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NUMERICAL SOLUTIONS FOR A SECOND ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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ABSTRACT: We study a second order differential equation with nonlinear three-point boundary conditions. The existence of solutions is proved using Fixed Point Theorems. Numerical simulations are shown by using iterative and shooting methods.

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1. INTRODUCTION

Three-point boundary value problems for second order ordinary differential equations of the type

$$u'' = f(x, u, u'), \qquad 0 < x < 1, \tag{1.1}$$

$$u(0) = 0, \qquad u(1) = \alpha u(\eta), \tag{1.2}$$

where $0 < \eta < 1$, $\alpha \in \mathbb{R}$ and $f \in C([0, 1] \times \mathbb{R}^2)$, have been extensively studied in the past two decades. The first results were given by Il'in and Moiseev [8], Il'in and Moiseev [9], which considered equation (1.1) with the m-point boundary condition

$$u(0) = 0 \quad \text{and} \quad u(1) = \sum_{j=1}^{m-2} \alpha_i u(\eta_i), \quad m \geq 3,$$

where $\alpha_i \in \mathbb{R}$ and $0 < \eta_i < 1$. Later, Gupta [5] developed a systematic study for this problem with $m = 3$, using integral representations and degree theory. Since then, many generalizations and variations of (1.1), (1.2) were done in several directions. Some interesting results concerned with the qualitative aspects such as positivity,

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multiplicity, resonance, and discrete equations, can be found in, for instance, Agarwalet al [1], Feng and Webb [2], García-Huidobro et al [3], Henderson [6], Henderson et al [7], Lan [10], Ma [11], Ma and Wang [12].

On the other hand, it seems that no numerical study was devoted to this class of boundary value problems. In order to contribute in this direction, we present some numerical algorithms for the three-point boundary value problem

$$\begin{cases} u'' = f(x, u, u'), & 0 < x < 1, \\ u(0) = 0, & u(1) = g(u(\eta)), \end{cases} \tag{1.3}$$

where f, g are continuous functions, possibly nonlinear.

The paper is organized as follows. In Section 2 we present an existence result for iterative solutions under some local assumptions on f and g. Section 3 is dedicated to numerical solutions with iterative methods. In Section 4 we discuss numerical solutions by using shooting methods. In both approaches, the value of $u(\eta)$ is computed through cubic spline interpolation.

2. EXISTENCE RESULTS

We begin this section by pointing out that the solutions of (1.3) can be written as

$$u(x) = \int_0^1 G(x,t) f(t, u(t), u'(t))\, dt + g(u(\eta))x, \tag{2.1}$$

where G is the Green's function for $u''(x) = f(x)$ with $u(0) = u(1) = 0$, namely,

$$G(x,t) = \begin{cases} x(t-1)\,, & x \le t\,, \\ t(x-1)\,, & t \le x\,. \end{cases} \tag{2.2}$$

Then we see that u is a solution of (1.3) if and only if it is a fixed point of the operator $T : C^1[0,1] \to C^1[0,1]$, defined by

$$(Tu)(x) = \int_0^1 G(x,t) f(t, u(t), u'(t))\, dt + g(u(\eta))x, \tag{2.3}$$

where $C^1[0,1]$ denotes the space of the continuously differentiable functions defined in $[0,1]$, equipped with the norm

$$\|u\|_{C^1} = \max\{\|u\|_\infty, \|u'\|_\infty\}, \quad \text{where} \quad \|w\|_\infty = \max_{0 \le t \le 1} |w(t)|. \tag{2.4}$$

As basic hypotheses on f and g, we assume there exist $\alpha, \beta, R > 0$ such that

$$|f(x,u,v)| \le \alpha \quad \text{and} \quad |g(u)| \le \beta, \tag{2.5}$$

for $t \in [0,1]$ and $u, v \in [-R, R]$. For iterative solutions, we also assume local Lipschitz condition near origin, that is, there exist $a, b, c > 0$ such that

$$|f(t,u,v) - f(t,\hat{u},\hat{v})| \le a|u - \hat{u}| + b|v - \hat{v}|, \tag{2.6}$$

$$|g(u) - g(\hat{u})| \le c|u - \hat{u}|, \tag{2.7}$$

for $t \in [0,1]$ and $u, v, \hat{u}, \hat{v} \in [-R, R]$.

We recall that iterative solutions are usually provided by the Contraction Principle, which in the present case, asserts that T has a unique fixed point if there exists $0 < \lambda < 1$ such that

$$\|Tu - Tv\|_{C^1} \le \lambda \|u - v\|_{C^1},$$

in some closed subset of $C^1[0,1]$. In the affirmative case, we can compute numerical solutions from the iterative formulae $u^{k+1} = Tu^k$.

Theorem 1. *If the condition* (2.5) *holds with*

$$\frac{\alpha}{2} + \beta \le R, \tag{2.8}$$

then problem (1.3) *has a solution. If in addition, conditions* (2.6) *and* (2.7) *hold with*

$$\frac{a+b}{2} + c < 1, \tag{2.9}$$

then problem (1.3) *has a unique solution* u *such that* $\|u\|_{C_1} \le R$. *Besides, this solution is the uniform limit of the iterative sequence* $u^{k+1} = Tu^k$.

Proof. We show that T has a fixed point in $C^1[0,1]$. Let

$$B = \{u \in C^1[0,1];\ u(0) = 0 \text{ and } \|u\|_{C^1} \le R\}.$$

Then for $u \in B$, we have $\|u\|_\infty \le \|u'\|_\infty$ (because $u(0) = 0$) and hence

$$\|u\|_{C^1} = \|u'\|_\infty.$$

Besides, B is a closed convex subset of $C^1[0,1]$. Now, since

$$\int_0^1 \left|\frac{\partial}{\partial x} G(x,t)\right| dt \le \frac{1}{2}, \quad 0 \le x \le 1,$$

we see that

$$\begin{aligned} |(Tu)'(x)| &= \left|\int_0^1 \frac{\partial}{\partial x} G(x,t) f(t, u(t), u'(t))\, dt + g(u(\eta))\right| \\ &\le \frac{1}{2} \max_{t \in [0,1]} |f(t, u(t), u'(t))| + |g(u(\eta))|. \end{aligned}$$

Then, using assumptions (2.5) and (2.8), we have for $u \in B$,

$$\|Tu\|_{C^1} = \max_{x \in [0,1]} |(Tu)'(x)| \le \frac{\alpha}{2} + \beta \le R,$$

that is, T applies B into B. Furthermore, from the Arzela-Ascoli Theorem, we see that T is a completely continuous operator in B. Hence T has a fixed point by the Schauder's Fixed Point Theorem.

To prove the second part of the theorem, we show that $T : B \to B$ is a contraction. Arguing as before,

$$|(Tu - Tv)'(x)| \le \frac{1}{2} \max_{t\in[0,1]} |f(t,u,u') - f(t,v,v')| + |g(u(\eta)) - g(v(\eta))|.$$

Then if $u, v \in B$, we have from (2.6) and (2.7),

$$\|Tu - Tv\|_{C^1} \le \left(\frac{a+b}{2} + c\right) \|u - v\|_{C^1},$$

which shows that T is a contraction since $\lambda = (a+b)/2 + c < 1$ from assumption (2.9). $\square$

Remarks. Nonlinear second order boundary value problems have very often multiple solutions. Since Contraction Principle establish the existence of a unique solution, very restrictive hypotheses are needed. For example, if the partial derivatives of f, with respect to u and v, vanish at $(t,0,0)$, and $g'(0)$ is small, then condition (2.6) is easily satisfied for a small R.

3. ITERATIVE SOLUTIONS

In this section we present some numerical simulations for second order three-point boundary value problems by means of iterative methods based on the formulae (2.3), that is,

$$\begin{aligned} u^{k+1}(x) &= \int_0^x t(x-1) f(t, u^k(t), u^{k\prime}(t))\, dt \\ &\quad + \int_x^1 x(t-1) f(t, u^k(t), u^{k\prime}(t))\, dt + g(u^k(\eta))x. \qquad (3.1) \end{aligned}$$

The integrals are computed through a Newton-Cotes method (e.g. trapezoidal rule) and $u(\eta)$ is computed via cubic spline interpolation. A basic algorithm is the following.

Algorithm 1.

1 - Define a uniformly spaced mesh $\{x_j\}$.
2 - Choose initial approximation $u_j^0 = u^0(x_j)$.
3 - For $k = 1, 2, 3, \cdots$
- Compute $u^k(\eta)$ by using cubic-spline interpolation.
- Compute $u_j'^k$ with central-differences.
- Compute u_j^{k+1} through formulae (3.1) with the trapezoidal rule.
- Test convergence.

4 - End iteration.

Iteration	E^k ($\eta = 0.12$)	E^k ($\eta = 0.58$)	E^k ($\eta = 0.98$)
1	.629106e-1	.271047e-0	.380470e-1
2	.377465e-2	.786099e-1	.198258e-1
3	.226494e-3	.228029e-1	.108974e-1
10	.165500e-7	.125188e-4	.237733e-2
20	.165500e-7	.858690e-5	.231920e-2
30	.165500e-7	.858690e-5	.231915e-2

TABLE 1. Iterative method with mesh size $h = 0.1$

Example 1. We begin with a simple linear problem

$$\begin{cases} u'' = -6x, & 0 < x < 1, \\ u(0) = 0, & u(1) = g(u(\eta)), \end{cases} \tag{3.2}$$

where $g(s) = s/2$, in order to see the dependence of the method with respect to η. We consider $\eta = 0.12$, $\eta = 0.58$ and $\eta = 0.98$. The exact solutions corresponding to these three η are respectively,

$$u(x) = \frac{31223}{29375}x - x^3,$$

$$u(x) = \frac{225611}{177500}x - x^3$$

and

$$u(x) = \frac{44117}{42500}x - x^3.$$

We obtain numerical solutions by setting mesh sizes $h = 0.1$ and $h = 0.05$, with initial guess $u^0 = 0$. The results are in the Table 1 and Table 2, where

$$E^k = \max |\, u_j^k - u(x_j) \,|$$

is the maximum absolute error in the k-th iteration.

This two tables show that numerical solutions for problem (1.3) are sensitive to the value of η. Better results are obtained when η is near to 0, since $u(0) = 0$ is fixed.

4. SHOOTING METHODS

Sometimes shooting methods (e.g. Golub and Ortega Golub and Ortega [4], Chapter 5) give more accurate results than the iterative method used above. As it is

Iteration	E^k $(\eta = 0.12)$	E^k $(\eta = 0.58)$	E^k $(\eta = 0.98)$
1	.629106e-1	.271047e-0	.380470e-1
2	.377463e-2	.786038e-1	.186396e-1
3	.226478e-3	.227951e-1	.913001e-2
10	.300000e-9	.393350e-5	.553160e-4
20	.300000e-9	.430000e-8	.651050e-5
30	.300000e-9	.520000e-8	.663300e-5

TABLE 2. Iterative method with mesh size $h = 0.05$

well-known, problem (1.3) must be transformed in the initial value problem

$$\begin{cases} u'' = f(x, u, u'), \qquad 0 < x < 1, \\ u(0) = 0, \\ u'(0) = v, \end{cases} \tag{4.1}$$

whose solution $u = u(x, v)$ is approximated by some initial problem solver, for instance, the fourth order Runge-Kutta method (RK). The problem is then reduced to finding a velocity v such that

$$u(1, v) = g(u(\eta, v)), \tag{4.2}$$

which can be solved by linear iteration. Accordingly, given initial approximations $v_0, v_1 \in \mathbb{R}$,

$$v_{k+1} = v_{k-1} + \frac{(g(u(\eta, v_k)) - u(1, v_{k-1}))\,(v_k - v_{k-1})}{u(1, v_k) - u(1, v_{k-1})}, \qquad k = 1, 2, 3, \ldots . \tag{4.3}$$

converges to a solution of (4.2). A basic algorithm for the shooting method is the following.

Algorithm 2.

1 - Define a uniformly spaced mesh $\{x_j\}$.
2 - Choose initial velocities v_0, v_1.
3 - Compute $u_j(v_1) = u(x_j, v_1)$ through RK.
4 - For $k = 1, 2, 3, \cdots$
  - Compute $u(\eta, v_k)$ via cubic-spline interpolation.
  - Compute v_{k+1} by linear iteration (4.3).
  - Compute $u_j(v_{k+1})$ through RK.

Iteration	E^k (iterative)	E^k (shooting)
1	.180491e-1	.283394e-1
2	.177919e-2	.105374e-1
3	.113632e-2	.103603e-2
10	.115503e-2	.149859e-4
20	.115503e-2	.149859e-4
30	.115503e-2	.149859e-4

TABLE 3. f depending on u' and $\eta = 0.37$

- Test convergence.

5 - End iteration.

Example 2. Let us consider the nonlinear problem

$$\begin{cases} u'' = u'^2 - 9x^4 + 3x^2 - 6x - 0.25, \qquad 0 < x < 1, \\ u(0) = 0, \qquad u(1) = g(u(\eta)), \end{cases} \tag{4.4}$$

where

$$g(s) = s^2 - 0.5180491164.$$

The exact solution is $u(x) = 0.5x - x^3$. We compare the numerical solutions of this problem by using Algorithms 1 and 2. For the iterative method (Algorithm 1) we take initial guess $u_j^0 = 0$ and approximate the derivatives u_j' by their central-difference formula. For the shooting method we take initial velocities $v_0 = 0$ and $v_1 = 1$. The results obtained by setting mesh size $h = 0.1$ and $\eta = 0.37$ are shown in the Table 3. As before, $E^k = \max |\, u_j^k - u(x_j) \,|$.

Example 2 shows that the shooting method may have advantages over the iterative method when $f(x, u, u')$ depends essentially on u'. In fact, in the Algorithm 1, the derivatives u_j' are given by the finite-difference approximation, which combined with the trapezoidal rule gives poor results.

On the other hand, we may ask ourselves if shooting methods have better results than iterative methods when η is close to 1. The next example shows that in this case both methods have comparable results.

Example 3. Let us consider the same problem of Example 1 by using the shooting method as in Example 2. We compare the accuracy of the method with several values of η.

We note that Tables 1 and 4 are essentially equal.

Iteration	E^k ($\eta = 0.12$)	E^k ($\eta = 0.58$)	E^k ($\eta = 0.98$)
1	.339718e-2	.707489e-1	.178432e-1
2	.203845e-3	.205226e-2	.980768e-2
3	.122457e-4	.595706e-2	.587025e-3
10	.147000e-7	.875500e-5	.211289e-2
20	.147000e-7	.772900e-5	.208725e-2
30	.147000e-7	.772900e-5	.208723e-2

TABLE 4. Shooting method with $h = 0.1$ and various η

5. CONCLUSION

Our work shows an existence result through iterative methods to a class of three-point boundary value problems for second order differential equations. The numerical solutions are studied by iterative and shooting methods. The Examples 1 and 3 show that both methods are equally dependent on the value of η. When η approaches to 1, the results are poor because the evaluation of $u(\eta)$, since $u(1)$ is also unknown. But if η is close to 0, the results are very accurate since $u(0)$ has fixed value 0. Now, if f is strongly dependent on u', then the shooting method has better accuracy than iterative method, as shown in Example 2. We notice that in the beginning of this work, we have considered the Richardson extrapolation in the iterative method. But in view of the overall performance, we prefer the shooting method. Finally, if linear interpolation is used instead cubic splines, then both methods have comparable accuracy.

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References

[1] R.P. Agarwal, H.B. Thompson and C.C. Tisdell, On three-point boundary value problems for second-order discrete equations, *Comput. Math. Appl.*, **45** (2003), 1437-1444.

[2] W. Feng and J.R.L. Webb, Solvability of m-point boundary value problems with nonlinear growth, *J. Math. Anal. Appl.*, **212** (1997), 467-480.

[3] M. García-Huidobro, C.P. Gupta and R. Manásevich, Solvability for a nonlinear three-point boundary value problem with p-Laplacian-like operator at resonance, *Abstr. Appl. Anal.*, **6** (2001), 191-213.

[4] G.H. Golub and J.M. Ortega, *Scientific Computing and Differential Equations*, Academic Press, San Diego, 1992.

[5] C.P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, *J. Math. Anal. Appl.*, **168** (1992), 540-551.

[6] J. Henderson, Uniqueness implies existence for three-point boundary value problems for second order differential equations, *Appl. Math. Lett.*, **18** (2005), 905-909.

[7] J. Henderson, B. Karna and C.C. Tisdell, Existence of solutions for three-point boundary value problems for second order equations, *Proc. Amer. Math. Soc.*, **133** (2005), 1365-1369.

[8] V.A. Il'in and E.I. Moiseev, Nonlocal boundary-value problem of the secod kind for a Sturm-Liouville operator, *Differential Equations*, **23** (1987), 979-987.

[9] V.A. Il'in and E.I. Moiseev, A nonlocal boundary value problem for the Sturm-Liouville operator in the differential and difference treatments, *Sov. Math. Dokl.*, **34** (1987), 507-511.

[10] K.Q. Lan, Properties of kernels and eigenvalues for three point boundary value problems, *Discrete Contin. Dyn. Syst. Suppl.* (2005), 546-555.

[11] R. Ma, Existence theorems for a second order m-point boundary value problem, *J. Math. Anal. Appl.*, **211** (1997), 545-555.

[12] R. Ma and H. Wang, Positive solutions of nonlinear three-point boundary-value problems, *J. Math. Anal. Appl.*, **279** (2003) 216-227.