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CONFORMAL EVOLUTION OF SPACETIME SOLUTIONS OF EINSTEIN'S EQUATIONS

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ABSTRACT: In this paper we study a spacially compact spacetime (M, g) evolved through a conformal Killing vector (CKV) field ξ such that: (a) the normal component of ξ is constant on each spacelike slice Σ and each Σ has constant mean curvature; (b) the stress energy tensor obeys the mixed energy condition; (c) the conformal scalar function is non-decreasing along the evolution CKV field ξ. We prove that: (i) ξ is homothetic and orthogonal to Σ: (ii) Σ is hyperbolic and totally umbilical in M; and (iii) M is a vacuum spacetime. We also discuss a physically important case of Killing horizon when ξ is a null Killing vector field and Σ degenerates to a null hypersurface.

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1. INTRODUCTION

Solutions of the highly nonlinear Einstein's field equations requires the assumption that they admit Killing or homothetic vector fields. Even though these solutions provide significant clues and insights into astrophysical and cosmological questions, it would be interesting to analyze solutions with weaker symmetries, such as a conformal Killing vector (CKV). Robertson-Walker spacetimes admit a G_6 of Killing vectors and a G_9 of CKVs, see Maartens and Maharaj [7]. A CKV preserves the causal character of the spacetime but does not preserve the Einstein tensor, and hence is not a natural symmetry. In spite of this, many solutions with a CKV are known (see Duggal and Sharma [4]).

Considering a solution of Einstein's equations as the time evolution of an initial spacelike hypersurface has proved useful in quantization of gravity, gravitational initial value problem, the computer evolution of colliding black holes and collapsing

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stars. Using this formalism Eardley et al [6] proved the following result "Let (M, g) be a globally hyperbolic spacetime which: (a) satisfies the Einstein's equations for a stress tensor T obeying the mixed and dominant energy conditions; (b) admits a homothetic vector field ξ of g; and (c) admits a compact hypersurface of constant mean curvature. Then either (M, g) is an expanding vacuum hyperbolic model or ξ is Killing". Intrigued by Sharma [8] obtained the following result: "Let (M, g) be a spacetime solution of Einstein's equations admitting a CKV field ξ and be evolved out of a complete spacelike hypersurface Σ such that: (a) Σ is totally umbilical in M; (b) the normal component of ξ is non-constant on Σ; and (c) the normal sectional curvature of M is independent of the tangential direction at each point of Σ. Then Σ is conformally diffeomorphic to (i) a 3-sphere, or (ii) Euclidean space E^3, or (iii) hyperbolic space H^3, or (iv) the Riemannian product of a complete 2-dimensional manifold and an open real interval. If Σ is compact, then only (i) holds." Also see some more results in Duggal and Sharma [5].

In this paper we examine the case when the normal component of the CKV ξ is constant on Σ. We also obtain a condition for ξ to be null when Σ degenerates to a null hypersurface as a Killing horizon.

2. PRELIMINARIES

We use the $3+1$ splitting (see Arnowitt et al [1]) of the spacetime manifold (M, g). This assumes a thin sandwich of (M, g) evolved from a spacelike hypersurface Σ_t at a coordinate time t to another spacelike hypersurface Σ_{t+dt} at coordinate time $t + dt$ with the metric g given by

$$ds^2 = -\lambda^2 dt^2 + g_{ab}(dx^a + S^a dt)(dx^b + S^b dt)\,, \tag{1}$$

where $x^0 = t$,and x^a are three spatial coordinates; λ is the lapse function and $S^a\partial/\partial x^a$ is shift vector. For brevity we denote Σ_t by Σ. Let $N = -\lambda\bar{\nabla}t$ be the unit future pointing normal vector to Σ, where $\bar{\nabla}$ is the spacetime covariant derivative operator. Denote arbitrary vector fields tangent to Σ by X, Y, Z, W. Let K denote the shape operator of Σ defined by the Weingarten formula $KX = \bar{\nabla}_X N$. If $i : \Sigma \to M$ is the embedding map, then the pull-back $\gamma = i^* g$ is the 3-metric induced on Σ. The pair (γ, K) constitutes the initial data for the evolution. Denote the traceless part of K by L, i.e. $L = K - \frac{\tau}{3}I$, where $\tau = \mathrm{Tr}K$, and I is the identity tensor. Using the Gauss-Codazzi-Mainardi equations and Einstein's field equations

$$\bar{Ric} - \frac{\bar{r}}{2}\bar{g} = 8\pi T \tag{2}$$

one obtains the constraint equations

$$r + \frac{2\tau^2}{3} - |L|^2 = 16\pi T_{nn} \quad \text{and} \quad (\mathrm{div}L)X - \frac{2}{3}X\tau = 8\pi T(X, N), \tag{3}$$

where $\bar{Ric}$ and $\bar{r}$ are the Ricci tensor and scalar curvature of g, and r is the scalar curvature of the 3-metric γ. Also, T is the stress-energy tensor and $T_{nn} = T(N, N)$. Assume that (M, g) admits a CKV field ξ, i.e. satisfies

$$\pounds_{\xi} g = \sigma g \tag{4}$$

for a smooth conformal scalar function σ. Decompose ξ orthogonally as $\rho N + V$, where V is the tangential part of ξ. Following Berger [2], set the evolution vector field $\partial/\partial t$ equal to the CKV field ξ so that $\lambda = \rho$. As derived in Eardley et al [6], Sharma [8], we write the following conformal evolution equations

$$(\pounds_V \gamma)(X, Y) = \sigma\gamma(X, Y) - 2\rho\gamma(LX, Y) - 2\frac{\rho\tau}{3}\gamma(X, Y), \tag{5}$$

$$\begin{aligned}(\pounds_V L)X &= -(\nabla_X D\rho - \frac{\nabla^2\rho}{3}X) - 8\pi\rho(TX - \frac{T^m_m}{3}X)\\ &\quad +(\rho\tau - \frac{\sigma}{2})LX + \rho(QX - \frac{r}{3}X),\end{aligned} \tag{6}$$

$$\pounds_V \tau = \frac{3N\sigma}{2} - \frac{\sigma\tau}{2} - \nabla^2\rho + \rho[\frac{\tau^2}{3} + |L|^2 + 4\pi(T^m_m + T_{nn})]. \tag{7}$$

3. RESULTS

Recall that a stress energy tensor T is said to obey mixed energy condition (e.g. Eardley et al [6]) if at any point x on Σ, the strong energy condition $T_{nn} + T^m_m|_x \geq 0$ holds and equality implies vanishing of all components of T at x.

Theorem 1. *Let (M, g) be a spatially compact spacetime evolved by a CKV field ξ whose normal component ρ is constant on each spacelike slice Σ ($t =$ a constant) and each Σ has constant mean curvature. Then:*

(a) *ρ satisfies the differential equation*

$$\frac{d^2}{dt^2}(\ln\rho) + \frac{\rho^2}{3}[|L|^2 + 4\pi(T^m_m + T_{nn})] = 0 \quad \textit{and} \quad \sigma = 2\frac{d\ln\rho}{dt}. \tag{8}$$

(b) *Moreover, if (1) the stress energy tensor obeys the mixed energy condition, (2) the conformal scale function is non-decreasing along ξ, then: (i) ξ is homothetic and orthogonal to Σ; (ii) Σ is hyperbolic and totally umbilical in (M, g); and (iii) (M, g) is a vacuum spacetime.*

Proof. As shown in Berger [2], the decomposition of the conformal Killing equation (4) into tangential-tangential, tangential-normal and normal-normal components yields equation (5) and the following equations

$$\bar{\nabla}_N V = KV + D\rho - \rho D\ln\lambda + (V\ln\lambda)N, \tag{9}$$

$$\sigma = 2V\ln\lambda + 2N\rho. \tag{10}$$

By hypothesis, $\lambda = \rho$ are constants on each Σ. So (10) reduces to

$$\sigma = 2N\rho \tag{11}$$

and (9) reduces to $\bar{\nabla}_N V = KV$. Hence $\pounds_\xi N = -\frac{\sigma}{2}N$. Operating (10) by $\bar{\nabla}_N$ and using the commutation relation (see Yano [9], p. 39)

$$\pounds_\xi \bar{\nabla}_N N - \bar{\nabla}_N \pounds_\xi N - \bar{\nabla}_{[\xi,N]} N = (\pounds_\xi \bar{\nabla})(N,N)\,,$$

we get $\sigma\bar{\nabla}_N N = 2(N\sigma)N + \bar{D}\sigma$. Taking its inner product with N shows that $\bar{D}\sigma = -(N\sigma)N$, i.e. σ is a function of only t. Now the γ-trace of (5) is $2\mathrm{div}V = 3\sigma - 2\rho\tau$. As Σ is compact, using divergence theorem in this equation and noting that τ is constant on Σ we have

$$3\sigma = 2\rho\tau. \tag{12}$$

In view of (12) and (11) we obtain $\tau = 3N \ln \rho$. Using equation (7) and noting that $\partial t = \rho N + V$ we obtain (8) which proves (a).

In view of (11), equation (8) assumes the form

$$\frac{d\sigma}{dt} + \frac{2\rho^2}{3}[|L|^2 + 4\pi(T^m_m + T_{nn})] = 0\,.$$

By hypothesis, the first term of the above equation is non-negative which implies that σ is constant on M, i.e. ξ is homothetic, and since T obeys mixed energy condition we also conclude that $L = 0$, i.e. Σ is totally umbilical and $T = 0$ on each Σ, i.e. (M,g) is a vacuum spacetime. Consequently, first equation of (3) reduces to $r = -2\frac{\tau^2}{3}$. Also equation (6) reduces to

$$Ric = -\frac{2\tau^2}{9}\gamma. \tag{13}$$

As (Σ,γ) is a 3-dimensional Riemannian space, its Weyl conformal tensor is zero. Hence (13) implies that γ has constant negative curvature. Only the standard hyperbolic metrics satisfy this condition, and they are specified by the choice of global topology and the choice of a single scale factor. Furthermore we see from (5) and (12) that V is a Killing vector on (Σ,γ). However we know from Yano [9], Theorem 6.1, p. 46, that a compact Riemannian manifold with negative definite Ricci tensor can not admit a CKV field other than the zero vector field. Thus, $V = 0$ so homothetic vector field ξ is orthogonal to Σ, which proves (b) and the proof is complete. □

Proposition 1. (see Duggal and Sharma [5]) *A null CKV field on a spacetime (M,g) is a geodesic vector field.*

Now we consider the case when the evolution vector field ξ is a null CKV field on (M,g). First we prove the following result.

Theorem 2. *Let (M,g) be a spacetime evolved through a 1-parameter family of spacelike hypersurfaces Σ_t such that the evolution vector field ξ is a null CKV field on (M,g). Then ξ reduces to a Killing vector field if and only if the part of ξ tangential* *to Σ_t is asymptotic everywhere on Σ_t for all t. Moreover, ξ is a geodesic vector field.*

Proof. First we write equation (5) as

$$(\pounds_V \gamma)(X,Y) = \sigma\gamma(X,Y) - 2\rho\gamma(KX,Y) \tag{14}$$

for any vector fields tangent to Σ_t. As ξ is null, we have

$$\gamma(V,V) = \rho^2, \tag{15}$$

which gives $\gamma(\nabla_X V, V) = \rho X\rho$. Substituting V for Y in (14) we get

$$\nabla_V V + \rho D\rho = \sigma V - 2\rho KV, \tag{16}$$

where D is the gradient operator of the 3-metric γ. Taking inner product of (9) with V gives $g(\bar{\nabla}_N V, V) + \rho V \ln\lambda - \gamma(KV,V) - V\rho = 0$, which in view of (15) assumes the form

$$\rho N\rho + \rho V \ln\lambda - \gamma(KV,V) - V\rho = 0. \tag{17}$$

As $\gamma(\nabla_V V, V) = \rho V\rho$, taking inner product of (16) with V yields

$$\rho\gamma(KV,V) = \frac{\sigma}{2}|V|^2 - \rho V\rho. \tag{18}$$

As ξ is the evolution vector field, we have $\lambda = \rho$, and hence $\rho > 0$. The use of (18) in (17) gives $\rho N\rho = V\rho + \frac{\sigma}{2\rho}|V|^2$. Using this last equation and (18) we get $\rho N\rho = \frac{\sigma}{\rho}|V|^2 - \gamma(KV,V)$. Now using (17) in this last equation we obtain $\gamma(KV,V) = \frac{\sigma}{2}\rho$. This shows that $\sigma = 0$ on M if and only if $\gamma(KV,V) = 0$, i.e. V is asymptotic everywhere on Σ_t for all t. Finally, ξ geodesic follows from Proposition 1, which completes the proof. $\square$

Observe that when ξ is null the coefficient of dt^2 in the metric of g vanishes i.e., $\lambda^2 = S^a S_a$. Now, using (15) and $\lambda = \rho$ we have $S^a = V^a$ so that $\lambda^2 = V^a V_a$. Thus, the metric g of M takes the form

$$ds^2 = 2g_{ab}V^a dx^b dt + g_{ab}dx^a dx^b. \tag{19}$$

Consequently, the absence of dt^2 in (19) implies that there exists a foliation of null hypersurfaces (Σ_0, γ^0), of M, defined by $\{x^1 = \text{constant}\}$ whose each induced degenerate metric γ^0 is given by

$$ds^2_{\gamma^0} = \gamma^0_{a\alpha}V^a dx^\alpha dt + \gamma^0_{\alpha\beta}dx^\alpha dx^\beta, \quad 2 \le \alpha, \beta \le 3, \tag{20}$$

such that the null Killing vector field ξ is tangent to the null hypersurface Σ_0. Moreover, as per Proposition 1, ξ is a geodesic vector field. At this point we follow Duggal and Sharma [4], Chapter 2, Section 2.7. Contrary to the Riemannian case, for null hypersurfaces both the tangent space $T_p(\Sigma_0)$ and the normal space $T_p(\Sigma_0)^\perp$, at every point p of Σ_0, are degenerate. Moreover,

$$T_p(\Sigma_0) \cap T_p(\Sigma_0)^\perp = T_p(\Sigma_0)^\perp, \qquad \dim(T_p(\Sigma_0)^\perp) = 1.$$

Dropping the suffix p we let $S(T\Sigma_0)$ be the 2-dimensional complementary spacelike distribution to $T(\Sigma_0)^\perp$ in $T(\Sigma_0)$. Then we have

$$T(M) = S(T\Sigma_0) \oplus S(T\Sigma_0)^\perp, \quad S(T\Sigma_0) \cap S(T\Sigma_0)^\perp = \{0\}, \tag{21}$$

where $S(\Sigma_0)^\perp$ is a 2-dimensional complementary orthogonal distribution of $T(M)$. Let $T(\Sigma_0)^\perp$ be spanned by $\{\ell\}$, where ℓ is a real null vector.

Theorem. (see Duggal and Sharma [4], p. 29) *Let $(\Sigma_0, \gamma^0, S(T\Sigma_0))$ be a null hypersurface of a 4-dimensional Lorentz manifold (M, g). Then, with respect to each coordinate neighborhood $\mathcal{U}$ of Σ_0, there exists a unique null distribution $E = \cup_{p\in\Sigma} E_p$, where E is spanned by a unique null vector field n such that*

$$g(\ell, n) = 1, \qquad g(n, n) = g(n, X) = 0, \quad \forall X \in \Gamma(S(T\Sigma_0)|_{\mathcal{U}}). \tag{22}$$

Using (21) and (22) we have the following decompositions:

$$T(M) = S(T\Sigma_0) \oplus (T(\Sigma)^\perp + E) \quad \text{and} \quad T(\Sigma) = S(T\Sigma_0) \oplus T(\Sigma)^\perp, \tag{23}$$

where $+$ denotes non-orthogonal complementary sum. Using above and (20) we say that the metric of a leaf L^i of $S(T\Sigma_0^i)$ is given by

$$ds^2|_{S(T\Sigma_0^i)} = \gamma^i_{\alpha\beta} dx^\alpha dx^\beta. \tag{24}$$

4. PHYSICAL INTERPRETATION

Denote the null evolution vector field on M by ξ_0. We first recall the concept of Killing horizon (see Carter [3]) which is defined as the union $\Sigma_0 = \bigcup \Sigma_0^i$, where Σ_0^i is a connected component of the set of points forming a null hypersurface on which a Killing vector field ξ_0 is null and is nowhere vanishing. Now we show how a Killing horizon acts as a link between the spacelike hypersurfaces of Theorem 1 and a foliation of null hypersurfaces $(\Sigma_0, \gamma^0, S(T\Sigma_0))$ of M.

Consider a pseudo-orthonormal basis $\{\ell, n, u, v\}$ at each point $p \in M$, where $\{\ell, n\}$, satisfying (21), are future directed null vectors, $\{u, v\}$ are unit spacelike vectors. Let H be an orientable spacelike 2-surface of (M, g) such that T_pH is generated by $\{u, v\}$. Thus, we have

$$TM = TH \oplus TH^\perp,$$

where the normal bundle $TH^\perp$ is generated by $\{\ell, n\}$. Let (M, g) be evolved through a family of spacelike hypersurfaces Σ_s, with ξ_s the evolution CKV field as described in Theorem 1. and containing H_s as a codimension one submanifold, where $s \in (0, \delta)$

and $\delta > 0$. Let C_s be a differentiable curve of Σ_s such that for each s, the unit normal vector field of N_s is given by

$$N_s = \frac{1}{\sqrt{2}}(s^{-1}n - s\ell),$$

such that the component of N_s in the ℓ-direction vanishes as $s \to 0$. This means that N_s approaches a null vector field N_0 which is entirely in the n-direction as $s \to 0$. From this data we construct a null hypersurface Σ_0 of M in the following way:

Suppose $\Omega(\Sigma_s)$ is an object defined on each spacelike hypersurface Σ_s. Then, the concept of null limit (explained above) can be used to define analogous object $\Omega(\Sigma_0)$, for the null hypersurface Σ_0, by defining

$$\Omega(\Sigma_0) = \ \mathrm{Lim}_{s\to o}\Omega(\Sigma_s)\,,$$

In this way, we say that

$$\Sigma_0 = \mathrm{Lim}_{s\to o}(\Sigma_s) \quad \text{such that} \quad \mathrm{Lim}_{s\to o}(\xi_s) = \xi_0$$

is the null Killing geodesic vector field of Σ_0 which is Killing horizon as a boundary, where ξ_s approaches ξ_0 and the spacelike hypersurface Σ_s degenerates to the null hypersurface Σ_0. Contrary to the non-degenerate case, ξ_0 can not be uniquely expressed as a sum of its tangential and normal components. Instead, using (23) we can decompose null ξ_0 of Σ_0 in terms of three componets as

$$\xi_0 = V + \frac{\lambda}{\sqrt{2}}(\ell - n).$$

Furthermore, it follows from (1) that the induced metric of each spacelike 2-surface $H^i \subset \Sigma^i \subset M$ can be expressed as

$$ds^2|_{H^i} = \gamma^i_{\alpha\beta}dx^\alpha dx^\beta, \quad 2 \le \alpha, \beta \le 3\,,$$

where we define H^i in Σ^i by x^1 = constant. Compairing above with (24) we conclude that a leaf L^i of a chosen screen distribution $S(\Sigma^i_0)$ can be identified with each corresponding spacelike 2-surface $H^i \subset \Sigma^i_0 \subset M$, that is each

$$\Sigma^i_0 \supset L^i = H^i \subset \Sigma^i$$

is a common 2-surface of both Σ_0 and Σ_t. For some more basic details on null hypersurfaces, see Duggal and Sharma [4], Chapter 2, Section 7.

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