Communications in Applied Analysis 11 (2007) 395-402

THEORY OF FRACTIONAL DIFFERENTIAL INEQUALITIES AND APPLICATIONS

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ABSTRACT. In this paper, we develop the theory of fractional differential inequalities involving Riemann-Loiuville differential operators of order $0 < q < 1$, use it for the existence of extremal solutions and global existence. Necessary tools are discussed and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

Keywords and Phrases: Fractional differential inequalities, comparison principle, basic existence results

AMS Subject Classification: 34A40, 34A12, 34A99

1. INTRODUCTION

In a recent paper [8], we investigated the basic theory of fractional differential equations involving Riemann-Liouville differential operators of order $0 < q < 1$, since such equations are important in the modeling of several physical phenomena [1]–[6], [9]–[11]. We followed the classical approach of differential equations [7] in order to compare and contrast differences as well as the intricacies that might result in the study. In our discussion, we employed the equivalent Volterra integral equation of fractional order, which demands an extra assumption of monotone character of the functions involved, as in the classical case. At that time, we were not sure how to develop the corresponding theory of fractional differential inequalities and the resulting comparison principle, that are fundamental to discuss the basic theory parallel to the classical approach.

In this paper, we shall first develop the theory of fractional differential inequalities, strict as well as nonstrict, use it for the existence of extremal solutions and global existence. Also, necessary modifications are incorporated in the Peano's Theorem, discussed in [8]. Naturally, the comparison principle proved will also be useful for further study of qualitative behavior of solutions of fractional differential equations, in a general set up.

Received July 10, 2007

2. FRACTIONAL DIFFERENTIAL INEQUALITIES

We consider the initial value problem (IVP) for fractional differential equations given by

$$D^q x = f(t,x), \quad x(0) = x_0, \tag{1}$$

where $f \in C([0,T] \times R, R)$, $D^q x$ is the fractional derivative of x and q is such that $0 < q < 1$. Since f is assumed continuous, the IVP (1) is equivalent to the following Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \quad 0 \le t \le T, \tag{2}$$

that is, every solution of (2) is also a solution of (1) and vice versa. Here and elsewhere Γ denotes the Gamma function.

Before we proceed further, we shall prove the following two lemmas and some needed notation.

Lemma 2.1: Let $m : R_+ \to R$ be locally Hölder continuous such that for any $t_1 \in (0, \infty)$, we have

$$m(t_1) = 0 \text{ and } m(t) \le 0 \text{ for } 0 \le t \le t_1. \tag{3}$$

Then it follows that

$$D^q m(t_1) \ge 0. \tag{4}$$

Proof: We know that

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{p-1} m(s) ds, \tag{5}$$

where $1 - q = p$. Let $H(t) = \int_0^t (t-s)^{p-1} m(s) ds$. Consider for $h > 0$,

$$\begin{aligned} H(t_1) - H(t_1 - h) = \int_0^{t_1 - h} & [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds \\ & + \int_{t_1 - h}^{t_1} (t-s)^{p-1} m(s) ds = I_1 + I_2, \end{aligned}$$

say. Since $[t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] < 0$ for $0 \le s \le t_1 - h$ and $m(s) \le 0$ by hypothesis, we have $I_1 \ge 0$. Also,

$$H(t_1) - H(t_1 - h) \ge \int_{t_1 - h}^{t_1} (t_1 - s)^{p-1} m(s) ds = I_2.$$

Since $m(t)$ is locally Hölder continuous and $m(t_1) = 0$, there exists a constant $K(t_1) > 0$, such that, for $t_1 - h \le s \le t_1 + h$,

$$-K(t_1)(t_1 - s)^\lambda \le m(s) \le K(t_1)(t_1 - s)^\lambda,$$

where $\lambda > 0$ is such that $\lambda + p - 1 > 0$ and $0 < \lambda < 1$. We then get

$$I_2 \geq -K(t_1)\int_{t_1-h}^{t_1}(t_1-s)^{p-1+\lambda}ds = \frac{K(t_1)}{\Gamma(p+\lambda)}h^{p+\lambda}.$$

Hence $H(t_1) - H(t_1-h) - \frac{K(t_1)}{\Gamma(p+\lambda)}h^{p+\lambda} \geq 0$, for sufficiently small $h > 0$. Letting $h \to 0$, we obtain $H'(t_1) \geq 0$, which implies $D^q m(t_1) = \frac{1}{\Gamma(p)}H'(t_1) \geq 0$ and the proof is complete.

Lemma 2.2: Let $\{x_\epsilon(t)\}$ be a family of continuous functions defined on $[0,T]$, for each $\epsilon > 0$, where $D^q x_\epsilon(t) = f(t, x_\epsilon(t))$, $x_\epsilon(0) = x_0$, and $|f(t, x_\epsilon(t))| \leq M$ for $0 \leq t \leq T$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $0 \leq t \leq T$.

Proof: For $0 \leq t_1 \leq t_2 \leq T$, consider

$$\begin{aligned}
|x_\epsilon(t_1) - x_\epsilon(t_2)| &= \frac{1}{\Gamma(q)}\left|\int_0^{t_1}(t_1-s)^{q-1}f(s,x_\epsilon(s))ds - \int_0^{t_2}(t_2-s)^{q-1}f(s,x_\epsilon(s))ds\right| \\
&\leq \frac{M}{\Gamma(q)}\left|\int_0^{t_1}\left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right]ds + \int_{t_1}^{t_2}(t_2-s)^{q-1}ds\right| \\
&\leq \frac{M}{\Gamma(q+1)}\left[t_1^q - t_2^q + 2(t_2-t_1)^q\right] \leq \frac{2M}{\Gamma(q+1)}(t_2-t_1)^q < \epsilon,
\end{aligned}$$

provided $|t_2 - t_1| < \delta = \left[\frac{\epsilon\Gamma(q+1)}{2M}\right]^{\frac{1}{q}}$, proving the claim.

Corresponding to each Dini derivative $D^\pm_\pm$, one can define fractional Dini derivatives from the relation, namely,

$$D^{\pm q}_\pm u(t) = \frac{1}{\Gamma(p)}D^\pm_\pm\int_0^t (t-s)^{p-1}u(s)ds,$$

where $p = 1-q$, $0 < q < 1$, as before. For example
$D_-u(t) = \lim_{h\to 0^-}\inf\frac{1}{h}[u(t+h)-u(t)]$. Also, it is not difficult to get the relation $D^{+q}|u(t)| \leq |D^q u(t)|$, when $D^q u(t)$ exists.

Let us first discuss a fundamental result relative to strict fractional differential inequalities.

Theorem 2.3: Let $v, w : [0,T] \to R$ be locally Hölder continuous, $f \in C([0,T] \times R, R)$ and

$$\text{(i) } D^q v(t) \leq f(t, v(t)), \quad \text{(ii) } D^q w(t) \geq f(t, w(t)), \quad 0 \leq t \leq T,$$

one of the inequalities being strict. Then

$$v(0) < w(0) \tag{6}$$

implies

$$v(t) < w(t), \quad 0 \leq t \leq T. \tag{7}$$

Proof: Suppose that the conclusion (7) is not true. Let us suppose that the inequality (ii) is strict. Then, setting $m(t) = v(t) - w(t)$, $0 \leq t \leq t_1$, we find that $m(t) \leq 0$,

$0 \le t \le t_1$ and $m(t_1) = 0$. Then by Lemma 2.1, we get $D^q m(t_1) \ge 0$, which yields, using (i) and (ii), and the definition of $m(t)$,

$$f(t_1, v(t_1)) \ge D^q v(t_1) \ge D^q w(t_1) > f(t_1, w(t_1)).$$

This is a contradiction since $v(t_1) = w(t_1)$. Hence the conclusion (7) is valid and the proof is complete.

The next result is for nonstrict fractional differential inequalities which requires a one sided Lipshitz type condition.

Theorem 2.4: Assume that the conditions of theorem 2.3 hold with nonstrict inequalities (i) and (ii). Suppose further that

$$f(t,x) - f(t,y) \le \frac{L}{1+t^q}(x-y), \quad \text{wherever } x \ge y \text{ and } L > 0. \tag{8}$$

Then $v(0) \le w(0)$ implies, provided $LT^q \le \frac{1}{\Gamma(1-q)}$,

$$v(t) \le w(t), \quad 0 \le t \le T. \tag{9}$$

Proof: We set $w_\epsilon(t) = w(t) + \epsilon(1+t^q)$, for small $\epsilon > 0$, so that we have

$$w_\epsilon(0) > w(0) \quad \text{and} \quad w_\epsilon(t) > w(t), \quad 0 \le t \le T. \tag{10}$$

Now

$$\begin{aligned}
D^q w_\epsilon(t) &= D^q w(t) + \epsilon D^q(1+t^q) \\
&\ge f(t, w(t)) + \epsilon\left[\frac{1}{t^q\Gamma(1-q)} + \Gamma(1+q)\right] \\
&> f(t, w_\epsilon(t)) - L\epsilon + \epsilon\frac{1}{t^q\Gamma(1-q)} \\
&> f(t, w_\epsilon(t)), \quad 0 \le t \le T.
\end{aligned}$$

Here we have used the relations (8), (10) and the assumptions $LT^q \le \frac{1}{\Gamma(1-q)}$. We can now apply Theorem 2.3 to v and $w_\epsilon(t)$ to get $v(t) < w_\epsilon(t)$, $0 \le t \le T$. Since $\epsilon > 0$ is arbitrary, we conclude that (9) is true and we are done.

3. LOCAL EXISTENCE AND EXTREMAL SOLUTIONS

Let us start with Peano's type existence result.

Theorem 3.1: Assume that $f \in C[R_0, R]$ where $R_0 = [(t,x) : 0 \le t \le a$, and $|x - x_0| \le b]$, and let $|f(t,x)| \le M$ on R_0. Then there exists at least one solution for the IVP (1) on $0 \le t \le \alpha$, where $\alpha = \min(a, \left[\frac{b}{M}\Gamma(q+1)\right]^{\frac{1}{q}})$, $0 < q < 1$.

Proof: Let $x_0(\epsilon)$ be a continuous function on $[-\delta, 0]$, $\delta > 0$ such that $x_0(0) = x_0$, $|x_0(t) - x_0| \le b$ and $|D^q x_0(t)| \le M$, where $D^q x_0(t)$ is the continuous fractional derivative. For $0 < \epsilon \le \delta$, we define the function $x_\epsilon(t) = x_0(t)$ on $[-\delta, 0]$ and

$$x_\epsilon(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x_\epsilon(s-\epsilon))ds \tag{11}$$

on $[0, \alpha_1]$, where $\alpha_1 = \min(\alpha, \epsilon)$. We observe that $D^q x_\epsilon(t)$ exists and

$$\begin{aligned} |x_\epsilon(t) - x_0| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_\epsilon(s-\epsilon))| ds \\ &\leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds = \frac{M\alpha^q}{\Gamma(q+1)} \leq b, \end{aligned} \tag{12}$$

because of the choice of α_1. If $\alpha_1 < \alpha$, we can employ (11) to extend $x_\epsilon(t)$ as a continuously fractional differentiable function on $[-\delta, \alpha_2]$, $\alpha_2 = \min(\alpha, 2\epsilon)$ such that $|x_\epsilon(t) - x_0| \leq b$ holds. Continuing this process, we can define $x_\epsilon(t)$ over $[-\delta, \alpha]$ so that $|x_\epsilon(t) - x_0| \leq b$, it has a continuous fractional derivative and satisfies (11) on the same interval $[-\delta, \alpha]$. Moreover, $|D^q x_\epsilon(t)| \leq M$, since $|f(t, x_\epsilon(t-\epsilon))| \leq M$ on R_0, and therefore the family $\{x_\epsilon(t)\}$ forms an equicontinuous and uniformly bounded functions by Lemma 2.2. As application of Ascoli-Arzela's theorem shows the existence of a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > ... > \epsilon_n \to 0$ as $n \to \infty$, and $x(t) = \lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly on $[-\delta, \alpha]$. Since f is uniformly continuous, we obtain $f(t, x_{\epsilon_n}(t-\epsilon_n))$ tends to uniformly to $f(t, x(t))$ as $n \to \infty$, and hence term by term integration of (11) with $\epsilon = \epsilon_n$, $\alpha_1 = \alpha$ yields

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds.$$

This proves that $x(t)$ is a solution of IVP (1) and the proof is complete.

Employing Theorems 3.1 and 2.3, we can now prove the existence of extremal solutions for the IVP (1).

Theorem 3.2: Under the assumptions of theorem 3.1, there exist extremal solutions for the IVP (1) on $0 \leq t \leq \alpha_0$, $\alpha_0 = \min\left(a, \left[\frac{b\Gamma(1+q)}{2M+b}\right]^{\frac{1}{q}}\right)$.

Proof: We shall prove the existence of the maximal solution only, since the case of the minimal solution is very similar. Let $0 < \epsilon \leq \frac{b}{2}$ and consider the IVP

$$D^q x = f(t, x) + \epsilon, \quad x(0) = x_0 + \epsilon. \tag{13}$$

Note that $f_\epsilon(t, x) = f(t, x) + \epsilon$ is defined and continuous on

$$R_\epsilon = \left[(t, x) : 0 \leq t \leq a \quad \text{and} \quad |x - (x_0 + \epsilon)| \leq \frac{b}{2}\right],$$

$R_\epsilon \subset R_0$ and $|f_\epsilon(t, x)| \leq M + \frac{b}{2}$ on R_ϵ. We then deduce from Theorem 3.1 that IVP (13) has a solution $x(t, \epsilon)$ on $0 \leq t \leq \alpha_0$.

Now for $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we have

$$\begin{aligned} x(0, \epsilon_2) &< x(0, \epsilon_1) \\ D^q x(t, \epsilon_2) &\leq f(t, x(t, \epsilon_2)) + \epsilon_2 \\ D^q x(t, \epsilon_1) &> f(t, x(t, \epsilon_1)) + \epsilon_2, \quad \text{on } 0 \leq t \leq \alpha_0. \end{aligned}$$

We apply Theorem 2.3 to get

$$x(t,\epsilon_2) < x(t,\epsilon_1), \quad 0 \le t \le \alpha_0.$$

Consider the family of continuous functions $\{x(t,\epsilon)\}$ on $0 \le t \le \alpha_0$. Then it follows that

$$|x(t,\epsilon) - x(0,\epsilon)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s,\epsilon))| ds \le \frac{2M+b}{2\Gamma(q)} \int_0^t (t-s)^{q-1} ds$$

$$\le \frac{2M+b}{2} \frac{\alpha_0^q}{\Gamma(q+1)} \le \frac{b}{2} \le b,$$

showing that the family is uniformly bounded. Since $|D^q x(t,\epsilon)| = |f(t, x(t,\epsilon))| \le M + \frac{b}{2}$, the family is also equicontinuous by Lemma 2.2. Hence there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ and the uniform limit $\eta(t) = \lim_{n\to\infty} x(t,\epsilon_n)$ exists on $[0,\alpha_0]$. Clearly $\eta(0) = x_0$. The uniform continuity of f, gives arguing as before (as in Theorem 3.1) that $\eta(t)$ is a solution of IVP (1).

Next we show that $\eta(t)$ is the required maximal solution of (1). Let $x(t)$ be any solution of (1) on $0 \le t \le \alpha_0$. then we have

$$\begin{aligned} x_0 &< x_0 + \epsilon = x(0,\epsilon), \\ D^q x(t) &< f(t, x(t)) + \epsilon, \\ D^q x(t,\epsilon) &\ge f(t, x(t,\epsilon)) + \epsilon. \end{aligned}$$

By Theorem 2.3, we obtain $x(t) < x(t,\epsilon)$, $0 \le t \le \alpha_0$ for every $\epsilon > 0$. The uniqueness of maximal solution shows that $x(t,\epsilon)$ tends to $\eta(t)$ on $[0,\alpha_0]$ as $\epsilon \to 0$. The proof is therefore complete.

4. **GLOBAL EXISTENCE**

We need the following comparison theorem before we proceed further.

Theorem 4.1: Assume that $m \in C([0,T], R_+)$, locally Hölder continuous, $g \in C([0,T] \times R_+, R_+)$, and

(14) $$D^q m(t) \le g(t, m(t)), \quad 0 \le t < T.$$

Let $\eta(t)$ be the maximal solution of

$$D^q u(t) = g(t, u(t)), \quad u(0) = u_0 \ge 0,$$

existing on $[0,T)$ such that $m(0) \le u_0$. then we have

(15) $$m(t) \le \eta(t), \quad 0 \le t < T.$$

Proof: In view of the definition of the maximal solution $\eta(t)$, it is enough to prove, to conclude (15), that

(16) $$m(t) < u(t,\epsilon), \quad 0 \le t < T,$$

where $u(t,\epsilon)$ is any solution of

$$D^q u = g(t,u) + \epsilon, \quad u(0) = u_0 + \epsilon, \quad \epsilon > 0. \tag{17}$$

Now it follows from (17) that

$$D^q u(t,\epsilon) > g(t, u(t,\epsilon)).$$

Then applying Theorem 2.3, we get immediately (16) and since $\lim_{\epsilon \to 0} u(t,\epsilon) = \eta(t)$ uniformly on each $0 \le t \le T_0 < T$, the proof is complete.

We are now in the position to prove the global existence result.

Theorem 4.2: Assume that $f \in C([0,\infty) \times R, R)$, $g \in C([0,\infty) \times R_+, R_+)$, $g(t,u)$ is nondecreasing in u for each t and

$$|f(t,x)| \le g(t,|x|). \tag{18}$$

Suppose that we have local existence of solutions $x(t,x_0)$ of

$$D^q x = f(t,x), \quad x(0) = x_0, \tag{19}$$

and the maximal solution of

$$D^q u = g(t,u), \quad u(0) = u_0 \ge 0, \tag{20}$$

exists on $[0,\infty)$. Then the largest interval of existence of any solution $x(t,x_0)$ of (19) such that $|x_0| \le u_0$ is $[0,\infty)$.

Proof: Let $x(t,x_0)$ be any solution of (19) with $|x_0| \le u_0$ which exists on $[0,\beta)$, $\beta < \infty$ and the value of β cannot be increased further. Set $m(t) = |x(t,x_0)|$ for $0 \le t < \beta$. Then using the assumption (18) we get

$$D^q m(t) \le |D^q x(t,x_0)| = |f(t,x(t,x_0))| \le g(t,|x(t,x_0)|) \le g(t,m(t)), \quad 0 \le t < \beta,$$

and $m(0) \le u_0$. Applying Theorem 4.1, we obtain

$$m(t) \le \eta(t), \quad 0 \le t < \beta,$$

and therefore $|D^q x(t,x_0)| \le g(t,m(t)) \le g(t,\eta(t)) \le M$ on $0 \le t \le \beta$, since $\eta(t)$ is assumed to exist on $[0,\infty)$, it follows that $g(t,\eta(t)) \le M$. Now for $0 \le t_1 \le t_2 < \beta$, we find by Lemma 2.2,

$$|x(t_1,x_0) - x(t_2,x_0)| \le \frac{2M}{\Gamma(q+1)}(t_2 - t_1)^q.$$

Letting $t_1, t_2 \to \beta^-$ and using Caushy criterion, it follows that $\lim_{t \to \beta^-} x(t,x_0)$ exists. We denote $x(\beta,x_0) = \lim_{t\to\beta^-} x(t,x_0)$ and consider the new IVP

$$D^q x = f(t,x), \quad x(\beta) = x(\beta,x_0).$$

By the assumed local existence, we find that $x(t,x_0)$ can be continued beyond β, contradicting our assumption. Hence every solution $x(t,x_0)$ of (19) exists on $[0,\infty)$ and the proof is complete.

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