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FUZZY SOLUTIONS FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, a fixed point theorem for absolute retract is used to investigate the existence of fuzzy solutions for first and second order impulsive ordinary differential equations.

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1. INTRODUCTION

This paper is concerned with the existence of fuzzy solutions for initial value problems for first and second order ordinary differential equations with impulsive effects. We consider the first order initial value problem (IVP for short)

$$y'(t) = f(t, y(t)), \quad t \in J = [0, T], \quad t \neq t_k, \quad k = 1, \ldots, m, \tag{1.1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.2}$$

$$y(0) = a \in E^n, \tag{1.3}$$

where we let E^n is the set of fuzzy real numbers and $f : J \times E^n \to E^n$, $I_k : E^n \to E^n$, $k = 1, \ldots, m$ are given functions, $t_0 = 0 < t_1 < \ldots < t_m < t_{m+1} = T$, $a \in E^n$ and $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

We also study the second order IVP

$$y''(t) = f(t, y(t)), \; t \in J := [0, T], \quad t \neq t_k, \quad k = 1, \ldots, m, \tag{1.4}$$

$$y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.5}$$

$$y'(t_k^+) = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.6}$$

$$y(0) = a, \; y'(0) = b, \tag{1.7}$$

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where f, I_k, and a are as in the problem (1.1)–(1.3), $\overline{I}_k : E^n \to E^n$ and $b \in E^n$.

Differential equations with impulses are a basic tool to study evolution processes that are subjected to abrupt changes in their state. Such equations arise naturally from a wide variety of applications, such as space-craft control, inspection processes in operations research, drug administration, and threshold theory in biology. See the monographs of Benchohra *et al.* [10], Lakshmikantham *et al.* [24], and Samoilenko and Perestyuk [38].

Kandel and Byatt [17] introduced the concept of fuzzy differential equations. Later it was applied in fuzzy processes and fuzzy dynamical systems. Until now, there are many works on fuzzy differential equations, see for instance, the monograph of Lakshmikantham and Mohapatra [26] and the references therein, and the papers [1, 3, 4, 5, 6, 11, 18, 29, 33, 39]. Lakshmikantham and Tolstonogov [28] (see also [23]) showed the connection between the solutions of fuzzy differential equation and the set differential equation that is generated from it. Park *et al* studied fuzzy differential equations with nonlocal conditions [34]. Recently, Lakshmikantham and McRae [25] have initiated the study of fuzzy impulsive differential equations, see also the paper by Vatsala [41].

There are several approaches to defining a solution for a fuzzy differential equation: Hukuhara approach [18, 20, 29], Differential Inclusions [16, 37], Quasiflows and Differential Equations in metric spaces [22, 27]. Other approaches can be found in [9, 12].

There are not too many papers on impulsive fuzzy differential equations, but some basic results on impulsive fuzzy differential equations can be found in [40, 15, 25, 30, 41].

For recent works on fuzzy differential equations, we refer, for instance, to [2, 7, 8, 13, 20, 30, 32, 35].

In this paper we study the existence of fuzzy solutions for impulsive differential equations. Our approach relies on the absolute retract fixed point theorem [14]. Our results complement the few existence results devoted to impulsive fuzzy differential equations.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. In the following $CC(\mathbb{R}^n)$ denotes the set of all nonempty compact, convex subsets of $\mathbb{R}^n$. Denote by

$$E^n = \{y : \mathbb{R}^n \to [0,1] \text{ such that satisfies } (i) \text{ to } (iv) \text{ mentioned below}\},$$

(i) y is normal, that there exists an $x_0 \in \mathbb{R}^n$ such that $y(x_0) = 1$;

(ii) y is fuzzy convex, that is for $x, z \in \mathbb{R}^n$ and $0 \le \lambda \le 1$,

$$y(\lambda x + (1-\lambda)z) \ge \min[y(x), y(z)];$$

(iii) y is upper semi-continuous;

(iv) $[y]^0 = \overline{\{x \in \mathbb{R}^n : y(x) > 0\}}$ is compact. Here $\overline{A}$ denotes the closure of the subset A.

For $0 < \alpha \le 1$, we denote $[y]^\alpha = \{x \in \mathbb{R}^n : y(x) \ge \alpha\}$. Then from (i) to (iv), it follows that the $\alpha-$level sets $[y]^\alpha \in CC(\mathbb{R}^n)$. If $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, then, according to Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the function defined by

$$g(y, \overline{y})(z) = \sup_{z=g(x,\bar{z})} \min\{y(x), \overline{y}(\bar{z})\}.$$

If g is continuous, then it is well known that

$$[g(y, \overline{y})]^\alpha = g([y]^\alpha, [\overline{y}]^\alpha) \text{ for all } y, \overline{y} \in E^n, \text{ and } 0 \le \alpha \le 1.$$

Especially for addition and scalar multiplication, we have

$$[y + \overline{y}]^\alpha = [y]^\alpha + [\overline{y}]^\alpha, \quad [ky]^\alpha = k[y]^\alpha,$$

where $y, \overline{y} \in E^n, \ k \in \mathbb{R}, \ \ 0 \le \alpha \le 1$.

Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} \|a-b\|, \sup_{b\in B}\inf_{a\in A} \|a-b\|\right\}$$

where $\| \, . \, \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then $(CC(\mathbb{R}^n), H_d)$ is a complete and separable metric space [36]. We define the supremum metric d_∞ on E^n by

$$d_\infty(u, \overline{u}) = \sup_{0<\alpha\le 1} H_d([u]^\alpha, [\overline{u}]^\alpha)$$

for all $u, \overline{u} \in E^n$. (E^n, d_∞) is a complete metric space. Also for all $u, v, w \in E^n$ and $\lambda \in \mathbb{R}$ we have

$$d_\infty(u + w, v + w) = d_\infty(u, v)$$

and

$$d_\infty(\lambda u, \lambda v) = |\lambda| d_\infty(u, v).$$

We define $\widehat{0} \in E^n$ as $\widehat{0}(x) = 1$ if $x = 0$ and $\widehat{0}(x) = 0$ if $x \neq 0$. It is well known that (E^n, d_∞) can be embedded isometrically as a cone in a Banach space X, i.e. there exists an embedding $j : E^n \to X$ (see also [19]) defined by

$$j(u) = \langle u, \widehat{0}\rangle \ \text{ where } u \in E^n;$$

here $\langle ., \rangle$ is defined in [36]. Notice also that

$$\|\langle u, v\rangle\|_X = d_\infty(u, v) \ \text{ for } u, v \in E^n,$$

so in particular

$$\|ju\|_X = d_\infty(u, \widehat{0}) \text{ for } u \in E^n.$$

The supremum metric H_1 on the space of continuous fuzzy valued functions from J into E^n denoted by $C(J, E^n)$ is defined by

$$H_1(w, \overline{w}) = \sup_{t\in J} d_\infty(w(t), \overline{w}(t)).$$

$(C(J,E^n), H_1)$ is a complete metric space. It is well known that $C(J,E^n)$ is a complete metric space. Now since $j : E^n \to C \subset X$ we can define a map $\bar{J} : C(J,E^n) \to C(J,X)$ by

$$[\bar{J}x](t) = j(x(t)) = jx(t) \text{ for } t \in J;$$

here $x \in C(J,E^n)$ (note if $x \in C(J,E^n)$ and $t_0,\ t \in J$, then by definition of j we have

$$\|[\bar{J}x](t) - [\bar{J}x](t_0)\|_{C(J,X)} = \sup_{t\in J} \|jx(t) - jx(t_0)\| = d_\infty(x(t), x(t_0)).$$

Also, it is easy to check that

$$\bar{J} : C(J,E^n) \to \bar{J}(C(J,E^n)),$$

is a homeomorphism. To see that $\bar{J}$ is continuous let $x_n, n \in \mathbb{N},\ x \in C(J,E^n)$ with

$$H_1(x_n, x) = \sup_{t\in J} d_\infty(x_n(t), x(t)) \text{ for } n \text{ large.}$$

Then

$$\|\bar{J}x_n - \bar{J}x\|_{C(J,X)} = \sup_{t\in J} \|jx_n(t) - jx(t)\| = \sup_{t\in J} d_\infty(x_n(t), x(t)) \to 0 \text{ for } n \text{ large,}$$

so $\bar{J}$ is continuous. To see that $\bar{J}^{-1}$ is continuous let $y_n \in \bar{J}(C(J,E^n))$ with $\|y_n - y\|_{C(J,X)} \to 0$ as $n \to \infty$. Then there exists $x_n,\ x \in C(J,E^n)$ with $\bar{J}x_n = y_n$ and $y = \bar{J}x$. Thus,

$$\begin{aligned} H_1(\bar{J}^{-1}y_n, \bar{J}^{-1}y) &= \sup_{t\in J} d_\infty(\bar{J}^{-1}y_n(t), \bar{J}^{-1}y(t)) \\ &= \sup_{t\in[0,T]} d_\infty(y_n(t), y(t)) \\ &= \sup_{t\in[0,T]} \|jy_n(t) - jy(t))\|_X \\ &= \sup_{t\in[0,T]} \|y_n(t) - y(t)\|_X \\ &\qquad \to 0 \text{ as } n \to \infty, \end{aligned}$$

since $y_n(t) = jx_n(t)$ and $y(t) = jx(t)$. Thus $\bar{J}^{-1}$ is continuous.

Definition 2.1. A map $f : J \to E^n$ is strongly measurable if, for all $\alpha \in [0,1]$, the multi-valued map $f_\alpha : J \to CC(\mathbb{R}^n)$ defined by

$$f_\alpha(t) = [f(t)]^\alpha$$

is Lebesgue measurable, when $CC(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 2.2. A map $f : J \to E^n$ is called levelwise continuous at $t_0 \in J$ if the multi-valued map $f_\alpha(t) = [f(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0, 1]$.

A map $f : J \to E^n$ is called integrably bounded if there exists an integrable function h such that $\|y\| \le h(t)$ for all $y \in f_0(t)$.

Definition 2.3. Let $f : J \to E^n$. The integral of f over J, denoted $\int_0^T f(t)dt$ is defined by the equation

$$\begin{aligned}\left(\int_0^T f(t)dt\right)^\alpha &= \int_0^T f_\alpha(t)dt \\ &= \left\{\int_0^T v(t)dt \mid v : J \to R^n \text{ is a measurable selection for } f_\alpha\right\}\end{aligned}$$

for all $\alpha \in (0, 1]$.

A strongly measurable and integrably bounded map $f : J \to E^n$ is said to be *integrable* over J, if $\int_0^T f(t)dt \in E^n$.

If $f : J \to E^n$ is measurable and integrably bounded, then f is integrable.

Definition 2.4. A map $f : J \to E^n$ is called differentiable at $t_0 \in J$ if there exists a $f'(t_0) \in E^n$ such that the limits

$$\lim_{h\to 0^+} \frac{f(t_0 + h) - f(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(t_0) - f(t_0 - h)}{h}$$

exist and are equal to $f'(t_0)$. Here the limit is taken in the metric space (E^n, H_d). At the end points of J, we consider only the one-side derivatives.

If $f : J \to E^n$ is differentiable at $t_0 \in J$, then we say that $f'(t_0)$ is the *fuzzy derivative* of $f(t)$ at the point t_0. For the concepts of fuzzy measurability and fuzzy continuity we refer to [21].

Definition 2.5. A map $f : J \times E^n \to E^n$ is called levelwise continuous at point $(t_0, x_0) \in J \times E^n$ provided, for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that

$$H_d\left([f(t, x)]^\alpha, [f(t, x_0)]^\alpha\right) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $H_d\left([x]^\alpha, [x_0]^\alpha\right) < \delta(\epsilon, \alpha)$ for all $t \in J, x \in E^n$.

We restate the fixed point result needed in Sections 3 and 4. Its proof can be found in [14].

Theorem 2.6. . *Let $X \in AR$ and $F : X \to X$ a continuous and completely continuous map. Then F has a fixed point.*

Remark 2.7. A space Z is called an absolute retract (written $Z \in AR$) if Z is metrizable and for any metrizable space W and any embedding $h : Z \to W$ the set $h(Z)$ is a retract of W.

3. FIRST ORDER FUZZY IMPULSIVE DIFFERENTIAL EQUATIONS

In this section we are concerned with the existence of fuzzy solutions for problem (1.1)–(1.3). In order to define the solution of (1.1)–(1.3) the following space will be used:

$$\begin{aligned} PC \ &= \{y : [0,T] \to E^n : y \in C(J_k, E^n), \lim_{t\to t_k-} y(t) = y(t_k), \\ &\quad \text{and } \ \lim_{t\to t_k+} y(t) \ \text{ exists}, \ k = 1, \ldots, m\}. \end{aligned}$$

Here $J_k = (t_k, t_{k+1}], \ k = 0, \ldots, m$ with $t_0 = 0$ and $t_{m+1} = T$.

Definition 3.1. A function $y \in C^1(J\backslash\{t_k\}, E^n) \cap PC$ is said to be a solution of (1.1)–(1.3) if y satisfies the equation $y'(t) = f(t, y(t))$ on $J, \ t \neq t_k, \ k = 1, \ldots, m$ and the conditions $y(t_k^+) = I_k(y(t_k^-)), \ k = 1, \ldots, m$ and $y(0) = a$.

Theorem 3.2. *Assume that*

(H1) *There exists a continuous nondecreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$ *and a continuous function* $p : J \to \mathbb{R}_+$ *such that*

$$d_\infty(f(t, y), \widehat{0}) \leq p(t)\psi(d_\infty(y, \widehat{0})) \ \ \textit{for} \ \ t \in J \ \ \textit{and each} \ \ y \in E^n$$

with

$$\int_{t_k}^{t_{k+1}} p(s)ds < \int_{d_\infty(I_k(y(t_k)),\widehat{0})}^{T} \frac{du}{\psi(u)}, \quad k = 0, \ldots, m \quad \textit{and} \quad I_0 = a;$$

(H2) *For each* $t \in J_k, \quad k = 0, \ldots, m$ *the set*

$$\Big\{I_k(y(t_k)) + \int_{t_k}^{t} f(s, y(s))ds : y \in \mathcal{A}_k\Big\},$$

is a totally bounded subset of E^n, *where*

$$\mathcal{A}_k = \{y \in C(J_k, E^n) : d_\infty(y(t), \widehat{0}) \leq a_k(t), \ t \in J_k\},$$

$$a_k(t) = M_k^{-1}\left(\int_{t_k}^{t} p(s)ds\right)$$

and

$$M_k(z) = \int_{d_\infty(I_k(y(t_k)),\widehat{0})}^{z} \frac{du}{\psi(u)}.$$

Then the IVP (1.1)-(1.3) has at least one fuzzy solution on $[0, T]$.

Proof. We shall proceed on each subinterval $[t_k, t_{k+1}], \ k = 0, \ldots, m$. The proof will be given in several steps.

Step 1: Consider the following problem

$$y'(t)) = f(t, y(t)), \quad t \in [0, t_1], \tag{3.1}$$

$$y(0) = a \in E^n. \tag{3.2}$$

Transform the problem (3.1)–(3.2) into a fixed point problem. Consider the operator $N : C([0, t_1], E^n) \to C([0, t_1], E^n)$ defined by:

$$N(y)(t) = a + \int_0^t f(s, y(s))ds.$$

Let

$$\mathcal{A}_0 \cong \mathcal{B}_0 \equiv \{\bar{J}y \in C(J_0, E^n) : y \in C(J_0, E^n) \text{ and } d_\infty(y(t), \widehat{0}) \leq a_0(t), \ t \in J_0\}.$$

Clearly, $\mathcal{B}_0$ is a convex subset of the Banach space $C(J_0, X)$, so in particular $\mathcal{B}_0$ is an absolute retract. As a result $\mathcal{A}_0$ is an absolute retract. We shall show that the operator N maps $\mathcal{A}_0$ into $\mathcal{A}_0$ and is continuous and completely continuous.

Claim 1: $N : \mathcal{A}_0 \to \mathcal{A}_0$

Let $y \in \mathcal{A}_0$ and $t \in [0, t_1]$. From (H2) we have

$$\begin{aligned}
d_\infty(Ny(t), \widehat{0}) &\leq d_\infty(Ny(t), Ny(0)) + d_\infty(a, \widehat{0}) \\
&= d_\infty\left(\int_0^t f(s, y(s))ds, \widehat{0}\right) + d_\infty(a, \widehat{0}) \\
&\leq \int_0^t d_\infty(f(s, y(s)), \widehat{0})ds + d_\infty(a, \widehat{0}) \\
&\leq \int_0^t p(s)\psi(d_\infty(y(s), \widehat{0}))ds + d_\infty(a, \widehat{0}) \\
&\leq \int_0^t p(s)\psi(a_0(s))ds + d_\infty(a, \widehat{0}) \\
&= \int_0^t a_0'(s)ds + d_\infty(a, \widehat{0}) = a_0(t),
\end{aligned}$$

since

$$\int_{d_\infty(a,\widehat{0})}^{a_0(t)} \frac{du}{\psi(u)} = \int_0^t p(s)ds.$$

Claim 2: *N is continuous.*

Let $\{y_n\} \in \mathcal{A}_0$ be a sequence such that $y_n \to y \in \mathcal{A}_0$ in $C([0, t_1], E^n)$.

$$\begin{aligned}
H_1(Ny_n(t), Ny(t)) &= H_1\left(a + \int_0^t f(s, y_n(s))ds, a + \int_0^t f(s, y(s))ds\right) \\
&\leq \int_0^t H_1(f(s, y_n(s)), f(s, y(s)))ds.
\end{aligned}$$

Hence

$$H_1(Ny_n, Ny) \leq \int_0^{t_1} H_1(f(s, y_n(s)), f(s, y(s)))ds.$$

Let

$$\rho_n(s) = d_\infty(f(s, y_n(s)), f(s, y(s))).$$

Since f is continuous then

$$\rho_n(t) \to 0 \text{ as } n \to \infty \text{ for } t \in [0, t_1].$$

From (H2) we have that

$$\begin{aligned}
\rho_n(t) &\leq d_\infty(f(t, y_n(t), \widehat{0}) + d_\infty(\widehat{0}, f(t, y(t)) \\
&\leq p(t)[\psi(d_\infty(y_n(t), \widehat{0})) + \psi(d_\infty(y(t), \widehat{0}))] \\
&\leq 2p(t)\psi(a_0(t)).
\end{aligned}$$

As a result

$$\lim_{n\to\infty} \int_0^{t_1} \rho_n(s)ds = \int_0^{t_1} \lim_{n\to\infty} \rho_n(s)ds = 0.$$

Then

$$H_1(Ny_n, Ny) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $N : \mathcal{A}_0 \to \mathcal{A}_0$ is continuous.

Claim 3: *$N(\mathcal{A}_0)$ is an equicontinuous set of $C([0, t_1], E^n)$.*

Let $l_1, l_2 \in [0, t_1]$, $l_1 < l_2$, and let $y \in \mathcal{A}_0$. Then

$$\begin{aligned}
d_\infty(Ny(l_2), Ny(l_1)) &= d_\infty\left(a + \int_0^{l_2} f(s, y(s))ds, a + \int_0^{l_1} f(s, y(s))ds\right) \\
&= d_\infty\left(\int_0^{l_2} f(s, y(s))ds, \int_0^{l_1} f(s, y(s))ds\right) \\
&= d_\infty\left(\int_{l_1}^{l_2} f(s, y(s))ds, \widehat{0}\right) \\
&\leq \int_{l_1}^{l_2} d_\infty(f(s, y(s), \widehat{0})ds \\
&\leq \int_{l_1}^{l_2} p(s)\psi(d_\infty(y(s), \widehat{0}))ds \\
&\leq \int_{l_1}^{l_2} p(s)\psi(a_0(s))ds \\
&= \int_{l_1}^{l_2} a_0'(s)ds = a_0(l_2) - a_0(l_1).
\end{aligned}$$

As a consequence of Claims 1 to 3 and (H2) together with the Arzela-Ascoli theorem we can conclude that $N : \mathcal{A}_0 \to \mathcal{A}_0$ is continuous and completely continuous and by Theorem 2.6 N has a fixed point y_1 which is solution of the problem (3.1)–(3.2).

Step 2: Consider now the following problem

$$y'(t) = f(t, y(t)), \quad t \in [t_1, t_2], \tag{3.3}$$

$$y(t_1^+) = I_1(y_1(t_1)). \tag{3.4}$$

Consider the operator $N_1 : C([t_1, t_2], E^n) \to C([t_1, t_2], E^n)$ defined by:

$$N_1(y)(t) = I_1(y_1(t_1)) + \int_{t_1}^{t} f(s, y(s))ds.$$

Set

$$\mathcal{A}_1 \cong \mathcal{B}_1 \equiv \{\bar{J}y \in C(J_1, E^n) : y \in C(J_1, E^n) \text{ and } d_\infty(y(t), \widehat{0}) \leq a_1(t),\ t \in J_1\}.$$

Clearly, $\mathcal{B}_1$ is a convex subset of the Banach space $C(J_1, X)$, so in particular $\mathcal{B}_1$ is an absolute retract. As a result $\mathcal{A}_1$ is an absolute retract. Now we prove that $N_1(\mathcal{A}_1) \subset \mathcal{A}_1$. Let $y \in \mathcal{A}_1$, then

$$N_1 y(t) = I_1(y_1(t_1)) + \int_{t_1}^{t} f(s, y(s))ds \quad \text{for} \quad t \in J_1.$$

From (H2) we have

$$\begin{aligned}
d_\infty(N_1 y(t), \widehat{0}) &\leq d_\infty\Big(N_1 y(t), N y(t_1)) + d_\infty(I_1(y_1(t_1)), \widehat{0}\Big) \\
&= d_\infty\left(\int_{t_1}^{t} f(s, y(s))ds, \widehat{0}\right) + d_\infty(I_1(y(t_1)), \widehat{0}) \\
&\leq \int_{t_1}^{t} d_\infty(f(s, y(s)), \widehat{0})ds + d_\infty(I_1(y_1(t_1)), \widehat{0}) \\
&\leq \int_{t_1}^{t} p(s)\psi(d_\infty(y(s), \widehat{0}))ds + d_\infty(I_1(y_1(t_1)), \widehat{0}) \\
&\leq \int_{t_1}^{t} p(s)\psi(a_1(s))ds + d_\infty(I_1(y_1(t_1)), \widehat{0}) \\
&= \int_{t_1}^{t} a_1'(s)ds + d_\infty(I_1(y_1(t_1)), \widehat{0}), \\
&= \int_{t_1}^{t} a_1'(s)ds + a_1(y_1(t_1)) = a_1(t),
\end{aligned}$$

since

$$\int_{d_\infty(I_1(y_1(t_1)), \widehat{0})}^{a_1(t)} \frac{du}{\psi(u)} = \int_{t_1}^{t} p(s)ds.$$

As in Step 1 we can show that N_1 is continuous and completely continuous and by Theorem 2.6 we deduce that N_1 has a fixed point y_2 which is a solution to problem (3.3)–(3.4).

Step 3: We continue this process and take into account that $y_m := y\Big|_{[t_m,T]}$ is a solution to the problem

$$y'(t) = f(t, y(t)), \quad t \in (t_m, T), \tag{3.5}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)). \tag{3.6}$$

The solution y of the problem (1.1)-(1.3) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_m(t), & \text{if } t \in (t_m, T]. \end{cases}$$

4. SECOND ORDER FUZZY IMPULSIVE DIFFERENTIAL EQUATIONS

In this section we give an existence result for the IVP (1.4)–(1.7).

Definition 4.1. A function $y \in C^2(J\backslash\{t_k\}, E^n) \cap PC$ is said to be a solution of (1.4)–(1.7) if y satisfies the equation $y''(t) = f(t, y(t))$ on $J,\ t \neq t_k,\ k = 1, \dots, m$ and the conditions $y(t_k^+) = I_k(y(t_k^-)),\ y'(t_k^+) = \overline{I}_k(y(t_k^-)),\ t = t_k,\ k = 1, \dots, m$ and $y(0) = a, \quad y'(0) = b$.

Theorem 4.2. *Assume that the conditions*

(A1) *There exists a continuous non-decreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and a continuous function $p : J \to \mathbb{R}_+$ such that*

$$d_\infty(f(t, y), \widehat{0}) \leq p(t)\psi(d_\infty(y, \widehat{0})) \quad \textit{for a.e.} \quad t \in J \quad \textit{and each} \quad y \in E^n$$

and $M_k > 0,\ k = 0, \dots, m \quad (I_0 = a,\ \bar{I}_0 = b)$ with

$$\frac{M_k}{d_\infty(I_k(y(t_k)) + (t_{k+1} - t_k)\bar{I}_k(y(t_k)), \widehat{0}) + \psi(M_k)\int_{t_k}^{t_{k+1}}(t_{k+1} - s)p(s)ds} \geq 1,$$

(A2) *For each $t \in J_k, \quad k = 0, \dots, m, \quad$ the set*

$$\Big\{I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k)) + \int_{t_k}^{t}(t - s)f(s, y(s))ds) : y \in \mathcal{A}_k\Big\},$$

is a totally bounded subset of E^n, where

$$\mathcal{A}_k^* = \{y \in C(J_k, E^n) : d_\infty(y(t), \widehat{0}) \leq M_k,\ t \in J_k\}$$

are satisfied. Then the IVP (1.4)-(1.7) has at least one fuzzy solution on J.

Proof. The proof will be given in several steps.

Step 1: Consider the following problem

$$y''(t) = f(t, y(t)), \quad t \in [0, t_1], \tag{4.1}$$

$$y(0) = a, \;\; y'(0) = b. \tag{4.2}$$

Transform the problem (4.1)–(4.2) into a fixed point problem. Consider the operator $N_* : C([0, t_1], E^n) \to C([0, t_1], E^n)$ defined by:

$$N_* y(t) = a + bt + \int_0^t (t - s) f(s, y(s)) ds.$$

Let

$$\mathcal{A}_0^* = \{y \in C(J_0, E^n) : d_\infty(y(t), \widehat{0}) \le M_0, \;\; t \in J_0\}$$

Clearly, $\mathcal{A}_0^*$ is a convex subset of the Banach space $C(J_0, E^n)$, so in particular $\mathcal{A}_0^*$ is an absolute retract. We shall show that the operator N_* maps $\mathcal{A}_0^*$ into $\mathcal{A}_0^*$ and is continuous and completely continuous.

Claim 1 $N_* : \mathcal{A}_0^* \to \mathcal{A}_0^*$

Let $y \in \mathcal{A}_0^*$ and $t \in [0, t_1]$. From (A1) we have

$$\begin{aligned}
d_\infty(N_* y(t), \widehat{0}) &\le d_\infty(N_* y(t), N_* y(0)) + d_\infty(a + tb, \widehat{0}) \\
&= d_\infty\left(\int_0^t (t - s) f(s, y(s)) ds, \widehat{0}\right) + d_\infty(a + tb, \widehat{0}) \\
&\le \int_0^t (t - s) d_\infty(f(s, y(s)), \widehat{0}) ds + d_\infty(a + tb, \widehat{0}) \\
&\le \int_0^t (t - s) p(s) \psi(d_\infty(y(s), \widehat{0})) ds + d_\infty(a + tb, \widehat{0}) \\
&\le \psi(M_0) \int_0^{t_1} (t_1 - s) p(s) ds + d_\infty(a + t_1 b, \widehat{0}) \\
&\le M_0.
\end{aligned}$$

So, $N_*(\mathcal{A}_0^*) \subset \mathcal{A}_0^*$.

Claim 2: N_* *is continuous.*

Let $\{y_n\} \in \mathcal{A}_0^*$ be a sequence such that $y_n \to y \in \mathcal{A}_0^*$ in $C([0, t_1], E^n)$.

$$\begin{aligned}
H_1(N_* y_n(t), N_* y(t)) &= H_1\left(a + tb + \int_0^t (t - s) f(s, y_n(s) ds, a + tb + \int_0^t f(s, y(s)) ds\right) \\
&\le \int_0^t (t - s) H_1(f(s, y_n(s)), f(s, y(s)) ds.
\end{aligned}$$

Hence

$$H_1(N_*y_n, N_*y) \le t_1 \int_0^{t_1} H_1(f(s, y_n(s)), f(s, y(s)))ds.$$

Let

$$\rho_n(s) = d_\infty(f(s, y_n(s)), f(s, y(s))).$$

Since f is continuous then

$$\rho_n(t) \to 0 \text{ as } n \to \infty \text{ for } t \in [0, t_1].$$

From (A1) we have that

$$\begin{aligned}
\rho_n(t) &\le d_\infty(f(t, y_n(t), \widehat{0}) + d_\infty(\widehat{0}, f(t, y(t))) \\
&\le p(t)[\psi(d_\infty(y_n(t), \widehat{0})) + \psi(d_\infty(y(t), \widehat{0}))] \\
&\le 2p(t)\psi(M_0).
\end{aligned}$$

As a result

$$\lim_{n\to\infty} \int_0^{t_1} \rho_n(s)ds = \int_0^{t_1} \lim_{n\to\infty} \rho_n(s)ds = 0.$$

Then

$$H_1(N_*y_n, N_*y) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $N_* : \mathcal{A}_0^* \to \mathcal{A}_0^*$ is continuous.

Claim 3: *$N(\mathcal{A}_0^*)$ is an equicontinuous set of $C([0, t_1], E^n)$* .

Let $l_1, l_2 \in [0, t_1]$, $l_1 < l_2$, and let $y \in \mathcal{A}_0^*$. Then

$$\begin{aligned}
d_\infty(N_*y(l_2), N_*y(l_1)) &= d_\infty\Bigg(a + l_2 b + \int_0^{l_2}(l_2 - s)f(s, y(s))ds, a + l_1 b \\
&\quad + \int_0^{l_1}(l_1 - s)f(s, y(s))ds\Bigg) \\
&= d_\infty\Bigg(l_2 b + \int_0^{l_2}(l_2 - s)f(s, y(s))ds, l_1 b \\
&\quad + \int_0^{l_1}(l_1 - s)f(s, y(s))ds\Bigg) \\
&= d_\infty\Bigg((l_2 - l_1)b + \int_0^{l_1}(l_2 - l_1)f(s, y(s))ds \\
&\quad + \int_{l_1}^{l_2}(l_2 - s)f(s, y(s))ds, \widehat{0}\Bigg) \\
&\le (l_2 - l_1)d_\infty(b, \widehat{0}) + \int_0^{l_1}(l_2 - l_1)d_\infty(f(s, y(s)), \widehat{0}))ds \\
cr &+ \int_{l_1}^{l_2} l_2 d_\infty(f(s, y(s)), \widehat{0}))ds
\end{aligned}$$

$$
\begin{aligned}
&\leq (l_2 - l_1)d_\infty(b, \widehat{0}) + \int_0^{l_1} (l_2 - l_1)d_\infty(f(s, y(s)), \widehat{0})ds \\
&+ \int_{l_1}^{l_2} l_2 p(s)\psi(y(s), \widehat{0}))ds \\
&\leq (l_2 - l_1)d_\infty(b, \widehat{0}) + \int_0^{l_1} (l_2 - l_1)p(s)\psi(M_0)ds \\
&+ \int_{l_1}^{l_2} l_2 p(s)\psi(M_0)ds. \\
&= \psi(M_0)\int_0^{l_1} (l_2 - l_1)p(s)ds + \psi(M_0)\int_{l_1}^{l_2} l_2 p(s)ds \\
&+ (l_2 - l_1)d_\infty(b, \widehat{0}).
\end{aligned}
$$

As a consequence of Claim 2 and (A2) together with the Arzela-Ascoli theorem we can conclude that $N_* : \mathcal{A}_0^* \to \mathcal{A}_0^*$ is continuous and completely continuous and by Theorem 2.6 N_* has a fixed point y_1 which a solution of the problem (4.1)–(4.2).

Step 2: Consider now the following problem

$$y''(t) = f(t, y(t)), \quad t \in [t_1, t_2], \tag{4.3}$$

$$y(t_1^+) = I_1(y_1(t_1)) \tag{4.4}$$

$$y'(t_1^+) = \bar{I}_1(y_1(t_1)). \tag{4.5}$$

Let the operator $\overline{N}_* : C([t_1, t_2], E^n) \to C([t_1, t_2], E^n)$ defined by:

$$\overline{N}_* y(t) = I_1(y_1(t_1)) + (t - t_1)\bar{I}_1(y_1(t_1)) + \int_{t_1}^{t} (t - s)f(s, y(s))ds.$$

Set

$$\mathcal{A}_1^* = \{y \in C(J_1, E^n) : \ d_\infty(y(t), \widehat{0}) \leq M_1, \ t \in J_1\}.$$

Clearly, that $\mathcal{A}_1$ is an absolute retract. Now we prove that $\overline{N}_*(\mathcal{A}_1^*) \subset \mathcal{A}_1^*$. Let $y \in \mathcal{A}_1^*$, then from (A2) we have

$$\begin{aligned}
d_\infty(\overline{N}_* y(t), \widehat{0}) &\leq d_\infty\left(\int_{t_1}^{t}(t-s)f(s,y(s))ds, \widehat{0}\right) \\
&\quad + d_\infty(I_1(y_1(t_1)) + (t-t_1)\bar{I}_1(y_1(t_1)), \widehat{0}) \\
&\leq \int_{t_1}^{t}(t-s)d_\infty(f(s,y(s)), \widehat{0})ds \\
&\quad + d_\infty(I_1(y_1(t_1)) + (t_2-t_1)\bar{I}_1(y_1(t_1)), \widehat{0}) \\
&\leq \int_{t_1}^{t}(t_2-s)p(s)\psi(d_\infty(y(s), \widehat{0}))ds \\
&\quad + d_\infty(I_1(y_1(t_1)) + (t_2-t_1)\bar{I}_1(y_1(t_1)), \widehat{0}) \\
&\leq \int_{t_1}^{t_2} t_2 p(s)\psi(M_1)ds + d_\infty(I_1(y_1(t_1)) \\
&\quad + (t_2-t_1)\bar{I}_1(y_1(t_1)), \widehat{0}) \\
&\leq M_1.
\end{aligned}$$

The same reasoning as in Step 1 shows that $\overline{N}_*$ has a fixed point y_2 which is a solution to problem (3.3)–(3.4).

Step 3: We continue this process and taking into account that $y_m := y\Big|_{[t_m,T]}$ is a solution to the problem

$$y''(t) = f(t, y(t)), \quad t \in (t_m, T), \tag{4.6}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)) \tag{4.7}$$

$$y'(t_m^+) = \bar{I}_m(y_{m-1}(t_m^-)). \tag{4.8}$$

The solution y of the problem (1.4)–(1.7) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, t_2], \\ \vdots & \\ y_m(t), & \text{if } t \in (t_m, T]. \end{cases}$$

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