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SUFFICIENT CONDITIONS FOR OSCILLATION OF THE SOLUTIONS OF IMPULSIVE LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT

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ABSTRACT: In the present paper, linear homogeneous differential equations with impulsive effects and retarded argument are investigated. The impulsive influences are realised in the moments where the integral curve of the equation intersects preliminary given curves in the equation's expanded phase space. The sufficient conditions for oscillation of the solutions of such equations are found.

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1. INTRODUCTION

In the recent paper linear homogeneous differential equations with impulses and retardation are considered. The impulses are realized in the moments, when the integral curve of the equation intersects some from the preliminary given curves (named impulsive curves too):

$$\sigma_i = \{(t,x); t = \tau_i(x), x \in D\}\,, \qquad i = 1,2,\dots\,, \tag{1}$$

where $\tau_i : D \to \mathbb{R}^+$, D is an open interval.

The following initial problem is the main subject of our investigation:

$$\dot{x}(t) + p(t)x(\Omega(t,x(t))) = 0\,, \qquad t \neq \tau_i(x(t))\,, \tag{2}$$

$$\Delta x(t) + p_i x(t) = 0\,, \qquad t = \tau_i(x(t))\,, \qquad i = 1,2,\dots\,, \tag{3}$$

$$x(0) = x_0\,, \tag{4}$$

where $p : \mathbb{R}^+ \to \mathbb{R}$, $\Omega : \mathbb{R}^+ \times D \to \mathbb{R}^+$, the constants p_i, $i = 1,2,\dots$, are real numbers, $x_0 \in D$ and $\Delta x(t) = x(t+0) - x(t)$.

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We denote by

$$t_i, \qquad i = 1, 2, \dots, \qquad 0 < t_1 < t_2, \dots, \tag{5}$$

the moments when the integral curve of the problem (2), (3), (4) intersects some of the curves (1).

We introduce the notations:

$$x_i = x(t_i), \qquad x_i^+ = x(t_i + 0), \qquad i = 1, 2, \dots.$$

Let j_i be the running number of the impulsive curve, which is intersected by the integral curve of the considered problem in moment t_i, i.e. the next equalities are valid

$$t_i = \tau_{j_i}(x_i), \qquad i = 1, 2, \dots.$$

We can describe the solution of the problem (2), (3), (4), as follows:

(i) When $0 \le t \le t_1$ the solution of the considered problem coincides with the solution of the initial problem (2), (4).

(ii) When $t_i < t \le t_{i+1}$, $i = 1, 2, \dots$ then it coincides with the solution of equation (2) with initial condition

$$x(t_i + 0) = x_i^+ = x(t_i) - p_i x(t_i) = (1 - p_i)x_i.$$

(iii) The integral curve $(t, x(t))$ of the considered problem intersects the curves (1) only in the moments t_i, $i = 1, 2, \dots$, which fulfill the inequality (5). We shall remark that in the common case it is fulfilled that $i \neq j_i$ (see Example 1 in Bainov and Dishliev [1]).

A preliminary requirement in investigation of asymptotic properties of the solutions of the differential equations is their unrestricted extension. For this purpose it is necessary the phenomenon *"beating"* to be excluded, in which the integral curves intersects multiple (it is possible countless many times) one and the same impulsive curve. If that phenomenon appears, it is possible that a given impulsive curve to be not *"surmountable"* from the integral curve. In this case the solution could be not to be extended further from a given moment. The sufficient conditions for absence of the phenomenon "beating" in the common case of impulsive functional differential equations are obtained in Bainov and Dishliev [1].

Because equation (2)-(3) is a particular case of impulsive functional differential equations, the conditions for absence of the phenomenon "beating" and unrestricted extension of the solutions are specific and weaker restricting then the respective conditions in Bainov and Dishliev [1].

We shall say that the conditions (H) are fulfilled, if the following conditions are valid:

(H1.) The interval D is restricted, i.e. there exists a constant $M_1 \in \mathbb{R}^+$, such that $D \subset (-M_1, M_1)$.

(H2.) The function $p \in C[\mathbb{R}^+, \mathbb{R}]$ is restricted, i.e. there exists a constant $M_2 \in \mathbb{R}^+$, such that $|p(t)| \le M_2$, $t \in \mathbb{R}^+$.

(H3.) The following inequalities are valid: $0 \le p_i \le 2$, $i = 1, 2, \ldots$.

(H4.) For each $x \in D$ it is fulfilled that $(1 - p_i)x \in D$, $i = 1, 2, \ldots$.

(H5.) For each point $(t_0, x_0) \in \mathbb{R}^+ \times D$ the equation (2) with initial condition $x(t_0 + 0) = x_0$ has a unique solution, which is could be extended in infinity.

(H6.)τ_i are Lipschiz functions in D with the respective Lipschiz constants L_i for which the inequalities $0 < L_i < \frac{1}{M_1 M_2}$, $i = 1, 2, \ldots$, are valid.

(H7.) $0 < \tau_1(x) < \tau_2(x) < \ldots$, $x \in D$.

(H8.) $\lim_{i\to\infty} \tau_i(x) = \infty$, uniformly at $x \in D$.

(H9.) For each $x \in D$ the following inequalities are valid

$$\tau_i((1 - p_i)x) < \tau_i(x)\,, \qquad i = 1, 2, \ldots .$$

Remark 1. We shall note that condition H3 is essential. Indeed, according to condition H4 it follows that the point $(1 - p_i)x \in D$ when $x \in D$. Applying multiple condition H4 we obtain that the points $(1 - p_i)^2 x$, $(1 - p_i)^3 x$, $\ldots$ are in D, too. If we prove that $p_i < 0$, or $p_i > 2$ for some i, it will follow that $|1 - p_i| > 1$. Hence, for each point $x \in D$, $x \ne 0$, it holds $\lim_{n\to\infty} |(1 - p_i)^n x| = \infty$. Then there exists natural number n such that $|(1 - p_i)^n x| > M_1$, i.e. according to condition H1 the point $(1 - p_i)^n x \notin D$, which is a contradiction.

Theorem 1. *Let conditions (H) be fulfilled. Then:*

(j) *The integral curve* $(t, x(t))$ *of the problem* (2), (3), (4) *intersects each of the curves* (1) *no more then one time.*

(jj) *The solution of the considered problem could be extended to infinity.*

Proof. (j). Let t_i is the first moment, when the integral curve $(t, x(t))$ of the problem with impulses (2), (3), (4) intersects the curve σ_i, i.e. $t_i = \tau_{j_i}(x_i)$ and $t \ne \tau_{j_i}(x(t))$ for $0 < t < \tau_i$ holds. We will show that if $t > t_i$ the considered integral curve does not intersect the curve σ_{j_i}.

Let us suppose the contrary, i.e. there exists point $t^* > t_i$ such that $t^* = \tau_{j_i}(x(t^*))$. We shall consider the following two cases:

Case 1. The integral curve $(t, x(t))$ does not intersect curves of (1) for $t_i < t < t^*$. That means that $t^* = t_{i+1}$ and $j_{i+1} = j_i$. In this case we receive the following contradiction:

$$t_{i+1}-t_i=\tau_{j_{i+1}}(x(t_{i+1}))-\tau_{j_i}(x(t_i))=\tau_{j_i}(x_{i+1})-\tau_{j_i}(x_i)\le\tau_{j_i}(x_{i+1})-\tau_{j_i}((1-p_i)x_i)$$
$$=\tau_{j_i}(x_{i+1})-\tau_{j_i}(x_i^+)\le L_{j_i}|x_{i+1}-x_i^*|$$
$$=L_{j_i}\left|\int_{t_i}^{t_{i+1}}p(t)x(\Omega(t,x(t)))dt\right|\le L_{j_i}M_1M_2(t_{i+1}-t_i)<t_{i+1}-t_i\,.$$

Case 2. The integral curve $(t,x(t))$ of the problem (2), (3), (4) intersects the curves $\sigma_{j_{i+1}},\sigma_{j_{i+2}},\dots$ for $t_i<t<t^*$. That means that $t_i<t_{i+1}<t_{i+2}<\cdots<t^*$. We will show that $j_i<j_{i+1}$. Let us assume that $j_i>j_{i+1}$ (the case when $j_i=j_{i+1}$ was considered already in the previous case). We consider the function

$$\varphi(t)=\begin{cases}\tau_{j_i}(x_i^+)-t_i\,, & t=t_i\,,\\ \tau_{j_i}(x(t))-t\,, & t_i<t\le t_{i+1}\,.\end{cases}$$

The function φ is continuous in the interval $[t_i,t_{i+1}]$. We obtain from condition H9 the next inequality

$$\varphi(t_i)=\tau_{j_i}(x_i^+)-t_i=t_{j_i}((1-p_i)x_i)-t_{j_i}(x_i)<0\,. \tag{6}$$

From the assumption and condition H7 it follows that

$$\varphi(t_{i+1})=\tau_{j_i}(x(t_{i+1}))-t_{i+1}>\tau_{j_{i+1}}(x(t_{i+1}))-t_{i+1}=t_{i+1}-t_{i+1}=0\,. \tag{7}$$

From (6) and (7) it follows that there exists point t^{**}, $t_i<t^{**}<t_{i+1}$ such that $\varphi(t^{**})=0$, i.e. $\tau^{**}=\tau_{j_i}(x(t^{**}))$. The last equality shows that the integral curve $(t,x(t))$ of the problem (2), (3), (4) intersects the hypersurface σ_{j_i} in the moment t^{**}, for which the inequality $t_i<t^{**}<t_{i+1}$ is valid. The last inequality contradicts to the way of definition of the moments $t_1,t_2,\dots$. By this approach we have showed that $j_i<j_{i+1}$. Analogously, the following inequalities can be obtained

$$j_i<j_{i+1}<j_{i+2}<\dots\,. \tag{8}$$

Let the integral curve $(t,x(t))$ of the considered problem intersects finite number of curves from (1) when $t_i<t<t^*$. Let $\sigma_{j_{i+k}}$ be the last curve which is intersected. The integral curve of the problem (2), (3), (4) intersects the curve σ_{j_k} in the moment t^*, immediately after the moment t_{i+k}. Hence, $t^*=t_{i+k+1}$ and $j_k=j_{i+k+1}$. The last equality contradicts to (8).

Let the integral curve of the problem (2), (3), (4) intersect infinitely many curves from (1) for $t_i<t<t^*$. The following inequalities are valid

$$\tau_{j_{i+k}}(x_{i+k})=\tau_{j_{i+k}}(x(t_{i+k}))=t_{i+k}<t^*\,,\qquad k=1,2,\dots\,. \tag{9}$$

It follows from condition H8 that there exists number i_0 such that

$$\tau_i(x)>t^*\,,\qquad x\in D\,,\qquad i\ge i_0\,. \tag{10}$$

The inequalities (8) show that there exists number k_0 such that $j_{i+k_0} > i_0$. Then from condition H7 and inequalities (9) we obtain

$$\tau_{i_0}(x_{i+k_0}) < \tau_{j_{i+k_0}}(x_{i+k_0}) = t_{i+k_0} < t^*,$$

which contradicts to (10).

(jj). We shall consider the following two cases.

Case 1. Let the moments of impulsive influence be finite number. Let t_k be the last impulsive moment. According to condition H4 it is fulfilled that $(t_k, x_k^+) = (t_k, (1-p_k)x_k) \in \mathbb{R} \times D$. Then from condition H5 we conclude that the solution of equation (2) with initial point (t_k, x_k^+) is unique and can be extended to infinity. Because that solution coincides with the solution of the considered problem when $t > t_k$, than corollary (jj) is proved in this case.

Case 2. Let the impulsive moments (5) are infinitely many. We shall prove that $\lim_{i\to\infty} t_i = \infty$, whence, having in mind condition H5, it would follow that the solution of the problem (2), (3), (4) is unique and can be extended to infinity. Indeed from (8) and conditions H7 and H8 we obtain

$$\lim_{i\to\infty} t_i = \lim_{i\to\infty} \tau_{j_i}(x_i) \geq \lim_{i\to\infty} \tau_i(x_i) = \infty .$$

So, the theorem is proven. □

2. MAIN RESULTS

In this section we shall consider a particular case of equation (2):

$$\dot{x}(t) + p(t)x(\omega(t)) = 0\,, \qquad t \neq \tau_i(x(t))\,. \tag{11}$$

We introduce the following conditions (HH):

(HH1.) The function $\omega \in C(\mathbb{R}^+, \mathbb{R})$ and for each $t \in \mathbb{R}^+$ the following holds

$$0 \leq \omega(t) \leq t\,.$$

(HH2.) The function ω is monotonously increasing in $\mathbb{R}^+$ and $\lim_{t\to\infty} \omega(t) = \infty$.

(HH3.) The inequality $p(t) \geq 0$, $t \in \mathbb{R}^+$ is valid.

(HH4.) For each point $(t_0, x_0) \in \mathbb{R}^+ \times D$ the equation (11) with initial condition $x(t_0 + 0) = x_0$ possesses a unique solution and can be extended to infinity.

(HH5.) There exists monotone increasing and unrestricted sequence $\{T_n\}$ such that for each natural number the following inequality is fulfilled

$$\int_{\omega(T_n)}^{T_n} p(s)ds + \sum_{\omega(T_n)\leq t_k \leq T_n} p_k \geq 1\,.$$

Theorem 2. *Let condition* H1-H4, H6-H9 *and* HH *hold. Then the solution of the problem* (11), (3), (4) *is oscillatory.*

Proof. We shall consider the following two cases.

Case 1. Let the solution $x(t)$ of the problem (11), (3), (4) is finally positive, i.e. there exists a constant $T_{01} > 0$ such that

$$x(t) > 0\,, \qquad x(\omega(t)) > 0 \qquad \text{for} \qquad t > T_{01}\,. \tag{12}$$

Then from (11) and conditions HH3 and H3 we obtain that $x'(t) \le 0$ and $\Delta x(t_k) \le 0$ for $t > T_{01}$ and $t_k > T_{01}$, respectively. Hence the solution $x(t)$ is monotone decreasing function for $t > T_{01}$. After integration of (11) in the boundaries from $\omega(t)$ to t we receive the following equality

$$x(t) - x(\omega(t)) + \int_{\omega(t)}^{t} p(s)x(\omega(s))ds + \sum_{\omega(t)\le t_k \le t} p_k x(\omega(t_k)) = 0\,. \tag{13}$$

From condition HH2 we conclude that there exists a constant $T^{01} > T_{01}$ such that for $t > T^{01}$ it is fulfilled that $\omega(t) > T_{01}$. Hence $x(\omega(s)) \ge x(\omega(t))$ for $\omega(t) \le s < t$. Having in mind the equation (13) we obtain

$$x(t) + x(\omega(t))\left[\int_{\omega(t)}^{t} p(s)ds + \sum_{\omega(t)\le t_k \le t} p_k - 1\right] \le 0\,.$$

According to (12) we conclude that

$$\int_{\omega(t)}^{t} p(s)ds + \sum_{\omega(t)\le t_k \le t} p_k < 1\,, \qquad t > T^{01}\,,$$

which contradicts to condition HH5.

Case 2. Let us assume that the solution $x(t)$ of the considered problem is finally negative, i.e. there exists a constant T_{02} such that

$$x(t) < 0\,, \qquad x(\omega(t)) < 0 \qquad \text{for} \qquad t > T_{02}\,. \tag{14}$$

As in the forthcoming case by the help of (11) and conditions HH3 and H3 we establish that $x(t)$ is monotone increasing function for $t > T_{02}$. There exists a constant $T^{02} > T_{02}$ such that for $t > T^{02}$ and $\omega(t) \le s < t$ the inequality $x(\omega(s)) \le x(\omega(t))$ holds. Then from (13) it follows that

$$x(t) + x(\omega(t))\left[\int_{\omega(t)}^{t} p(s)ds + \sum_{\omega(t)\le t_k \le t} p_k - 1\right] \ge 0\,,$$

whence with the help of (14) we receive a contradiction to condition HH5.

From the considered two cases it follows that the solution of the problem (11), (3), (4) is neither finally positive nor finally negative. Hence the solution is oscillatory.

So, the theorem is proved. □

Remark 2. In the theorem above we can replace condition HH5 by the following condition.

(HH5′.) There exists monotone increasing and not restricted sequence $\{T_n\}$ such that

$$\lim_{t\to\infty}\sup\left\{\int_{\omega(t)}^{t} p(s)ds + \sum_{\omega(t)\le t_k<t} p_k\right\} > 1\,.$$

We shall consider the following problem

$$\dot{x}(t) + px(t-\lambda) = 0\,, \qquad t \neq \tau_i(x(t))\,, \tag{15}$$

$$\Delta x(t) + p_0 x(t) = 0\,, \qquad t = \tau_i(x(t))\,, \quad i = 1, 2, \dots\,. \tag{16}$$

$$x(t) = x_0\,, \qquad -\lambda \le t \le 0\,, \tag{17}$$

where p, p_0 and λ are real constants.

We denote the number of the impulsive moments which belong to the interval $[a, b]$ with $i[a, b]$.

As consequence of the forthcoming theorems we obtain the following theorem.

Theorem 3. *Let the following coditions hold:*

1. *The conditions* H6*,* H7*,* H8 *and* H9 *hold.*
2. $p \ge 0$*,* $p_0 \ge 0$*,* $\lambda > 0$*.*
3. *There exists monotone increasing and not restricted sequence* $\{T_n\}$ *such that for each natural number it holds*

$$p_\lambda + p_0 i[T_n - \lambda, T_n] \ge 1\,.$$

Then the solution of the problem (15)*,* (16)*,* (17) *is oscillatory.*

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