ is Hukuhara differentiable solution of (13) over $[0,\infty)$. This solution is illustrated in Figure 4.

Consider the initial value problems

$$\begin{cases} u' = u + (1,2,3)t \\ u(0) = (2,3,4) \end{cases}, \tag{14}$$

$$\begin{cases} u' + (-1)(1,2,3)t = u \\ u(0) = (2,3,4), \end{cases} \tag{15}$$

and

$$\begin{cases} u' - u = (1,2,3)t \\ u(0) = (2,3,4) \end{cases}. \tag{16}$$

In this case, by the variation of constants formula provided in [3], we get that

$$u_1(t) = \left(3e^t - t - 1, 5e^t - 2t - 2, 7e^t - 3t - 3\right),$$

is Hukuhara differentiable and is a solution of (14) on $[0,\infty)$. Also,

$$u_2(t) = \left(5e^t - 3t - 3, 5e^t - 2t - 2, 5e^t - t - 1\right)$$

is Hukuhara differentiable and is a solution of (15) on $[0,\infty)$.

The solution of the equation (16) is obtained on $(\ln 2,\infty)$ under Hukuhara differentiability concept and is given by

$$u_3(t) = (5e^t - 3t - 2e^{-t} - 1, 5e^t - 2t - 2, 5e^t - t + 2e^{-t} - 3).$$

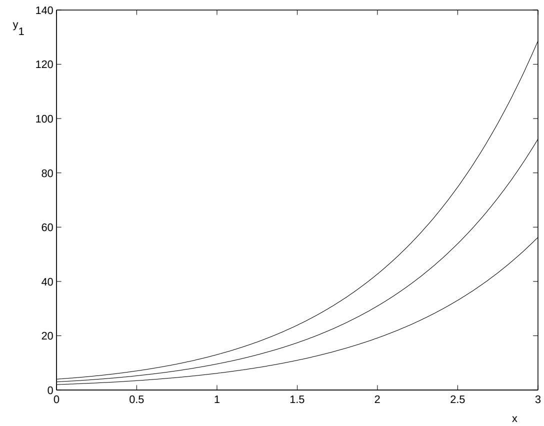

FIGURE 5. Graphical representation of the solution u_1 presented

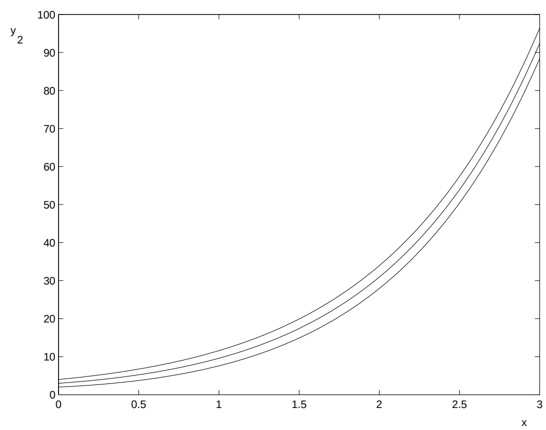

FIGURE 6. Graphical representation of the solution u_2 presented.

Since this is not a solution near the origin we do not consider it a proper solution of the problem (16).

The graphical representation of the solutions of (14) and (15) can be seen in figures 5 and 6 respectively. We observe that these solutions exhibit different behaviors. Indeed, the second solution is "relatively stable", i.e. the uncertainty is small compared with the core, for large values of t. Thus, while the solution of homogeneous problem is not relatively stable, by adding a forcing term and choosing a convenient formulation we obtain a relatively stable solution.

Let us remark here that if we have the full information about the correlation of the fuzzy variables, only one from the FIVPs (14), (15), (16) remains valid. This remark remains valid in the next sections as well.

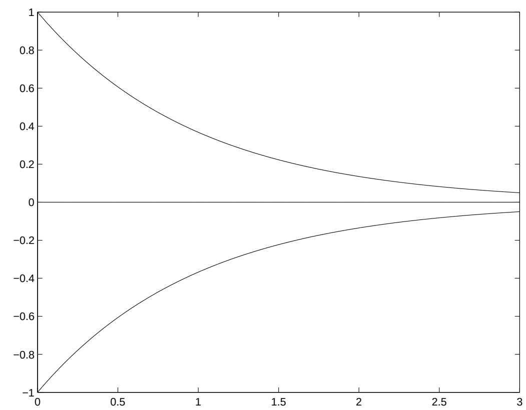

FIGURE 7. Solution of (6) under (ii)-differentiability

3. SOLUTIONS UNDER STRONGLY GENERALIZED DIFFERENTIABILITY

The recent study undertaken in [3] on linear FDEs further brings out the richness of the theory of FDEs.

In thi section we consider under the notion of strongly generalized differentiability the different formulations obtained by fuzzification of the crisp ODEs and demonstrate that the solutions exhibit dissimilar behaviors. We also demonstrate the effect of a forcing term on the behavior of the solutions.

Example 3.1. We revisit under generalized differentiability the IVPs studied in [9]. Since (i)-differentiability is in fact Hukuhara differentiability, the solution of the IVP (6) is same as before and is shown in figure 1. Under (ii)-differentiability notion we get the solution $u(t) = e^{-t}(-1, 0, 1)$ and is shown in figure 7.

We consider the three inequivalent FIVP (3), (4), (5), with the forcing term $\sigma(t) = e^{-t}(-1, 0, 1)$. With this $\sigma(t)$ for (3), under (i)-differentiability we get by Lemma 1.3

$$\begin{cases} x' = -z - e^{-t} \\ y' = -y \\ z' = -x + e^{-t} \end{cases}$$

and it follows that $u(t) = \left(\frac{1}{2}e^{-t} - \frac{3}{2}e^{t}, 0, \frac{3}{2}e^{t} - \frac{1}{2}e^{-t}\right)$. Since $u(t)$ is (i)-differentiable it is a solution of (3). Under (ii)-differentiability, (3) becomes

$$\begin{cases} z' = -z - e^{-t} \\ y' = -y \\ x' = -x + e^{-t}. \end{cases}$$

It is easy to check that $u(t) = e^{-t}(1-t)(-1, 0, 1)$ is (ii) differentiable on $(0, 1)$ and is a solution of (3). In figures 8 and 9 the (i)-differentiable and (ii)-differentiable solutions are presented.

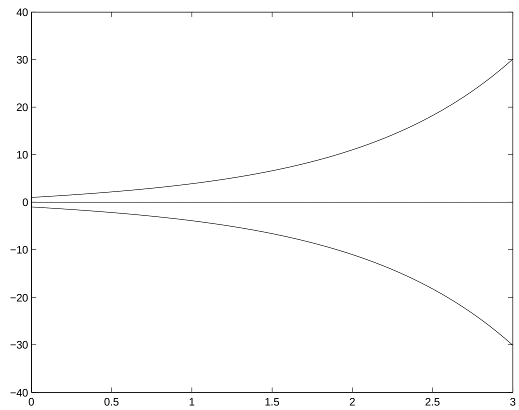

FIGURE 8. (i)-differentiable solution of equation (3)

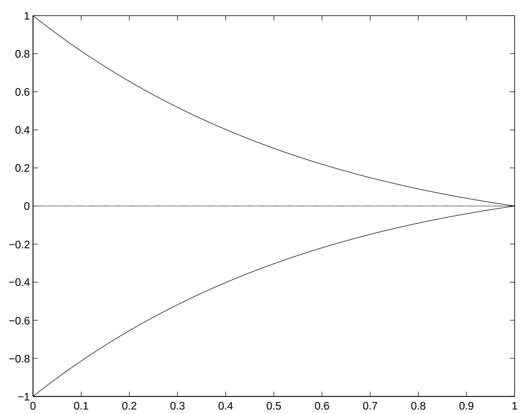

FIGURE 9. (ii)-differentiable solution of (3)

Now, consider (4). Under (i)-differentiability we get the system (17) having the (i)-differentiable solution $u(t) = (-\cosh t, 0, \cosh t)$ over $[0, \infty)$. Under (ii)-differentiability condition, again by Lemma 1.3 we get the system

$$\begin{cases} x' = -x - e^{-t} \\ y' = -y \\ z' = -z + e^{-t} \end{cases},$$

leading to the (ii)-differentiable solution $u(t) = e^{-t}(1+t)(-1, 0, 1)$. The solutions of the equation (5) are presented in figures 10 and 11.

Now, we consider the equation (5). If we assume that u is (i)-differentiable then we do not get any solution for $t > 0$, as shown in [12]. In fact, same is the case with the (ii)-differentiability. Thus, we get the system

$$\begin{cases} x' = -z + e^{-t} \\ y' = -y \\ z' = -x - e^{-t} \end{cases} \tag{17}$$

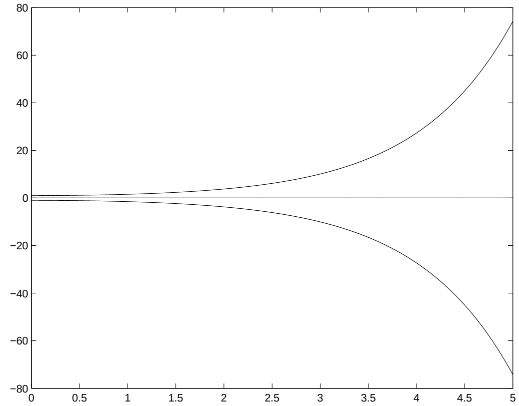

FIGURE 10. (i)-differentiable solution of (5)

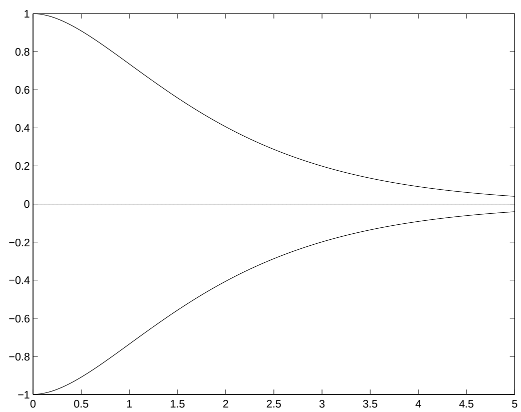

FIGURE 11. (ii)-differentiable solution of (5)

and we obtain $u(t) = (-\cosh t, 0, \cosh t) = (-1, 0, 1)\cosh t$. In this case, $u(t+h) = (-1, 0, 1)\cosh(t+h)$ and since cosh is increasing on $(0, \infty)$ we get that $u(t) \ominus u(t+h)$ does not exist. This contradicts the (ii)-differentiability assumption. So, again no solution is obtained.

So, in the case of generalized differentiability, for the inequivalent fuzzy versions (3), (4), (5) of equivalent crisp ODEs, we get four solutions, one of them being asymptotically stable. Also, we observe the influence of a forcing term on the behavior of the solutions of the FDEs under strongly generalized differentiability.

4. DIFFERENTIAL INCLUSIONS

The solutions of the fuzzy initial value problem

$$x'(t) = H(t, x(t)), \quad x(0) = x_0 \in \mathbb{R}_{\mathcal{F}} \tag{18}$$

where $H : \mathbb{R} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$, in the usual formulation, as we know, suffer from the disadvantage that the $\mathrm{diam}(x(t))$ is nondecreasing in t.

Considering the above FIVP, but with a $H : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_{\mathcal{F}}$, Hullermeier ([10]) proposed a different formulation of the FIVP based on a family of differential inclusions at each α level, $0 \le \alpha \le 1$,

$$y'_\alpha(t) \in [H(t, y_\alpha(t))]^\alpha \quad y_\alpha(0) \in [x_0]^\alpha. \tag{19}$$

Under suitable assumptions, it can be established that the solution sets of (19) are the level sets of a fuzzy set. However, this does not address the issue of solutions to the FIVP (18).

Here in this section, we wish to take a look at the different possible differential inclusion formulations of FIVPs, in the spirit of the earlier sections of this paper. We wish to investigate if any of the alternative formulations of the FIVP when considered with the corresponding version of the differential inclusion formulation overcome the known disadvantages associated with these problems.

Let us consider again the fuzzy differential equations

$$y' = -at + b, \tag{20}$$

$$y' + at = b, \tag{21}$$

and

$$y' - b = -at, \tag{22}$$

with the initial condition $y(0) = 0$.

In the original interpretation of Hullermeier the (21) and (22) are not included but in applications such situations may appear. So we will rewrite (21) such that the interpretation with the differential inclusions becomes possible. Indeed, we have

$$y' = b \ominus at,$$

provided that the H-difference $b \ominus at$ exists. It is easy to see that if b is triangular, for sufficiently small $t > 0$ the H-difference $b \ominus at$ exists (see [2]).

So, we get the following families of inequivalent differential inclusions:

$$y'_\alpha \in [-a]^\alpha t + [b]^\alpha$$

and

$$y'_\alpha \in [b \ominus at]^\alpha$$

with the same initial condition $y(0) = 0$.

Now let $a = (1, 2, 3)$ and $b = (2, 3, 4)$. So, for equation (20) we get

$$y'_\alpha \in [-a]^\alpha t + [b]^\alpha = [-3 + \alpha, -1 - \alpha]t + [2 + \alpha, 4 - \alpha].$$

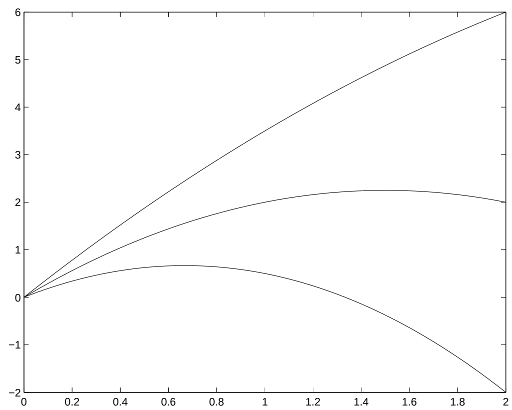

FIGURE 12. Solution with the interpretation using differential inclusions, given by (23)

We obtain,

$$y_\alpha \in \left[(-3+\alpha)\frac{t^2}{2} + (2+\alpha)t, (-1-\alpha)\frac{t^2}{2} + (4-\alpha)\,t\right].$$

that is

$$y = \left(-3\frac{t^2}{2} + 2t, -t^2 + 3t, -\frac{t^2}{2} + 4t\right), \tag{23}$$

$t \in (0, \infty)$.

For (21) we have

$$y'_\alpha \in [b \ominus at]^\alpha = [(2-t, 3-2t, 4-3t)]^\alpha .$$

That is

$$y'_\alpha \in [2 - t + \alpha - \alpha t, 4 - 3t - \alpha + \alpha t]$$

and finally we obtain

$$y = \left(2t - \frac{t^2}{2}, 3t - t^2, 4t - 3\frac{t^2}{2}\right) \tag{24}$$

which defines a fuzzy number for $t \in (0, 2)$. We thus have two solutions.

Under the interpretation that this is a motion with uncertain initial velocity and acceleration and that between these quantities there is no correlation, only the first interpretation remains valid.

But if we know from the physical interpretation that y is a positive fuzzy quantity, and that at should be a part of b such as if at is velocity of the decrease of something and we know in advance from the physical model that it cannot exceed the initial velocity b then the second interpretation in (21) is more appropriate. For example, if a and b are acceleration and velocity of a robot, and if we know that the robot is working well, then it should stop when the target position is reached.

Comparing the results in Figures 12 and 13. we get as in the previous sections,

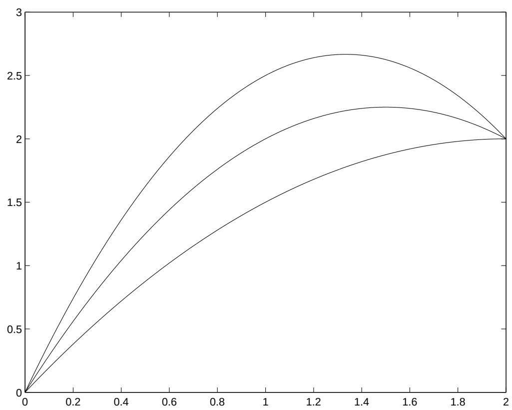

FIGURE 13. Solution under the interpretation with differential inclusions given by (24)

two solutions with quite different properties in the case of fuzzy differential inclusions too.

5. THE INTERPRETATION BASED ON ZADEH'S EXTENSION PRINCIPLE

Here the solution of the FIVP is generated by using the Zadeh's extension principle on the classical solution obtained by solving the crisp ODE corresponding to the fuzzy equation considered. Thus, different fuzzy formulations of the same ODE will lead to the same solution. So, this approach differs from the previous interpretations and uniqueness is ensured.

However, let us point out in the next simple example where the lack of knowledge about the correlation between variables can lead to some difficulties. This example could of course be considered in earlier approaches as well.

Consider the following FIVP

$$\begin{cases} y' = y - a \\ y(0) = y_0 \end{cases}$$

where $a = y_0 = (1, 2, 3) \in \mathbb{R}_{\mathcal{F}}$ are fuzzy numbers.

If the fact that $a = y_0$ is considered a simple coincidence then the solution of the ODE obtained symbolically from the problem writes

$$y(t) = y_0 e^t + a(1 - e^t).$$

Using Zadeh's extension principle we obtain the fuzzy solution

$$\widetilde{y}_1(t) = (4 - 3e^t, 4 - 2e^t, 4 - e^t).$$

If we a and y_0 represent fuzzy numbers modeling the same physical quantity, the corresponding classical ODE will be

$$\begin{cases} y' = y - a \\ y(0) = a \end{cases},$$

having its solution

$$y(t) = a.$$

This classical solution, by Zadeh's extension principle leads to the fuzzy solution

$$\widetilde{y}_2(t) = (1, 2, 3)$$

which exists for $t \in [0, \infty)$.

If an FDE corresponds to a real world phenomena, and the correlation between the fuzzy parameters ([7]) (for example, one of the parameters is a "part" of the other parameter) is known from the model then among the above-mentioned interpretations the conclusions based on only one of the approaches remain consistent with the real-world model. In that case one has to use the constrained fuzzy arithmetic ([13], [14], [17]). Unfortunately, enough information about the correlation between the fuzzy numbers is not ususally available and one has to resort to other options such as an expert's knowledge of the system under study to decide the best possible approach.

6. CONCLUSION

We studied different formulations of a FIVP and obtained solutions exhibiting different behaviors clearly indicating that the theory of FDEs is much richer than the theory of ODEs. Hence it should be pursued as an independent discipline.

Remark 6.1. One may have to choose a particular FDE formulation that better reflects the behavior of the situation at hand, in a given real life application. The fact that, various options are available is not a disadvantage. In fact, introducing uncertainties leads to various possible outcomes, and one may have to investigate whether there are other features of the physical problem that favor a particular FDE formulation. Our point is that this opens up several directions in which the FDE's may be studied not all of which are necessarily along the lines of the study of crisp ODEs. This makes it necessary to study fuzzy differential equations as an independent discipline, and exploring it further in different directions to facilitate its use in modelling entirely different physical and engineering problems satisfactorily. In this sense, the different approaches and different formulations are complimentary to each other.

Remark 6.2. From the comparison of all the above interpretations of a FIVPs it is easy to see that under any interpretation, if we have complete information about the correlation of the variables and fuzzy quantities that appear in the system, then one has to use the correct fuzzy arithmetic in order to eliminate the formulations that are inconsistent with the known correlation. Under interpretations based on Hukuhara differentiability and Zadeh's extension principle the uniqueness of the solution (if it exists) would be ensured. Whereas, with the notion of generalized differentiability one may obtain several solutions. Let us remark here that the generalized differentiability seems to be the most promising from the computational point of view, since the properties of the generalized differential are very near to those of its classical counterpart (see [2], [3]). Also, from Lemma 1.3, since a FIVP written in triangular numbers translates into a system of classical ODEs and we can compute easily the fuzzy solutions with the warning that we have to check always if the result of this procedure is indeed a fuzzy solution or not.

Remark 6.3. The knowledge of the correlations between the variables is rarely available in real-world problems. The multitude of solutions and interpretations of an FIVP then really turns into an advantage if we have some knowledge about the behavior of the solution based on the physical properties of the system. One can choose an appropriate formulation, for example, if we know that a solution is periodic or asymptotically stable, etc.

Moreover, the existence of several solutions allows us to integrate expert knowledge about the system with our information about the dynamics of the system, and so, we can choose the solutions that are consistent with the expert opinion. We propose for further research an alternative approach to modeling dynamical systems under uncertainty, in which we build our model by using fuzzy If-Then rules and FDEs. This allow us to take into account expert knowledge in a dynamical environment.

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