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On Asymptotic Stability of Solutions to Third Order Nonlinear Differential Equations with Retarded Argument

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Abstract: In this paper, we are concerned with the asymptotic stability of the trivial solution of third order nonlinear delay differential equations of the form

$$x'''(t)+\varphi(x(t),x'(t))x''(t)+\psi(x(t-r(t)),x'(t-r(t))+h(x(t-r(t)))=0\,.$$

By constructing a Lyapunov functional, we establish some new sufficient conditions which insure that the trivial solution of this equation is the asymptotically stable. In particular, an example is given to illustrate the importance of our result.

Keywords: Stability, Lyapunov functional, third order nonlinear differential equations with retarded argument.
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1. INTRODUCTION

It is well-known that the systems with aftereffect, with time lag or with delay are of great theoretical interest and form an important class as regards their applications. This class of systems is described by functional differential equations, which are also called differential equations with deviating arguments. Among functional differential equations one may distinguish some special classes of equations, retarded functional differential equations, neutral functional differential equations and advanced functional differential equations. In particular, retarded functional differential equations describe those systems or processes whose rate of change of state is determined by their past and present states. Such equations are frequently encountered as mathematical modes of most dynamical process in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Especially, since 1960s many good books, most of them are in Russian literature, have been published concerning to the delay differential equations (see for example the books of Burton ([1], [2]), Èl'sgol'ts [3], Èl'sgol'ts and Norkin [4], Gopalsamy [5], Hale [6], Hale and Verduyn Lunel [7], Kolmanovskii and Myshkis [8], Kolmanovskii and Nosov [9], Krasovskii [10] and Yoshizawa [20] and the references listed in these books). As it is also known, the investigation of qualitative properties of solutions, in particular, the stability of solutions is a very important problem in the theory and applications of the differential equations. The most efficient tool for the study of the stability of a given nonlinear system is provided by Lyapunov theory [11]. The Lyapunov's theory [11] is based on the use of positive functions that are non-increasing along the solutions of the considered differential system. But, finding an appropriate positive definite Lyapunov function is a difficult task for higher order nonlinear differential equations. However, up to now, the second method of Lyapunov [11] for asymptotic stability has been very successful when applied third order

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nonlinear differential equations satisfying the Routh-Hurwitz criteria. For the works achieved on third order nonlinear ordinary differential equations without delay one can refer to the book of Reissig et al. [13] as a survey and the papers of Tunç ([17], [18]) and the references citied in these sources. Since the use of Lyapunov's second method [11] for investigation of stability criteria of equations with delay encountered some principal difficulties, Krasovskii [10] achieved the use of functionals, which are now called Lyapunov functionals, defined on equations' trajectories instead of Lyapunov functions. It is worthy mentioning that, with respect to our observation, finding an appropriate positive definite Lyapunov functional for higher order nonlinear delay differential equations is a more difficult task than that of Lyapunov function for nonlinear differential equations without delay. At the same time, one can recognize that so far only a few significant theoretical results concerning stability of trivial solution of third order nonlinear differential equations with delay have been achieved, see for example the papers of Sadek [14], Sinha [15], Tejumola and Tchegnani [16], Tunç [19], Zhu [21] and the references listed in these papers. Meanwhile, it should be noted that, in 1969, Palusinski et al. [12] applied an energy metric algorithm for the generation of a Lyapunov function for third order ordinary nonlinear differential equation of the form:

$$x'''(t)+a_1x''(t)+f_2(x'(t))x'(t)+a_3x(t)=0\,.$$

They found some conditions for the stability of trivial solution of this equation as follows:

$$a_1>0\,,\ f_2(x')>a_3>0\,.$$

In this paper we consider the third order ordinary nonlinear delay differential equations of the type

$$x'''(t)+\varphi(x(t),x'(t))x''(t)+\psi(x(t-r(t)),x'(t-r(t))+h(x(t-r(t)))=0 \tag{1}$$

whose associated system is

$$\begin{aligned}
&x'(t)=y(t)\,,\ y'(t)=z(t)\,,\\
&z'(t)=-\varphi(x(t),y(t))z(t)-\psi(x(t),y(t))-h(x(t))+\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds\\
&\qquad+\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds+\int_{t-r(t)}^{t}h'(x(s))y(s)ds\,,
\end{aligned} \tag{2}$$

where r is a bounded delay, $0\le r(t)\le\gamma$, $r'(t)\le\sigma$, $0<\sigma<1$, γ and σ are some positive constants, γ will be determined later; the functions φ, ψ and h depend only on the arguments displayed explicitly and the primes in equation (1) denote differentiation with respect to t, $t\in[0,\infty)$. It is principally assumed that the functions φ, ψ and h are continuous for all values their respective arguments on $\Re^2$ and $\Re$, respectively. Besides, it is also supposed that $\psi(x,0)=h(0)=0$, and the derivatives $\varphi_x(x,y)\equiv\frac{\partial}{\partial x}\varphi(x,y)$ $\psi_x(x,y)\equiv\frac{\partial}{\partial x}\psi(x,y)$, $\psi_y(x,y)\equiv\frac{\partial}{\partial y}\psi(x,y)$ and $h'(x)\equiv\frac{dh}{dx}$ exist and are continuous; throughout the paper $x(t)$, $y(t)$ and $z(t)$ are, respectively, abbreviated as x, y and z. All solutions considered are also assumed to be real valued.

The motivation for the present work has been inspired basically by the paper of Palusinski et al. [12] and the papers mentioned above. Our aim here is to improve the results verified by Palusinski et al. [12] to nonlinear delay equation (1) for the asymptotic stability of trivial solution of this equation. We also give an explanatory example related to the asymptotic stability of the trivial solution of (1). All of the papers mentioned above were published without including an explanatory example on the stability of solutions of third order nonlinear differential equations with delay or without delay.

2. PRELIMINARIES

In order to reach the main result of this paper, we will give some important basic information for the general autonomous delay differential system. Now, we consider the general autonomous delay differential system

$$x' = f(x_t),\ x_t(\theta) = x(t+\theta),\ -r \le \theta \le 0,\ t \ge 0, \tag{3}$$

where $f : C_H \to \Re^n$ is a continuous mapping, $f(0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \Re^n$ with supremum norm, $r > 0$, C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r,0], \Re^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [1], shows that if $\phi \in C_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote the norm in $\Re^n$ with $|x| = \max_{1 \le i \le n} |x_i|$.

Definition 1. (See [1].) Let $f(0) = 0$. The zero solution of equation (3) is:

(a) stable if for each $t_1 \ge t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $[\ \|\phi\| \le \delta,\ t \ge t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$.

(b) asymptotically stable if it is stable and if for each $t_1 \ge t_0$ there is an η such that $\|\phi\| \le \eta$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

Definition 2. (See [1].) A continuous positive definite function $W : \Re^n \to [0, \infty)$ is called a wedge.

Definition 3. (See [1].) A continuous function $W : [0, \infty) \to [0, \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 4. (See [1].) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : D \to [0, \infty)$ is called

(a) positive definite if $V(0) = 0$ and if there is a wedge W_1 with $V(x) \ge W_1(|x|)$,

(b) decresent if there is a wedge W_2 with $V(x) \le W_2(|x|)$.

Definition 5. (See [1].) If V is a continuous scalar function in C_H, we define the derivative of V along the solutions of (3) by the following relation

$$\dot{V}_{(3)}(\phi) = \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi))}{h}.$$

Lemma. (See [15].) Suppose $f(0)=0$. Let V be a continuous functional defined on $C_H = C$ with $V(0)=0$, and let $u(s)$ be a function, non-negative and continuous for $0 \le s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0)=0$. If for all $\phi \in C$, $u(|\phi(0)|) \le V(\phi)$, $V(\phi) \ge 0$, $\dot{V}_{(3)}(\phi) \le 0$, then the solution $x_t = 0$ of (3) is stable.

If we define $Z = \{\phi \in C_H : \dot{V}_{(3)}(\phi) = 0\}$, then the solution $x_t = 0$ of (3) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3. MAIN RESULT

In this section we state and prove a theorem, which is our main result.

Theorem. In addition to the basic assumptions imposed on the functions φ, ψ and h that appeared in equation (1), we assume that there are positive constants a_1, a_2, a_3, λ, α, β, ε, μ, γ, L and M, such that the following conditions hold for all x, y and z:

(i) $a_1 a_2 - a_3 \ge \varepsilon > 0$.

(ii) $\varphi(x,y) \ge a_1 + 2\lambda$ and $y\varphi_x(x,y \le 0$.

(iii) $\psi(x,0)=0$, $\dfrac{\psi(x,y)}{y} \ge a_2 + 2\mu$, $(y \ne 0)$, $-L \le \psi_x(x,y) \le 0$ and $|\psi_y(x,y)| \le M$.

(iv) $h(0)=0$ and $0 < h'(x) \le a_3$.

Then, the trivial solution of equation (1) is asymptotically stable provided that

$$\gamma < \min\left\{ \frac{4\mu a_3}{2\alpha + a_3 L + a_3 M + a_3^2}, \frac{2\varepsilon + 4\lambda a_2}{a_2(L+M) + a_2 a_3 + 2\beta} \right\}.$$

Proof: To achieve the proof of the theorem, we define a new Lyapunov functional. $V = V(x_t, y_t, z_t)$. Namely, we impose some assumptions on Lyapunov functional V and its time derivative $\dfrac{d}{dt}V(x_t, y_t, z_t)$ which both imply the asymptotic stability of trivial solution of equation (1). We define our Lyapunov functional V as the following:

$$V(x_t, y_t, z_t) = a_3 \int_0^x h(\eta)d\eta + a_2 y h(x) + \frac{1}{2}a_2 z^2 + a_3 yz$$

$$+a_2\int_0^y \psi(x,\xi)d\xi + a_3\int_0^y \varphi(x,\xi)\xi d\xi$$

$$+\alpha\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \beta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds\,, \tag{4}$$

where a_2, a_3, α and β are some positive constants and the constants α and β will be determined later in the proof.

Making use of the assumptions $h(0)=0$ and $0<h'(x)\le a_3$, it follows that

$$h^2(x) = 2\int_0^x h(\eta)h'(\eta)d\eta \le 2a_3\int_0^x h(\eta)d\eta\,.$$

Hence

$$a_2 y h(x) \ge -\sqrt{2a_2^2 a_3\int_0^x h(\eta)d\eta}\;|y|\,.$$

Now, taking into consideration this last inequality, one can rearrange the Lyapunov functional $V=V(x_t,y_t,z_t)$, which is defined by (4), in the form:

$$V(x_t,y_t,z_t) = \frac{1}{2}a_2\left(z+\frac{a_3}{a_2}y\right)^2 + \frac{1}{2}\left(\sqrt{2a_3\int_0^x h(\eta)d\eta} - a_2|y|\right)^2$$

$$+\int_0^y\left[a_3\varphi(x,\xi) - a_2^2 - \frac{a_3^2}{a_2} + a_2\frac{\psi(x,\xi)}{\xi}\right]\xi d\xi$$

$$+\alpha\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \beta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds\,. \tag{5}$$

The assumptions $\varphi(x,y)\ge a_1+2\lambda$ and $\dfrac{\psi(x,y)}{y}\ge a_2+2\mu$ imply that

$$\int_0^y\left[a_3\varphi(x,\xi) - a_2^2 - \frac{a_3^2}{a_2} + a_2\frac{\psi(x,\xi)}{\xi}\right]\xi d\xi \ge \int_0^y\left[a_1a_3 - \frac{a_3^2}{a_2} + 2(a_3\lambda + a_2\mu)\right]\xi d\xi$$

$$=\left(\frac{a_1a_2a_3 - a_3^2 + 2a_2(a_3\lambda + a_2\mu)}{2a_2}\right)y^2 > 0\,. \tag{6}$$

By (5) and (6) we observe that

$$V(x_t,y_t,z_t) \ge \frac{1}{2}a_2\left(z+\frac{a_3}{a_2}y\right)^2 + \frac{1}{2}\left(\sqrt{2a_3\int_0^x h(\eta)d\eta} - a_2|y|\right)^2$$

$$+\left(\frac{a_1a_2a_3 - a_3^2 + 2a_2(a_3\lambda + a_2\mu)}{2a_2}\right)y^2$$

$$+\alpha\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds+\beta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds.$$

Note that one may show from the terms of this inequality that there exist sufficiently small positive constants $D_i, (i=1,2,3)$, such that

$$V(x_t,y_t,z_t)\geq D_1x^2+D_2y^2+D_3z^2+\alpha\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds+\beta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds. \quad (7)$$

Therefore, subject to the above discussion, the existence of a continuous function $u(|\phi(0)|)$ with $u(|\phi(0)|)\geq 0$, which satisfies the inequality $u(|\phi(0)|)\leq V(\phi)$, can be easily verified, since the integrals $\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ and $\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative.

Now, calculating the time derivative of the functional $V(x_t,y_t,z_t)$ in (4) along the system (2), we have

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t)=&-\left(a_3\frac{\psi(x,y)}{y}-a_2h'(x)\right)y^2-\left(a_2\varphi(x,y)-a_3\right)z^2\\
&+a_3y\int_0^y\varphi_x(x,\xi)\xi d\xi+a_2y\int_0^y\psi_x(x,\xi)d\xi\\
&+a_3y\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds+a_3y\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds\\
&+a_2z\int_{t-r(t)}^{t}\psi_x(x(s),y(s))y(s)ds+a_2z\int_{t-r(t)}^{t}\psi_y(x(s),y(s))z(s)ds\\
&+a_3y\int_{t-r(t)}^{t}h'(x(s))y(s)ds+a_2z\int_{t-r(t)}^{t}h'(x(s))y(s)ds+\alpha r(t)y^2\\
&-\alpha(1-r'(t))\int_{t-r(t)}^{t}y^2(s)ds+\beta r(t)z^2-\beta(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds. \quad (8)\end{aligned}$$

Employing the assumptions $\varphi(x,y)\geq a_1+2\lambda$, $y\varphi_x(x,y\leq 0$, $\frac{\psi(x,y)}{y}\geq a_2+2\mu$, $-L\leq\psi_x(x,y)\leq 0$, $|\psi_y(x,y)|\leq M$, $0<h'(x)\leq a_3$, $0\leq r(t)\leq\gamma$, $r'(t)\leq\sigma$ and the inequality $2|ab|\leq a^2+b^2$, we obtain the following inequalities for all terms contained in (8):

$$-\left(a_2\varphi(x,y)-a_3\right)z^2\leq-\left(a_1a_2-a_3\right)z^2-2\lambda a_2z^2\leq-(\varepsilon+2\lambda a_2)z^2,$$

$$-\left(a_3\frac{\psi(x,y)}{y}-a_2h'(x)\right)y^2\leq-2\mu a_3y^2,$$

$$a_3 y\int_0^y \varphi_x(x,\xi)\xi d\xi \le 0\,,\ a_2 y\int_0^y \psi_x(x,\xi)d\xi \le 0\,,$$

$$a_3 y\int_{t-r(t)}^t \psi_x(x(s),y(s))y(s)ds \le \frac{a_3 L}{2}\,r(t)y^2(t)+\frac{a_3 L}{2}\int_{t-r(t)}^t y^2(s)ds$$

$$\le \frac{a_3\gamma L}{2}\,y^2(t)+\frac{a_3 L}{2}\int_{t-r(t)}^t y^2(s)ds\,,$$

$$a_3 y\int_{t-r(t)}^t \psi_y(x(s),y(s))z(s)ds \le \frac{a_3 M}{2}\,r(t)y^2(t)+\frac{a_3 M}{2}\int_{t-r(t)}^t z^2(s)ds$$

$$\le \frac{a_3\gamma M}{2}\,y^2(t)+\frac{a_3 M}{2}\int_{t-r(t)}^t z^2(s)ds\,,$$

$$a_2 z\int_{t-r(t)}^t \psi_x(x(s),y(s))y(s)ds \le \frac{a_2 L}{2}\,r(t)z^2(t)+\frac{a_2 L}{2}\int_{t-r(t)}^t y^2(s)ds$$

$$\le \frac{a_2\gamma L}{2}\,z^2(t)+\frac{a_2 L}{2}\int_{t-r(t)}^t y^2(s)ds\,,$$

$$a_2 z\int_{t-r(t)}^t \psi_y(x(s),y(s))z(s)ds \le \frac{a_2 M}{2}\,r(t)z^2(t)+\frac{a_2 M}{2}\int_{t-r(t)}^t z^2(s)ds$$

$$\le \frac{a_2\gamma M}{2}\,z^2(t)+\frac{a_2 M}{2}\int_{t-r(t)}^t z^2(s)ds\,,$$

$$a_3 y\int_{t-r(t)}^t h'(x(s))y(s)ds \le \frac{a_3^2}{2}\,r(t)y^2(t)+\frac{a_3^2}{2}\int_{t-r(t)}^t y^2(s)ds$$

$$\le \frac{\gamma a_3^2}{2}\,y^2(t)+\frac{a_3^2}{2}\int_{t-r(t)}^t y^2(s)ds\,,$$

$$a_2 z\int_{t-r(t)}^t h'(x(s))y(s)ds \le \frac{a_2 a_3}{2}\,r(t)z^2(t)+\frac{a_2 a_3}{2}\int_{t-r(t)}^t y^2(s)ds$$

$$\le \frac{\gamma a_2 a_3}{2}\,z^2(t)+\frac{a_2 a_3}{2}\int_{t-r(t)}^t y^2(s)ds\,,$$

$$\alpha r(t)y^2 - \alpha(1-r'(t))\int_{t-r(t)}^t y^2(s)ds \le \alpha\gamma y^2 - \alpha(1-\sigma)\int_{t-r(t)}^t y^2(s)ds\,,$$

$$\beta r(t)z^2 - \beta(1-r'(t))\int_{t-r(t)}^t z^2(s)ds \le \beta\gamma z^2 - \beta(1-\sigma)\int_{t-r(t)}^t z^2(s)ds\,.$$

Summing up these inequalities into (8), we get

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -\left[(2\mu a_3) - \left(\frac{2\alpha + a_3L + a_3M + a_3^2}{2}\right)\gamma\right]y^2$$
$$-\left[(\varepsilon + 2\lambda a_2) - \left(\frac{a_2L + a_2M + a_2a_3 + 2\beta}{2}\right)\gamma\right]z^2$$
$$+\frac{1}{2}\left[\left(a_2L + a_3L + a_3^2 + a_2a_3\right) - 2\alpha(1-\sigma)\right]\int_{t-r(t)}^{t} y^2(s)ds$$
$$+\frac{1}{2}\left[(a_2 + a_3)M - 2\beta(1-\sigma)\right]\int_{t-r(t)}^{t} z^2(s)ds. \tag{9}$$

By choosing $\alpha = \dfrac{(a_2 + a_3)L + a_3^2 + a_2a_3}{2(1-\sigma)}$ and $\beta = \dfrac{(a_2 + a_3)M}{2(1-\sigma)}$ in (9), we have

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -\left[(2\mu a_3) - \left(\frac{2\alpha + a_3L + a_3M + a_3^2}{2}\right)\gamma\right]y^2$$
$$-\left[(\varepsilon + 2\lambda a_2) - \left(\frac{a_2L + a_2M + a_2a_3 + 2\beta}{2}\right)\gamma\right]z^2. \tag{10}$$

Clearly, it follows from (10) for some positive constants k_1 and k_2 that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -k_1y^2 - k_2z^2 \le 0.$$

provided

$$\gamma < \min\left\{\frac{4\mu a_3}{2\alpha + a_3L + a_3M + a_3^2}, \frac{2\varepsilon + 4\lambda a_2}{a_2(L+M) + a_2a_3 + 2\beta}\right\}.$$

It is also obvious that the largest invariant set in Z is $Q = \{0\}$, where $Z = \{\phi \in C_H : \dot{V}(\phi) = 0\}$. Namely, the only solution of equation (1) for which $\frac{d}{dt}V(x_t, y_t, z_t) = 0$ is the solution $x_t \equiv 0$. Thus, under the above discussion, one can say that the trivial solution of equation (1) is asymptotically stable. The proof of the theorem is now complete.

Example: We consider the following third order nonlinear delay differential equation

$$x'''(t) + \left(4 + \frac{1}{1 + (x'(t))^2}\right)x''(t) + 4x'(t - r(t)) + \sin x'(t - r(t)) + 2arctgx(t - r(t)) = 0, \tag{11}$$

whose associated system is

$$x'(t)= y(t),\ y'(t)= z(t),$$

$$z'(t) = -\left(4+\frac{1}{1+y^2}\right)z(t) - (4y(t)+\sin y(t)) - 2arctgx(t)$$

$$+2\int_{t-r(t)}^{t}\frac{1}{1+(x(s))^2}y(s)ds + \int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds, \tag{12}$$

where $0 \le r(t) \le \gamma$, $r'(t) \le \sigma$, $0<\sigma<1$, γ and σ are some positive constants, γ will be determined later.
Now, it is clear that

$$\varphi(y) = 4+\frac{1}{1+y^2} \ge 4,$$

$$\psi(y) = 4y(t)+\sin y(t),\ \psi(0)=0,\ \frac{\psi(y)}{y} = 4+\frac{Siny}{y},\ (y \ne 0\ ,\ |y|<\pi),$$

$$4+\frac{\sin y(t)}{y(t)} \ge 3,\ h(x)=2arctgx,\ h(0)=0,\ h'(x)=\frac{2}{1+x^2}$$

and

$$0<h'(x) \le 2.$$

We introduce the following Lyapunov functional

$$V_1(x_t,y_t,z_t) = 2a_3\int_0^x arctg\eta d\eta + 2a_2 yarctgx + \frac{1}{2}a_2z^2 + a_3yz$$

$$+2a_2y^2 + a_2(1-\cos y) + 2a_3y^2 + \frac{a_3}{2}\ln(1+y^2)$$

$$+\alpha\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \beta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds. \tag{13}$$

It may be observed that the functional $V_1(x_t,y_t,z_t)$ is a special case of the functional $V(x_t,y_t,z_t)$ in (4). By an elementary calculation, one can show that there exist sufficiently small positive constants D_i, $(i=4,5,6)$, such that

$$V_1(x_t,y_t,z_t) \ge D_4x^2 + D_5y^2 + D_6z^2 + \beta\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds,$$

and hence $V_1(\phi) \ge u(|\phi(0)|) \ge 0$.

Now, calculating the time derivative of the functional $V_1(x_t,y_t,z_t)$ along the system (12), we obtain

$$\frac{d}{dt}V_1(x_t,y_t,z_t) = -\left(4a_3 + a_3\frac{\sin y}{y} - \frac{2a_2}{1+x^2} - \alpha r(t)\right)y^2$$

$$-\left(4a_2+\frac{a_2}{1+y^2}-a_3-\beta r(t)\right)z^2$$

$$+a_2 z\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds+a_3 y\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds$$

$$+2a_2 z\int_{t-r(t)}^{t}\frac{1}{1+x^2(s)}y(s)ds+2a_3 y\int_{t-r(t)}^{t}\frac{1}{1+x^2(s)}y(s)ds$$

$$-\alpha(1-r'(t))\int_{t-r(t)}^{t}y^2(s)ds-\beta(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds. \qquad \textbf{(14)}$$

Making use of the facts $0\le r(t)\le\gamma$, $r'(t)\le\sigma$, $0<\sigma<1$, $|4+\cos y|\le 5$, $\left|\frac{\sin y}{y}\right|\le 1, \frac{1}{1+x^2}\le 1$ and the inequality $2|ab|\le a^2+b^2$, we obtain the following inequalities for all terms included in (14):

$$-\left(4a_3+a_3\frac{\sin y}{y}-\frac{2a_2}{1+x^2}-\alpha r(t)\right)y^2\le-(3a_3-2a_2-\alpha\gamma)y^2,$$

$$-\left(4a_2+\frac{a_2}{1+y^2}-a_3-\beta r(t)\right)z^2\le-(4a_2-a_3-\beta\gamma)z^2,$$

$$a_2 z\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds\le\frac{5a_2}{2}r(t)z^2(t)+\frac{5a_2}{2}\int_{t-r(t)}^{t}z^2(s)ds$$

$$\le\frac{5a_2\gamma}{2}z^2(t)+\frac{5a_2}{2}\int_{t-r(t)}^{t}z^2(s)ds,$$

$$a_3 y\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds\le\frac{5a_3}{2}r(t)y^2(t)+\frac{5a_3}{2}\int_{t-r(t)}^{t}z^2(s)ds$$

$$\le\frac{5a_3\gamma}{2}y^2(t)+\frac{5a_3}{2}\int_{t-r(t)}^{t}z^2(s)ds,$$

$$2a_2 z\int_{t-r(t)}^{t}\frac{1}{1+x^2(s)}y(s)ds\le a_2 r(t)z^2(t)+a_2\int_{t-r(t)}^{t}y^2(s)ds$$

$$\le a_2\gamma z^2(t)+a_2\int_{t-r(t)}^{t}y^2(s)ds,$$

$$2a_3 y\int_{t-r(t)}^{t}\frac{1}{1+x^2(s)}y(s)ds\le a_3 r(t)y^2(t)+a_3\int_{t-r(t)}^{t}y^2(s)ds,$$

$$\le a_3 \gamma y^2(t) + a_3 \int_{t-r(t)}^{t} y^2(s)ds,$$

$$-\alpha(1-r'(t)) \int_{t-r(t)}^{t} y^2(s)ds \le -\alpha(1-\sigma) \int_{t-r(t)}^{t} y^2(s)ds,$$

$$-\beta(1-r'(t)) \int_{t-r(t)}^{t} z^2(s)ds \le -\beta(1-\sigma) \int_{t-r(t)}^{t} z^2(s)ds.$$

On gathering the above whole discussion into (14), we have

$$\begin{aligned}\frac{d}{dt}V_1(x_t, y_t, z_t) \le &-\left(3a_3 - 2a_2 - \left(\alpha + \frac{7a_3}{2}\right)\gamma\right)y^2 \\ &-\left(4a_2 - a_3 - \left(\beta + \frac{7a_2}{2}\right)\gamma\right)z^2 \\ &-(\alpha(1-\sigma) - (a_2 + a_3)) \int_{t-r(t)}^{t} y^2(s)ds. \\ &-\left(\beta(1-\sigma) - \frac{5}{2}(a_2 + a_3)\right) \int_{t-r(t)}^{t} z^2(s)ds. \end{aligned} \tag{15}$$

Let us choose $\alpha = \dfrac{a_2 + a_3}{1-\sigma}$ and $\beta = \dfrac{5(a_2 + a_3)}{2-2\sigma}$. Then, it follows from (15) that

$$\begin{aligned}\frac{d}{dt}V_1(x_t, y_t, z_t) \le &-\left(3a_3 - 2a_2 - \left(\alpha + \frac{7a_3}{2}\right)\gamma\right)y^2 \\ &-\left(4a_2 - a_3 - \left(\beta + \frac{7a_2}{2}\right)\gamma\right)z^2.\end{aligned}$$

Now, taking into account equation (11), we can choose $a_1 = 3$, $a_2 = 2$ and $a_3 = 2$. On the other hand, it is easy to check that $\alpha = \dfrac{4}{1-\sigma} > 0$, $\beta = \dfrac{10}{1-\sigma} > 0$, $a_1 a_2 - a_3 = 4 > 0$ and

$$\frac{d}{dt}V_1(x_t, y_t, z_t) \le -\left(2 - \left(\frac{4}{1-\sigma} + 7\right)\gamma\right)y^2 - \left(6 - \left(\frac{17-7\sigma}{1-\sigma}\right)\gamma\right)z^2. \tag{16}$$

Consequently, in view of (16), it follows for some positive constants k_3 and k_4 that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -k_3 y^2 - k_4 z^2$$

provided

$$\gamma < \min\left\{\frac{2-2\sigma}{11-7\sigma}, \frac{6-6\sigma}{17-7\sigma}\right\}.$$

The rest of the proof is the same as in the above theorem, and hence is omitted.

This shows that the trivial solution of equation (11) is asymptotically stable.

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