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FIRST-ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Problems of existence of solutions and quasi–solutions of first order impulsive functional differential equations with nonlinear two–point boundary conditions are discussed in this paper. Also impulsive differential inequalities with positive linear operators are investigated. The results are very general and some known results can be obtained from ours as special cases. Two examples are added to illustrate the obtained results.

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1. INTRODUCTION

Let $J = [0,T]$, $E = C(J,\mathbb{R})$ and $Q \in C(E,E)$. We shall say that Q is a causal operator, or nonanticipative, if the following property holds: for each couple of elements of E such that $u(s) = v(s)$ for $0 \le s \le t$, there results $(Qu)(s) = (Qv)(s)$ for $0 \le s \le t$ with $t < T$ arbitrary, for details see [3]. Note that $(Q_1x)(t) = \int_0^t W(t,s,x(s))ds, \quad t \in [0,c)$ and $(Q_2x)(t) = h(t,x(t)), \quad t \in [0,c)$ are examples of causal operators. Indeed, W and h are continuous functions with values in $\mathbb{R}^p$. In the literature operator Q_1 is known under the name "Volterra operator" and Q_2 is known as "Niemytskii operator".

Let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. Put $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$. In this paper, we investigate nonlinear two–point boundary value problems for impulsive functional differential equations with a causal operator Q of the form

$$(1) \qquad \begin{cases} x'(t) &= (Qx)(t), \quad t \in J', \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1,2,\cdots,m, \\ 0 &= g(x(0),x(T)), \end{cases}$$

where as usual $\Delta x(t_k) = x(t_k^+) - x(t_k^-); x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of x at t_k, respectively, and

$H_1 : Q \in C(E,E),\ I_k \in C(\mathbb{R},\mathbb{R})$ for $k = 1,2,\cdots,m$ and $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$.

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Functional equations with causal operators are discussed in the book [3]; see also papers [5]–[7],[14],[15]. To obtain approximate solutions of differential equations we can apply the monotone iterative technique, for details, see for example the book [16]. There exists a vast literature devoted to the applications of this method to differential equations with initial and boundary conditions. However, only a few papers have appeared where the monotone iterative technique is applied to differential equations with deviating arguments, see for example [2], [4], [6], [8]–[15], [18], [19], [21], [22]. Usually, the authors assumed a one sided Lipschitz condition on a function f (appearing on the right–hand–side of differential equations) with corresponding constants coefficients. Replacing constants by corresponding functions we can obtain less restrictive conditions for the existence of solutions, see papers [9]–[13]. Recently, the iterative method is applied to differential problems with causal operators, see papers [6],[14],[15]. It is important to observe that for problems with causal operators it is assumed that the causal operator Q satisfies a one sided Lipschitz condition with a corresponding positive linear operator $\mathcal{L}$. It is a first paper when this technique is applied to impulsive differential equations with causal operators. Note that impulsive differential equations are also discussed in the books [17],[20], see also paper [1].

The plan of this paper is as follows. In Section 2, we investigate impulsive differential inequalities with positive linear operators to obtain comparison results. In Section 3, we discuss a linear impulsive differential equation with the positive linear operator $\mathcal{L}$ giving sufficient conditions under which it has a one solution. The existence of extremal solutions of problem (1) is investigated in Section 3. We use the notation of lower and upper solutions of (1) to show that extremal solutions of problem (1) exist in a corresponding sector. This problem is discussed in Section 4. The case when problem (1) has a unique solution is investigated in Section 5. The last section is devoted to applications of coupled lower–upper solutions of (1) to discuss the problems when (1) has quasi–solutions or a solution. Two examples illustrate obtained results.

2. LINEAR IMPULSIVE DIFFERENTIAL INEQUALITIES

Put $J_0 = [0, t_1],\ J_k = (t_k, t_{k+1}],\ k = 1, 2, \cdots, m$. Let us introduce the spaces:

$$PC(J) = PC(J, \mathbb{R}) = \left\{ \begin{array}{l} x : J \to \mathbb{R},\ x|_{J_k} \in C(J_k, \mathbb{R}), \quad k = 0, 1, \cdots, m \\ \text{and there exist } x(t_k^+) \text{ for } k = 1, 2, \cdots, m \end{array} \right\},$$

and

$$PC^1(J) = PC^1(J, \mathbb{R}) = \left\{ \begin{array}{l} x \in PC(J),\ x|_{J_k} \in C^1(J_k, \mathbb{R}), \quad k = 0, 1, \cdots, m \\ \text{and there exist } x'(t_k^+) \text{ for } k = 1, 2, \cdots, m \end{array} \right\}.$$

We need the following

Lemma 1. *Let $\sigma \in C(J,\mathbb{R})$, $0 \le L_k < 1$, $k = 1,2,\cdots,m$. Assume that $p \in PC^1(J,\mathbb{R})$ and*

$$\begin{cases} p'(t) \le \sigma(t), & t \in J', \\ \Delta p(t_k^+) \le -L_k p(t_k), & k = 1,2,\cdots,m. \end{cases}$$

Then

$$p(t) \le p(0) \prod_{i=1}^{k}{}'(1-L_i) + \sum_{i=1}^{k}{}' \int_{t_{i-1}}^{t_i} \sigma(s)ds \prod_{j=i}^{k}{}'(1-L_j) + \int_{t_k}^{t} \sigma(s)ds, \quad t \in J_k$$

for $k = 0,1,\cdots,m$, where $\sum_{i=a}^{b}{}' \cdots = 0$, $\prod_{i=a}^{b}{}' \cdots = 1$ if $a > b$.

Proof. Note that the assertion holds for $k = 0$. If we assume that it holds for some fixed k, then we obtain

$$\begin{aligned} p(t) &\le p(t_{k+1}^+) + \textstyle\int_{t_{k+1}}^{t} \sigma(s)ds \\ &\le (1-L_{k+1}) \Bigg[p(0) \prod_{i=1}^{k}{}'(1-L_i) + \sum_{i=1}^{k}{}' \int_{t_{i-1}}^{t_i} \sigma(s)ds \prod_{j=i}^{k}{}'(1-L_j) \\ &\quad + \textstyle\int_{t_k}^{t_{k+1}} \sigma(s)ds \Bigg] + \int_{t_{k+1}}^{t} \sigma(s)ds \\ &= p(0) \prod_{i=1}^{k+1}{}'(1-L_i) + \sum_{i=1}^{k+1}{}' \int_{t_{i-1}}^{t_i} \sigma(s)ds \prod_{j=i}^{k+1}{}'(1-L_j) + \int_{t_{k+1}}^{t} \sigma(s)ds. \end{aligned}$$

This ends the proof. □

We shall first concentrate our attention to differential inequalities with positive linear operators. We shall say that a linear operator $\mathcal{L} \in C(E,E)$ is a positive linear operator if $(\mathcal{L}m)(t) \ge 0$ provided that $m(t) \ge 0$, $t \in J$.

Lemma 2. *Let $\mathcal{L} \in C(E,E)$ be a positive linear operator. Let $p \in PC^1(J,\mathbb{R})$ and*

$$\begin{cases} p'(t) \le -(\mathcal{L}p)(t), & t \in J', \\ \Delta p(t_k) \le -L_k p(t_k), & k = 1,2,\cdots,m, \\ p(0) \le rp(T), & 0 \le r \le 1. \end{cases}$$

In addition, we assume that

$$\int_0^T (\mathcal{L}1)(s)ds + \sum_{i=1}^{m} L_i \le 1 \quad \text{with} \quad 1(t) = 1,\ t \in J. \tag{2}$$

Then $p(t) \le 0$, $t \in J$.

Proof. Case 1. Assume that $p(0) \le 0$. Note that if $r = 0$, then also $p(0) \le 0$. We need to show that $p(t) \le 0$, $t \in J$. Suppose that this inequality is not true. Then, we can find $t_1^* \in (0,T]$ such that $p(t_1^*) > 0$. Put

$$p(t_0^*) = \inf_{[0,t_1^*]} p(t) = -\rho, \quad \rho \ge 0.$$

It means that $t_0^* \in J_\nu$ for some fixed ν such that $p(t_0^*) = -\rho$ or $p(t_\nu^+) = -\rho$. Below, we discuss only the situation when $p(t_0^*) = -\rho$ because in the case when $p(t_\nu^+) = -\rho$, the proof is similar.

Let $t_1^* \in J_\mu$ for some μ. Indeed, $\nu \le \mu$. Then,

$$\begin{aligned}
p(t_1^*) - p(t_0^*) &= p(t_1^*) - p(t_\mu^+) + p(t_\mu^+) - p(t_\mu) \\
&\quad + \sum_{i=\nu+2}^{\mu}{}' [p(t_i) - p(t_{i-1}^+) + p(t_{i-1}^+) - p(t_{i-1})] + p(t_{\nu+1}) - p(t_0^*) \\
&= \textstyle\int_{t_\mu}^{t_1^*} p'(s)ds + \Delta p(t_\mu) + \displaystyle\sum_{i=\nu+2}^{\mu}{}' \left[\int_{t_{i-1}}^{t_i} p'(s)ds + \Delta p(t_{i-1})\right] \\
&\quad + \textstyle\int_{t_0^*}^{t_{\nu+1}} p'(s)ds \\
&= \textstyle\int_{t_0^*}^{t_1^*} p'(s)ds + \displaystyle\sum_{i=\nu+1}^{\mu}{}' \Delta p(t_i) \\
&\le -\textstyle\int_{t_0^*}^{t_1^*} (\mathcal{L}p)(s)ds - \displaystyle\sum_{i=\nu+1}^{\mu}{}' L_i p(t_i).
\end{aligned}$$

It yields

$$\rho < \rho \left[\int_0^T (\mathcal{L}1)(s)ds + \sum_{i=1}^m L_i\right]$$

because

$$-(\mathcal{L}p)(s) \le \rho(\mathcal{L}1)(s) \quad \text{and} \quad -p(t_i) \le \rho.$$

Hence, if $\rho > 0$, then

$$1 < \int_0^T \mathcal{L}1)(s)ds + \sum_{i=1}^m L_i.$$

It is a contradiction. If $\rho = 0$, then $0 < 0$, so it is a contradiction too.

Case 2. Let $p(0) > 0$. Then also $p(T) > 0$. Let $0 < r \le 1$.

Subcase 2(i). Let $p(t) \ge 0,\ t \in J$ and $p(t) \not\equiv 0$. Then, in view of Lemma 1, we have

$$p(t) \le p(0) \prod_{i=1}^{k}{}'(1 - L_i) - \sum_{i=1}^{k}{}' \int_{t_{i-1}}^{t_i} (\mathcal{L}p)(s)ds \prod_{j=i}^{k}{}'(1 - L_j) - \int_{t_k}^{t} (\mathcal{L}p)(s)ds$$

for $t \in J_k,\ k = 0, 1, \cdots, m$. Now, in view of the boundary condition, we obtain

$$\begin{aligned}
p(0) &\le r\left[p(0)\prod_{i=1}^m (1 - L_i) - \sum_{i=1}^m \int_{t_{i-1}}^{t_i} (\mathcal{L}p)(s)ds \prod_{j=i}^m (1 - L_j) - \int_{t_m}^T (\mathcal{L}p)(s)ds\right] \\
&\le p(0) - r\left[\sum_{i=1}^m \int_{t_{i-1}}^{t_i} (\mathcal{L}p)(s)ds \prod_{j=i}^m (1 - L_j) + \int_{t_m}^T (\mathcal{L}p)(s)ds\right].
\end{aligned}$$

Hence

$$\sum_{i=1}^m \int_{t_{i-1}}^{t_i} (\mathcal{L}p)(s)ds \prod_{j=i}^m (1 - L_j) + \int_{t_m}^T (\mathcal{L}p)(s)ds \le 0.$$

It is a contradiction.

Subcase 2(ii). Let $p(t_2^*) < 0$. Put

$$p(t_0^*) = \inf_{t\in J} p(t) = -\lambda, \quad \lambda > 0.$$

It means that $t_0^* \in J_\nu$ for some fixed ν such that $p(t_0^*) = -\lambda$ or $p(t_\nu^+) = -\lambda$. Below, we discuss only the case when $p(t_0^*) = -\lambda$ because in the case when $p(t_\nu^+) = -\lambda$, the proof is similar.

Then,

$$\begin{array}{rcl} p(t_0^*) - p(T) & = & p(t_0^*) - p(t_{\nu+1}) + \sum\limits_{i=\nu+1}^{m}{}' \left[p(t_i) - p(t_i^+) + p(t_i^+) - p(t_{i+1})\right] \\ & = & \int_{t_{\nu+1}}^{t_0^*} p'(s)ds + \sum\limits_{i=\nu+1}^{m}{}' \left[\displaystyle\int_{t_{i+1}}^{t_i} p'(s)ds - \Delta p(t_i))\right] \\ & = & -\int_{t_0^*}^{T} p'(s)ds - \sum\limits_{i=\nu+1}^{m}{}' \Delta p(t_i) \\ & \geq & \int_{t_0^*}^{T} (\mathcal{L}p)(s)ds + \sum\limits_{i=\nu+1}^{m}{}' L_i p(t_i). \end{array}$$

Hence

$$-\lambda > -\lambda \left[\int_{t_0^*}^{T} (\mathcal{L}1)(s)ds + \sum_{i=\nu+1}^{m}{}' L_i\right].$$

Now, dividing by $-\lambda$ we have

$$1 < \int_0^T (\mathcal{L}1)(s)ds + \sum_{i=1}^{m} L_i.$$

It is a contradiction. The proof is complete. □

Lemma 3. *Let $\mathcal{L} \in C(E, E)$ be a positive linear operator. Let $K \in C(J, \mathbb{R}), L_k \in [0, 1)$, $k = 1, 2, \cdots, m$. Let $p \in PC^1(J, \mathbb{R})$ and*

$$(3) \qquad \left\{ \begin{array}{rcl} p'(t) & \leq & -K(t)p(t) - (\mathcal{L}p)(t), \quad t \in J', \\ \Delta p(t_k) & \leq & -L_k p(t_k), \quad k = 1, 2, \cdots, m, \\ p(0) & \leq & rp(T), \;\; 0 \leq r \leq e^{\int_0^T K(s)ds}. \end{array} \right.$$

In addition, we assume that

$$(4) \qquad \int_0^T e^{\int_0^s K(\tau)d\tau} (\mathcal{L}\bar{p})(s)ds + \sum_{i=1}^{m} L_i \leq 1 \quad \textit{with} \quad \bar{p}(t) = e^{-\int_0^t K(\tau)d\tau}.$$

Then $p(t) \leq 0$, $t \in J$.

Proof. Put

$$q(t) = e^{\int_0^t K(s)ds} p(t).$$

Note that

$$\begin{array}{rcl} q'(t) & = & e^{\int_0^t K(s)ds}[K(t)p(t) + p'(t)] \\ & \leq & -e^{\int_0^t K(s)ds}(\mathcal{L}\bar{q})(t) \quad \text{with} \quad \bar{q}(t) = e^{-\int_0^t K(s)ds} q(t). \end{array}$$

Then system 3 is replaced by

$$\begin{cases} q'(t) & \leq & -(\mathcal{L}_1 q)(t), \quad t \in J', \\ \Delta q(t_k) & \leq & -L_k q(t_k), \quad k = 1, 2, \cdots, m, \\ q(0) & \leq & r_0 q(T) \end{cases}$$

with

$$r_0 = re^{-\int_0^T K(s)ds}, \quad (\mathcal{L}_1 q)(t) = e^{\int_0^t K(\tau)d\tau}(\mathcal{L}\bar{q})(t).$$

Indeed, $0 \leq r_0 \leq 1$. Now, in view of Lemma 2, we see that $q(t) \leq 0$, $t \in J$. It means that also $p(t) \leq 0$, $t \in J$ and we have the assertion. This ends the proof. □

Remark 4. If $L_k = 0$, $k = 1, 2, \cdots, m$, then we may consider problem (3) as a problem without impulses when $\Delta p(t_k) = 0$, $k = 1, 2, \cdots, m$.

Remark 5. Let

$$(\mathcal{L}p)(t) = M(t)p(\alpha(t)), \quad M \in C(J, \mathbb{R}_+), \;\; \alpha \in C(J, J) \;\; 0 \leq \alpha(t) \leq t. \tag{5}$$

Then condition (4) takes the form

$$\begin{aligned} &\textstyle\int_0^T e^{\int_0^t K(\tau)d\tau} M(t) e^{-\int_0^{\alpha(t))} K(\tau)d\tau)} dt + \sum_{i=1}^m L_i \\ &= \textstyle\int_0^T e^{\int_{\alpha(t)}^t K(\tau)d\tau} M(t)dt + \sum_{i=1}^m L_i \leq 1. \end{aligned}$$

If we extra assume that K is a nonnegative function, then the above condition holds provided that

$$\int_0^T e^{\int_0^t K(\tau)d\tau} M(t)dt + \sum_{i=1}^m L_i \leq 1. \tag{6}$$

Observe that the last condition does not depend on α.

If $K(t) = K$, $M(t) = M \geq 0$, then condition (6) reduces to

$$\frac{M}{K}\left(e^{KT} - 1\right) + \sum_{i=1}^m L_i \leq 1 \quad \text{if} \quad K > 0 \tag{7}$$

and

$$TM + \sum_{i=1}^m L_i \leq 1 \quad \text{if} \quad K = 0. \tag{8}$$

Remark 6. Let the operator $\mathcal{L}$ be defined by

$$(\mathcal{L}p)(t) = \sum_{i=1}^{\nu} M_i(t) \int_0^t N_i(s)p(\alpha_i(s))ds,$$

where $M_i, N_i \in C(J, \mathbb{R}_+)$, $\alpha_i \in C(J, J)$ and $\alpha_i(t) \leq t$, $i = 1, 2, \cdots, \nu$.

Then condition (4) takes the form

$$\int_0^T e^{\int_0^t K(\tau)d\tau} \sum_{i=1}^{\nu} M_i(t) \int_0^t N_i(s) e^{-\int_0^{\alpha_i(s))} K(u)du} ds dt + \sum_{i=1}^{m} L_i \le 1.$$

Remark 7. In Lemma 3, it is assumed that $K \in C(J, \mathbb{R})$ and r is bounded by $0 \le r \le \rho \equiv e^{\int_0^T K(s)ds}$. Note that ρ depends on K, and ρ may be bigger or less than 1. Condition (4) is important in Lemma 3. If we assume that $K \in C(J, \mathbb{R}_+)$ and $0 \le r \le 1$, then, in the place of condition (4), we can also obtain another condition. This case is discussed in the next lemma.

Lemma 8. *Let $\mathcal{L} \in C(E, E)$ be a positive linear operator. Let $K \in C(J, \mathbb{R}_+)$, $L_k \in [0, 1)$, $k = 1, 2, \cdots, m$. Let $p \in PC^1(J, \mathbb{R})$ and*

$$\begin{cases} p'(t) & \le & -K(t)p(t) - (\mathcal{L}p)(t), \quad t \in J', \\ \Delta p(t_k) & \le & -L_k p(t_k), \quad k = 1, 2, \cdots, m, \\ p(0) & \le & rp(T), \ \ 0 \le r \le 1. \end{cases}$$

In addition, we assume that

$$\int_0^T [K(s) + (\mathcal{L}1)(s)]ds + \sum_{i=1}^{m} L_i \le 1. \tag{9}$$

Then $p(t) \le 0$, $t \in J$.

Proof. In the proof of Lemma 2, replace $(\mathcal{L}p)(s)$ by $(\mathcal{L}p)(s) + K(s)$ to obtain the assertion. □

Remark 9. Let the operator $\mathcal{L}$ be defined by (5). Then condition (9) takes the form

$$\int_0^T [K(t) + M(t)]dt + \sum_{i=1}^{m} L_i \le 1. \tag{10}$$

Remark 10. Let $M(t) = M \ge 0$, $K(t) = K \ge 0$, $t \in J$. Then condition (10) has the form

$$(K + M)T + \sum_{i=1}^{m} L_i \le 1. \tag{11}$$

Put $\gamma = \max_i(t_{i+1} - t_i)$, $i = 0, 1, \cdots, m$. Then $T < \gamma(m + 1)$. In this case, we have

$$(K + M)T + \sum_{i=1}^{m} L_i < (K + M)\gamma(m + 1) + \sum_{i=1}^{m} L_i.$$

Indeed, if

$$(K + M)\gamma(m + 1) + \sum_{i=1}^{m} L_i \le 1,$$

then also condition (11) holds. In paper ([4], Lemma 2.3), instead of the last condition we have

$$(K + M)\gamma(m + 2) + \sum_{i=1}^{m} L_i \le 1$$

with the assumption that $K > 0$.

Remark 11. Below we compare conditions (7) and (11) for some values of K, M and T. For example, if $K = M = T = 1$, then

$$\frac{M}{K}\left(e^{KT} - 1\right) < (K+M)T,$$

so condition (7) is better than condition (11).

Let $M = T = 1$, $K = 2$. Then

$$(K+M)T < \frac{M}{K}\left(e^{KT} - 1\right),$$

and in this case condition (11) is better than condition (7) provided that $0 \le r \le 1$. If $1 < r \le e^2 \approx 7.389$, then we have only condition (7) because Lemma 8 is not true in this case, see Lemmas 3 and 8.

3. LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

Now we consider the following impulsive problem

$$\text{(12)} \qquad \begin{cases} v'(t) &= -K(t)v(t) - (\mathcal{L}v)(t) + \eta(t), \quad t \in J', \\ v(t_k^+) &= (1 - L_k)v(t_k) + \gamma_k, \quad k = 1, 2, \cdots, m, \\ v(0) &= rv(T) + \beta, \ \beta \in \mathbb{R}, \ 0 \le r. \end{cases}$$

The next theorem concerns conditions under which problem (12) has a unique solution.

Theorem 12. *Let $K \in C(J, \mathbb{R})$, $\eta \in PC(J)$, $L_k \in [0, 1)$, $\gamma_k \in \mathbb{R}$, $k = 1, 2, \cdots, m$. Let $\mathcal{L} \in C(E, E)$ be a positive linear operator and let $r_1 = re^{-\int_0^T K(s)ds} \neq 1$. In addition, we assume that $\rho < 1$ with*

$$\text{(13)} \qquad \begin{aligned} \rho \;=\; \sup_t \frac{e^{-\int_0^t K(s)ds}}{|1 - r_1|} \Bigg[& r_1 \int_t^T e^{\int_0^s K(\tau)d\tau}(\mathcal{L}1)(s)ds + \int_0^t e^{\int_0^s K(\tau)d\tau}(\mathcal{L}1)(s)ds \\ & + r_1 \sum_{i=k+1}^{m}{}' L_i \, e^{\int_0^{t_i} K(s)ds} + \sum_{i=1}^{k}{}' L_i \, e^{\int_0^{t_i} K(s)ds} \Bigg] \end{aligned}$$

Then problem (12) has a unique solution $v \in PC^1(J)$.

Proof. Put

$$u(t) = e^{\int_0^t K(s)ds} v(t), \quad t \in J.$$

We see that

$$\text{(14)} \qquad \begin{aligned} u'(t) &= e^{\int_0^t K(s)ds}[K(t)v(t) + v'(t)] \\ &= e^{\int_0^t K(s)ds}\left[\eta(t) - (\mathcal{L}v)(t)\right] \equiv (\mathcal{L}^* v)(t). \end{aligned}$$

Moreover

$$\begin{aligned} u(t_k^+) &= e^{\int_0^{t_k} K(s)ds}[(1 - L_k)v(t_k) + \gamma_k] \\ &= u(t_k) + e^{\int_0^{t_k} K(s)ds}[-L_k v(t_k) + \gamma_k] \equiv u(t_k) + B_k(v). \end{aligned}$$

Then integrating (14) we have

$$u(t) = u(0) + \int_0^t (\mathcal{L}^* v)(s) ds, \quad t \in J_0.$$

Again integrating (14) we obtain

$$\begin{aligned} u(t) &= u(t_1^+) + \int_{t_1}^t (\mathcal{L}^* v)(s) ds \\ &= u(0) + \int_0^t (\mathcal{L}^* v)(s) ds + B_1 v, \quad t \in J_1. \end{aligned}$$

Repeating this process we see that

$$u(t) = u(0) + \int_0^t (\mathcal{L}^* v)(s) ds + {\sum_{i=1}^{k}}' B_i v, \quad t \in J_k, \ k = 0, 1, \cdots, m. \tag{15}$$

Now we need to use the boundary condition from (12). Note that

$$v(0) = u(0), \quad v(T) = e^{-\int_0^T K(s) ds} u(T).$$

This and (15) yield

$$\begin{aligned} v(0) &= r v(T) + \beta = r_1 u(T) + \beta \\ &= r_1 \left[v(0) + \textstyle\int_0^T (\mathcal{L}^* v)(s) ds + \displaystyle\sum_{i=1}^{m} B_i v \right] + \beta, \end{aligned}$$

so

$$v(0) = \frac{r_1}{1 - r_1} \left[\int_0^T (\mathcal{L}^* v)(s) ds + \sum_{i=1}^{m} B_i v \right] + \frac{\beta}{1 - r_1}.$$

Finally, any solution v of problem (12) satisfies the following impulsive integral equation

$$\begin{aligned} v(t) &= e^{-\int_0^t K(s) ds} u(t) \\ &= \frac{e^{-\int_0^t K(s) ds}}{1 - r_1} \left[r_1 \int_t^T (\mathcal{L}^* v)(s) ds + \int_0^t (\mathcal{L}^* v)(s) ds + \beta \right. \\ &\quad \left. + r_1 {\sum_{i=k+1}^{m}}' B_i v + {\sum_{i=1}^{k}}' B_i v \right] \end{aligned} \tag{16}$$

for $t \in J_k$, $k = 0, 1, \cdots, m$.

Denote by A the operator defined by the right–hand–side of (16). Let $x, y \in PC(J)$. Then

$$\begin{aligned}
\|Ax - Ay\| &= \sup_{t\in J} |Ax(t) - Ay(t)| \\
&= \sup_t \frac{e^{-\int_0^t K(s)ds}}{|1-r_1|} \bigg| r_1 \int_t^T e^{\int_0^s K(\tau)d\tau} (\mathcal{L}(x-y))(s)ds \\
&\quad + \int_0^t e^{\int_0^s K(\tau)d\tau} (\mathcal{L}(x-y))(s)ds + r_1 \sum_{i=k+1}^{m}{}' L_i\, e^{\int_0^{t_i} K(s)ds}[x(t_i) - y(t_i)] \\
&\quad + \sum_{i=1}^{k}{}' L_i\, e^{\int_0^{t_i} K(s)ds}[x(t_i) - y(t_i)] \bigg| \\
&\le \rho\|x - y\|.
\end{aligned}$$

This and the Banach fixed point theorem prove that problem (12) has the unique solution. ☐

Remark 13. If $K(t) = 0,\ t \in J$, then condition (13) takes the form

$$\rho = \frac{1}{|1-r|}\left[\int_0^T (\mathcal{L}1)(s)ds + \sum_{i=1}^m L_i\right] < 1.$$

Remark 14. Assume that $r \in [0,1]$, $K \in C(J, \mathbb{R}_+)$ and $K(t) \not\equiv 0$ for $t \in J$. Note that in this case $r_1 = re^{-\int_0^T K(s)ds} < 1$, so

$$\begin{aligned}
\rho_0 &= e^{-\int_0^t K(\tau)d\tau}\left[r_1 \int_t^T e^{\int_0^s K(\tau)d\tau} (\mathcal{L}1)(s)ds + \int_0^t e^{\int_0^s K(\tau)d\tau}(\mathcal{L}1)(s)ds\right] \\
&\le e^{-\int_0^t K(\tau)d\tau}\left[r_1 e^{\int_0^T K(s)ds} \int_t^T (\mathcal{L}1)(s)ds + e^{\int_0^t K(s)ds}\int_0^t (\mathcal{L}1)(s)ds\right] \\
&\le e^{-\int_0^t K(\tau)d\tau}\left[\int_t^T (\mathcal{L}1)(s)ds + e^{\int_0^t K(s)ds}\int_0^t (\mathcal{L}1)(s)ds\right] \\
&= e^{-\int_0^t K(s)ds}\int_t^T (\mathcal{L}1)(s)ds + \int_0^t (\mathcal{L}1)(s)ds \le \int_0^T (\mathcal{L}1)(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
\rho_1 &= \max_k e^{-\int_0^t K(s)ds}\left[r_1 \sum_{i=k+1}^m{}' L_i e^{\int_0^{t_i} K(s)ds} + \sum_{i=1}^k{}' L_i e^{\int_0^{t_i} K(s)ds}\right] \\
&\le \max_k e^{-\int_0^t K(s)ds}\left[r_1 e^{\int_0^T K(s)ds} \sum_{i=k+1}^m{}' L_i + e^{\int_0^t K(s)ds}\sum_{i=1}^k{}' L_i\right] \\
&\le \sum_{i=1}^m L_i.
\end{aligned}$$

Now, if we assume that

$$r_1 + \int_0^T (\mathcal{L}1)(s)ds + \sum_{i=1}^m L_i < 1, \tag{17}$$

then also $\rho < 1$. Let the operator $\mathcal{L}$ be defined by

$$(\mathcal{L}p)(t) = \sum_{i=1}^q M_i(t)p(\alpha_i(t)),\ M_j \in C(J, \mathbb{R}_+),\ \ 0 \le \alpha_j(t) \le t,\ t \in J,\ j = 1, 2, \cdots, q.$$

Then condition (17) takes the form

$$r_1 + T\sum_{i=1}^{q}\int_0^T M_i(s)ds + \sum_{i=1}^{m} L_i < 1. \tag{18}$$

Note that if $K(t) = K > 0$, $q = 1$ and $M_1(t) = M \geq 0$, then we have Lemma 5 of [2] as a special case of Theorem 12.

4. EXISTENCE OF EXTREMAL SOLUTIONS OF PROBLEM (1)

Let us introduce the following definition.

We say that $u \in PC^1(J)$ is a lower solution of (1) if

$$\begin{cases} u'(t) & \leq & (Qu)(t),\ t \in J', \\ \Delta u(t_k) & \leq & I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ g(u(0), u(T)) & \leq & 0, \end{cases} \tag{19}$$

and it is an upper solution of (1) if the above inequalities are reversed.

We assume that $y_0(t) \leq z_0(t)$, $t \in J$ and define the sector

$$[y_0, z_0]_* = \{v \in PC^1(J, \mathbb{R}) : y_0(t) \leq v(t) \leq z_0(t),\ t \in J\}.$$

A solution $x \in PC^1(J)$ of problem (1) is called minimal if $x(t) \leq y(t)$ on J for each solution y of (1), and it is maximal if the reverse inequality holds. If both minimal and maximal solutions exist we call them extremal solutions.

We prove a main result concerning the existence of the extremal solutions of problem (1) in a sector bounded by lower and upper solutions.

Theorem 15. *Let assumption H_1 hold. Moreover, assume that*

H_2 : *$y_0, z_0 \in PC^1(J)$ are lower and upper solutions of problem (1), respectively and $y_0(t) \leq z_0(t)$ on J,*

H_3 : *there exist a function $K \in C(J, \mathbb{R})$ and a positive linear operator $\mathcal{L} \in C(E, E)$ such that*

$$(Qu)(t) - (Q\bar{u})(t) \leq K(t)(\bar{u} - u) + (\mathcal{L}(\bar{u} - u))(t)$$

for $y_0(t) \leq u \leq \bar{u} \leq z_0(t)$, $t \in J$,

H_4 : *there exist constants $L_k \in [0, 1)$, $k = 1, 2, \cdots, m$ such that*

$$I_k(w(t_k)) - I_k(\bar{w}(t_k)) \leq L_k[\bar{w}(t_k) - w(t_k)], \quad k = 1, 2, \cdots, m$$

for any $w, \bar{w}$ with $y_0(t_k) \leq w(t_k) \leq \bar{w}(t_k) \leq z_0(t_k)$, $k = 1, 2, \cdots, m$,

H_5 : *there exist constants $0 \leq b$, $a > 0$ and such that*

$$g(\bar{u}, v) - g(u, v) \leq a(\bar{u} - u) \quad \textit{for} \quad y_0(0) \leq u \leq \bar{u} \leq z_0(0),$$

$$g(u, \bar{v}) - g(u, v) \leq -b(\bar{v} - v) \quad \textit{for} \quad y_0(T) \leq v \leq \bar{v} \leq z_0(T),$$

H_6 : *for* $r = \frac{b}{a}$, *condition (4) or (9) holds if* $0 \le r \le 1$ *while if* $1 < r < e^{\int_0^T K(s)ds}$ *only condition (4) holds,*

H_7 : *condition (13) holds with* $r_1 = \frac{b}{a}\, e^{-\int_0^T K(s)ds} \neq 1$.

Then, in the sector $[y_0, z_0]_*$, *there exist extremal solutions* y, z *of problem (1) such that* $y(t) \le z(t)$, $t \in J$.

Proof. Let us define sequences $\{y_n, z_n\}$ by relations

$$\begin{cases} y'_{n+1}(t) &= (Qy_n)(t) - K(t)[y_{n+1}(t) - y_n(t)] - (\mathcal{L}(y_{n+1} - y_n))(t), \quad t \in J', \\ \Delta y_{n+1}(t_k) &= I_k(y_n(t_k)) - L_k[y_{n+1}(t_k) - y_n(t_k)], \quad k = 1, 2, \cdots, m, \\ y_{n+1}(0) &= y_n(0) - \frac{1}{a} g(y_n(0), y_n(T)) + r[y_{n+1}(T) - y_n(T)] \end{cases}$$

and

$$\begin{cases} z'_{n+1}(t) &= (Qz_n)(t) - K(t)[z_{n+1}(t) - z_n(t)] - (\mathcal{L}(z_{n+1} - z_n))(t), \quad t \in J', \\ \Delta z_{n+1}(t_k) &= I_k(z_n(t_k)) - L_k[z_{n+1}(t_k) - z_n(t_k)], \quad k = 1, 2, \cdots, m, \\ z_{n+1}(0) &= z_n(0) - \frac{1}{a} g(z_n(0), z_n(T)) + r[z_{n+1}(T) - z_n(T)]. \end{cases}$$

Indeed, y_1, z_1 are well defined by Theorem 12, see assumption H_7. First, we show that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \ t \in J. \tag{20}$$

Put $p = y_0 - y_1$. Then

$$\begin{aligned} p'(t) &\le (Qy_0)(t) - (Qy_0)(t) + K(t)[y_1(t) - y_0(t)] + (\mathcal{L}(y_1 - y_0)(t) \\ &= -K(t)p(t) - (\mathcal{L}p)(t), \quad t \in J', \\ \Delta p(t_k) &\le I_k(y_0(t_k)) - I_k(y_0(t_k)) + L_k[y_1(t_k) - y_0(t_k)] \\ &= -L_k p(t_k), \quad k = 1, 2, \cdots, m, \\ p(0) &= y_0(0) - y_0(0) + \frac{1}{a} g(y_0(0), y_0(T)) - r[y_1(T) - y_0(T)] \le rp(T). \end{aligned}$$

This, Lemmas 3 or 8 (see assumption H_6), show that $y_0(t) \le y_1(t)$, $t \in J$. Similarly we can show that $z_1(t) \le z_0(t)$, $t \in J$. Now, we put $p = y_1 - z_1$.Then

$$\begin{aligned} p'(t) &= (Qy_0)(t) - (Qz_0)(t) - K(t)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\quad -(\mathcal{L}(y_1 - y_0 - z_1 + z_0))(t) \\ &\le K(t)[z_0(t) - y_0(t)] + (\mathcal{L}(z_0 - y_0))(t) - K(t)[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \\ &\quad -(\mathcal{L}(y_1 - y_0 - z_1 + z_0))(t) \\ &\quad = -K(t)p(t) - (\mathcal{L}p)(t), \quad t \in J', \\ \Delta p(t_k) &= I_k(y_0(t_k)) - I_k(z_0(t_k)) - L_k[y_1(t_k) - y_0(t_k) - z_1(t_k) + z_0(t_k)] \\ &\le L_k[z_0(t_k) - y_0(t_k)] - L_k[y_1(t_k) - y_0(t_k) - z_1(t_k) + z_0(t_k)] \\ &= -L_k p(t_k), \quad k = 1, 2, \cdots, m, \\ p(0) &= y_0(0) - z_0(0) + \frac{1}{a}[g(z_0(0), z_0(T)) - g(y_0(0), y_0(T))] \\ &\quad + r[y_1(T) - y_0(T) - z_1(T) + z_0(T)) \\ &\le rp(T) \end{aligned}$$

by assumptions H_3, H_4 and H_5. This, Lemmas 3 or 8 show $y_1(t) \le z_1(t)$, $t \in J$. It means that (20) holds.

Moreover, in view of assumptions H_2 until H_5 we see that

$$\begin{aligned}
y_1'(t) &= (Qy_0)(t) - (Qy_1)(t) + (Qy_1)(t) - K(t)[y_1(t) - y_0(t)] - (\mathcal{L}(y_1 - y_0))(t) \\
&\le (Qy_1)(t), \\
\Delta y_1'(t_k) &= I_k(y_0(t_k)) - I_k(y_1(t_k)) + I_k(y_1(t_k)) - L_k[y_1(t_k) - y_0(t_k)] \\
&\le I_k(y_1(t_k)),\ k = 1, 2, \cdots, m, \\
y_1(0) &= y_0(0) + \tfrac{1}{a}[-g(y_1(0), y_1(T)) + g(y_1(0), y_1(T)) - g(y_0(0), y_0(T)) \\
&\quad + r[y_1(T) - y_0(T)] \le y_1(0) - \tfrac{1}{a} g(y_1(0), y_1(T)),
\end{aligned}$$

so $g(y_1(0), y_1(T)) \le 0$. It shows that y_1 is a lower solution of problem (1). Similarly we can show that z_1 is an upper solution of (1).

Now, using the mathematical induction, we can show that

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le y_{n+1}(t) \le z_{n+1}(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t)$$

for $t \in J$ and $n = 0, 1, \cdots$.

Note that the sequences $\{y_n, z_n\}$ are bounded and equicontinuous on J. Since Q, I_k, g are continuous, so using the Arzela–Ascoli theorem we see that the sequences $\{y_n, z_n\}$ converge to their limit functions $y, z \in PC^1(J)$. Indeed, y, z are solutions of problem (1) and $y_0(t) \le y(t) \le z(t) \le z_0(t)$, $t \in J$.

It remains to show that y, z are extremal solutions of problem (1) in the sector $[y_0, z_0]_*$. To prove it we assume that u is any solution of (1) in this sector. Then, by induction in n we can show that

$$y_k(t) \le u(t) \le z_n(t),\ t \in J,\ n = 0, 1, \cdots .$$

Finally, if $n \to \infty$ then the assertion results from the last relation. □

Remark 16. According to assumption H_7, we see that $r_1 \ne 1$. Note that if the first condition in assumption H_5 holds with $a > 0$, then it is also true for any $\bar{a}$ bigger than a. It means that the relation $r_1 < 1$ can be always satisfied. Moreover, choosing $\bar{a}$ we can obtain a very small number for r, so also in this case the value of r_1 is a very small number.

Example 17. Put $J = [0, 1]$, $t_1 = \frac{1}{2}$, $J' = J \setminus \{t_1\}$. Consider the following problem

$$\left\{\begin{aligned}
x'(t) &= -A(t)x(t) + Be^{\sin x\left(\frac{1}{2}t\right)} - C\int_0^t x(s)ds \equiv (Qx)(t), \quad t \in J', \\
\Delta x(t_1) &= -Lx(t_1),\ \ 0 \le L < 1, \\
0 &= 2x^2(0) - x(1) - 1.
\end{aligned}\right. \tag{21}$$

Assume that $A \in C(J, \mathbb{R})$, $B > 0$, $C > 0$ and

$$\frac{1}{4} < e^{\int_0^1 A(s)ds}, \quad 5B \le 2A(t),\ t \in J. \tag{22}$$

Put $y_0(t) = 0$, $z_0(t) = 1$, $t \in J$. Then

$$\begin{aligned}
(Qy_0)(t) &= B > 0 = y_0'(t),\\
(Qz_0)(t) &= -A(t) + Be^{\sin 1} - Ct < -A(t) + \tfrac{5}{2}B \le 0 = z_0'(t),\\
\Delta y_0(t_1) &= 0 = -Ly_0(t_1),\\
\Delta z_0(t_1) &= 0 \ge -L = -Lz_0(t_1),\\
g(y_0(0), y_0(1)) &= g(0,0) = -1 < 0,\\
g(z_0(0), z_0(1)) &= g(1,1) = 0.
\end{aligned}$$

It proves that y_1, z_1 are lower and upper solutions of problem (21), respectively. Moreover,

$$K(t) = A(t), \quad (\mathcal{L}u)(t) = C\int_0^t u(s)ds, \quad L = L_1, \quad a = 4, \quad b = 1$$

and

$$r_1 = \frac{1}{4}e^{-\int_0^1 A(s)ds} < 1.$$

If we extra assume that $\rho < 1$ [ρ from condition (13)] and

$$C\int_0^1 e^{\int_0^t A(\tau)d\tau}\int_0^t e^{-\int_0^s A(\tau)d\tau}dsdt + L \le 1, \tag{23}$$

or

$$\int_0^1 [A(s) + Cs]ds + L \le 1, \tag{24}$$

then problem (21) has, in the sector $[y_0, z_0]_*$ the extremal solutions, by Theorem 15.

Let $A(s) = A > 0$ and assume that

$$\frac{1}{4}e^{-A} + \frac{1}{2}C + L < 1,$$

then $\rho < 1$, see Remark 14. For $A = 1, C = 1, B \le \frac{2}{5}$, problem (21) has a solution provided that $L \le 0.408$.

Note that, in assumption H_5, we can put $b = 0$. It means that g is nonincreasing with respect to the second variable. In this case $r_1 = 0$, so condition (13) is weaker in comparing with the case when $b > 0$. Below, we try to discuss this problem. Theorem 15 says that sequences $\{y_n, z_n\}$ converge to limit functions y, z, respectively, and y, z are solutions of problem (1). It means that elements y_n, z_n are approximate solutions of problem (1). Note that y_{n+1} and z_{n+1} are solutions of corresponding linear impulsive problems with the boundary conditions

$$y_{n+1}(0) = ry_{n+1}(T) + \beta_n \quad \text{or} \quad z_{n+1}(0) = rz_{n+1}(T) + \gamma_n.$$

If we put $b = 0$, then $r = 0$, so the boundary conditions reduce to the initial conditions. Therefore, we construct the next two sequences $\{v_n, w_n\}$ by relations: $v_0 = y_0$, $w_0 =$

z_0, $t \in J$, and

$$\begin{cases} v'_{n+1}(t) &=& (Qy_n)(t) - K(t)[v_{n+1}(t) - y_n(t)] - (\mathcal{L}(v_{n+1} - y_n))(t), \quad t \in J', \\ \Delta v_{n+1}(t_k) &=& I_k(y_n(t_k)) - L_k[v_{n+1}(t_k) - y_n(t_k)], \quad k = 1, 2, \cdots, m, \\ v_{n+1}(0) &=& y_n(0) - \frac{1}{a} g(y_n(0), y_n(T)) \end{cases}$$

and

$$\begin{cases} w'_{n+1}(t) &=& (Qz_n)(t) - K(t)[w_{n+1}(t) - z_n(t)] - (\mathcal{L}(w_{n+1} - z_n))(t), \quad t \in J', \\ \Delta w_{n+1}(t_k) &=& I_k(z_n(t_k)) - L_k[w_{n+1}(t_k) - z_n(t_k)], \quad k = 1, 2, \cdots, m, \\ w_{n+1}(0) &=& z_n(0) - \frac{1}{a} g(z_n(0), z_n(T)). \end{cases}$$

Elements y_n, z_n are defined as in the proof of Theorem 15.

In the next theorem we show that the element v_n is between y_{n-1} and y_n; similarly w_n is between z_n and z_{n-1}.

Theorem 18. *Assume that all assumptions of Theorem 15 are satisfied. Then*

$$y_{n-1}(t) \le v_n(t) \le y_n(t) \le z_n(t) \le w_n(t) \le z_{n-1}(t) \tag{25}$$

for $t \in J$ and $n = 1, 2, \cdots$.

Proof. It is easy to see that (25) holds for $n = 1$. Assume that (25) holds for $n = i$. Put $p = y_i - v_{i+1}$, $q = v_{i+1} - y_{i+1}$. Then, knowing that y_i is a lower solution of (1), we have

$$\begin{aligned} p'(t) &\le (Qy_i)(t) - (Qy_i)(t) + K(t)[v_{i+1}(t) - y_i(t)] + (\mathcal{L}(v_{i+1} - y_i))(t) \\ &= -K(t)p(t) - (\mathcal{L}p)(t), \ t \in J', \\ \Delta p(t_k) &\le I_k(y_i(t_k)) - I_k(y_i(t_k)) + L_k[v_{i+1}(t_k) - y_i(t_k)] \\ &= -L_k p(t_k), \ k = 1, 2, \cdots, m, \\ p(0) &= y_i(0) - y_i(0) + \frac{1}{a} g(y_i(0), y_i(T)) \le 0, \end{aligned}$$

and

$$\begin{aligned} q'(t) &= (Qy_i)(t) - (Qy_i)(t) - K(t)[v_{i+1}(t) - y_i(t)] - (\mathcal{L}(v_{i+1} - y_i)(t) \\ &\quad + K(t)[y_{i+1}(t) - y_i(t)] + (\mathcal{L}(y_{i+1} - y_i))(t) \\ &= -K(t)q(t) - (\mathcal{L}q)(t), \ t \in J', \\ \Delta q(t_k) &= I_k(y_i(t_k)) - I_k(y_i(t_k)) - L_k[v_{i+1}(t_k) - y_i(t_k)] + L_k[y_{i+1}(t) - y_i(t)] \\ &= -L_k q(t_k), \ k = 1, 2, \cdots, m, \\ q(0) &= y_i(0) - \frac{1}{a} g(y_i(0), y_i(T)) - y_i(0) + \frac{1}{a} g(y_i(0), y_i(T)) - r[y_{i+1}(T) - y_i(T)] \\ &= r[y_i(T) - y_{i+1}(T)] \le 0. \end{aligned}$$

By Lemmas 3 or 8, $y_i(t) \le v_{i+1}(t) \le y_{i+1}(t)$, $t \in J$. Similarly, we can show that $z_{i+1}(t) \le w_{i+1}(t) \le z_i(t)$, $t \in J$. This and mathematical induction prove that the assertion holds. This ends the proof. $\square$

5. EXISTENCE OF A UNIQUE SOLUTION OF PROBLEM (1)

The result of Theorem 15 ensures the existence of the extremal solutions y, z of problem (1) in the sector $[y_0, z_0]_*$, and $y(t) \le z(t)$ on J. Basing on this result we give sufficient conditions for the existence of a unique solution of problem (1). This is the content of the next theorem.

Theorem 19. *Assume that all assumptions of Theorem 15 are satisfied. In addition, we assume that*

H_3' : *there exists a function* $K_1 \in C(J, \mathbb{R}),\ K(t) + K_1(t) \ge 0,\ t \in J$ *and such that*

$$(Qu)(t) - (Q\bar{u})(t) \ge -K_1(t)(\bar{u} - u)$$

for $y_0(t) \le u \le \bar{u} \le z_0(t),\ t \in J,$

H_4' : *there exist constants* $M_k,\ L_k + M_k \ge 0,\ k = 1, 2, \cdots, m$ *such that*

$$I_k(w(t_k)) - I_k(\bar{w}(t_k)) \ge -M_k[\bar{w}(t_k) - w(t_k)], \quad k = 1, 2, \cdots, m$$

for any $w, \bar{w}$ *with* $y_0(t_k) \le w(t_k) \le \bar{w}(t_k) \le z_0(t_k),\ k = 1, 2, \cdots, m,$

H_5' : *there exist constants* $0 < a_1 \le a,\ b_1 \ge b$ *and such that*

$$g(\bar{u}, \bar{v}) - g(u, v) \ge a_1(\bar{u} - u) - b_1(\bar{v} - v)$$

for $y_0(0) \le u \le \bar{u} \le z_0(0),\ y_0(T) \le v \le \bar{v} \le z_0(T)$, *and*

$$b_1 e^{\int_0^T K_1(s)ds} \prod_{i=1}^{m} (1 + M_i) < a_1. \tag{26}$$

Then, in the sector $[y_0, z_0]_*$, *problem (1) has a unique solution.*

Proof. By Theorem 15, we know that problem (1) has, in the sector $[y_0, z_0]_*$ the extremal solutions y, z and $y(t) \le z(t),\ t \in J$. To show that $y = z$, we put $p = z - y$. In view of assumptions H_3' until H_5', we have

$$\begin{aligned} p'(t) &= (Qz)(t) - (Qy)(t) \le K_1(t)p(t),\ t \in J', \\ \Delta p(t_k) &= I_k(z(t_k)) - I_k(y(t_k)) \le M_k p(t_k),\ k = 1, 2, \cdots, m, \\ 0 &= g(z(0), z(T)) - g(y(0), y(T)) \ge a_1 p(0) - b_1 p(T). \end{aligned}$$

Now, by induction in n, we can show that

$$p(t) \le e^{\int_0^t K_1(s)ds} p(0) \prod_{i=1}^{k}{}' (1 + M_i), \quad t \in J_k,\ k = 0, 1, \cdots, m. \tag{27}$$

Adding to this the boundary condition $b_1 p(T) \ge a_1 p(0)$ assuming first that $b_1 > 0$, we see that

$$p(0)\left[\frac{a_1}{b_1} - e^{\int_0^T K_1(s)ds} \prod_{i=1}^{m} (1 + M_i)\right] \le 0.$$

Hence $p(0) \le 0$, by condition (26). If $b_1 = 0$, then $p(0) \le 0$. It proves that $p(t) \le 0$, $t \in J$, so $y = z$. This ends the proof. ∎

Remark 20. Observe that if g is a function of the first variable only, then $b_1 = b = 0$.

6. EXISTENCE OF QUASI–SOLUTIONS OF PROBLEM (1)

Let us introduce the following definition.

We say that $u, w \in PC^1(J)$ are coupled lower–upper solutions of (1) if

$$\begin{cases} u'(t) & \leq & (Qu)(t), \ t \in J', \\ \Delta u(t_k) & \leq & I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ g(u(0), w(T)) & \leq & 0, \end{cases}$$

$$\begin{cases} w'(t) & \geq & (Qw)(t), \ t \in J', \\ \Delta w(t_k) & \geq & I_k(w(t_k)), \quad k = 1, 2, \cdots, m, \\ g(w(0), u(T)) & \geq & 0. \end{cases}$$

We say that functions $y, z \in PC^1(J)$ are quasi–solutions of problem (1) if they are solutions of the system

$$\begin{cases} \begin{cases} y'(t) & = & (Qy)(t), \quad t \in J', \\ \Delta y(t_k) & = & I_k(y(t_k)), \quad k = 1, 2, \cdots, m, \\ g(y(0), z(T)) & = & 0, \end{cases} \\ \begin{cases} z'(t) & = & (Qz)(t), \quad t \in J', \\ \Delta z(t_k) & = & I_k(z(t_k)), \quad k = 1, 2, \cdots, m, \\ g(z(0), y(T)) & = & 0. \end{cases} \end{cases}$$

Now we give sufficient conditions when problem (1) has quasi–solutions.

Theorem 21. *Let assumption H_1 hold. Moreover, assume that*

H_2'' : *$y_0, z_0 \in PC^1(J)$ are coupled lower–upper solutions of problem (1), respectively and $y_0(t) \leq z_0(t)$ on J.*

Let assumptions H_3 and H_4 hold. In addition, we assume that

H_8 : *there exists a constant $a > 0$ such that*

$$g(\bar{u}, v) - g(u, v) \leq a(\bar{u} - u) \quad \text{for} \quad y_0(0) \leq u \leq \bar{u} \leq z_0(0),$$

$$g(u, v) - g(u, \bar{v}) \leq 0 \quad \text{for} \quad y_0(T) \leq v \leq \bar{v} \leq z_0(T),$$

H_9 : *condition (13) holds with $r_1 = 0$,*

H_{10} *condition (4) or (9) holds.*

Then,

(a) *in the sector $[y_0, z_0]_*$, there exist quasi–solutions y, z of problem (1) such that $y(t) \leq z(t)$, $t \in J$,*

(b) *if $u \in [y_0, z_0]_*$ is any solution of (1) then $y(t) \leq u(t) \leq z(t)$ on J.*

Proof. Let us define sequences $\{y_n, z_n\}$ by relations

$$\begin{cases} y'_{n+1}(t) &=& (Qy_n)(t) - K(t)[y_{n+1}(t) - y_n(t)] - (\mathcal{L}(y_{n+1} - y_n))(t), \quad t \in J', \\ \Delta y_{n+1}(t_k) &=& I_k(y_n(t_k)) - L_k[y_{n+1}(t_k) - y_n(t_k)], \quad k = 1, 2, \cdots, m, \\ y_{n+1}(0) &=& y_n(0) - \frac{1}{a} g(y_n(0), z_n(T)) \end{cases}$$

and

$$\begin{cases} z'_{n+1}(t) &=& (Qz_n)(t) - K(t)[z_{n+1}(t) - z_n(t)] - (\mathcal{L}(z_{n+1} - z_n))(t), \quad t \in J', \\ \Delta z_{n+1}(t_k) &=& I_k(z_n(t_k)) - L_k[z_{n+1}(t_k) - z_n(t_k)], \quad k = 1, 2, \cdots, m, \\ z_{n+1}(0) &=& z_n(0) - \frac{1}{a} g(z_n(0), y_n(T)). \end{cases}$$

Similarly as in Theorem 15 we can prove part (a).

It remains to prove part (b). Let $u \in [y_0, z_0]_*$ be any solution of problem (1). By induction in n, we can show that

$$y_n(t) \le u(t) \le z_n(t), \ \ t \in J, \ n = 0, 1, \cdots .$$

Now, if $n \to \infty$, then from the above inequality we have the assertion. $\square$

Remark 22. Note that here $b = 0$ (see assumption H_8) so $r = 0$ in the definition of sequences $\{y_n, z_n\}$ in comparing with sequences $\{y_n, z_n\}$ from Theorem 15 where r could be bigger than zero.

Example 23. For $J = [0, 1], \ t_1 = \frac{1}{2}$, consider the problem

$$\text{(28)} \quad \begin{cases} x'(t) &=& A\cos^2 x(t) + B\cos^2 x\left(\frac{1}{2}t\right) \equiv (Qx)(t), \quad t \in J' = J \setminus \{t_1\}, \\ \Delta x(t_1) &=& \frac{1}{6}\sin^2 \frac{x(t_1)}{6}, \\ 0 &=& ex(0) - x^2(0) + \frac{1}{5}x(1) - 1 \equiv g(x(0), x(1)), \end{cases}$$

where $A, B \ge 0, \ A + B < \frac{35}{36}$.

Put

$$y_0(t) = 0, \ t \in J, \qquad z_0(t) = \begin{cases} 2t + 1, & t \in [0, t_1], \\ 2t + 2, & t \in (t_1, 1]. \end{cases}$$

Then

$$\begin{aligned} (Qy_0)(t) &= A + B \ge 0 = y'_0(t), \quad t \in J', \\ (Qz_0)(t) &\le A + B < 2 = z'_0(t), \quad t \in J', \\ \Delta y_0(t_1) &= 0 = \frac{1}{6}\sin^2 \frac{y_0(t_1)}{6}, \\ \Delta z_0(t_1) &= 1 > \frac{1}{6}\sin^2 \frac{z_0(t_1)}{6} \approx 0.02, \\ g(y_0(0), z_0(1)) &= g\,(0, 4) = \frac{4}{5} - 1 < 0, \\ g(z_0(0), y_0(1)) &= g(1, 0) = e - 1 - 1 > 0. \end{aligned}$$

It shows that y_0, z_0 are coupled lower–upper solutions of problem (28).

We can show that

$$K(t) = A, \quad (\mathcal{L}p)(t) = Bp\left(\frac{1}{2}t\right), \quad L_1 = \frac{1}{36} \quad .$$

Function g satisfies assumption H_8 with $a = e$. Conditions (9) and (13) are satisfied, see Remark 14. It proves that problem (28) has the quasi–solutions y, z in the sector $[y_0, z_0]_*$, by Theorem 21. Moreover, any solution u of problem (28) satisfies the relation $y(t) \le u(t) \le z(t)$, $t \in J$.

Next we are going to concentrate our attention on a result which ensures the existence of a solution of problem (1) knowing that problem (1) has the quasi–solutions. By a similar way as Theorem 19, we can prove the following

Theorem 24. *Let all assumptions of Theorem 21 hold. Let assumptions H_3', H_4' hold. In addition, we assume that condition (26) holds for a_1, b_1 defined below:*

H_5'' : *there exist constants $0 < a_1 \le a$, $b_1 \ge 0$ and such that*

$$g(\bar{u}, v) - g(u, \bar{v}) \ge a_1(\bar{u} - u) - b_1(\bar{v} - v)$$

for $y_0(0) \le u \le \bar{u} \le z_0(0)$, $y_0(T) \le v \le \bar{v} \le z_0(T)$.

Then, in the sector $[y_0, z_0]_*$, problem (1) has a solution.

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