Communications in Applied Analysis 11 (2007) 441-450

ON SOLVABILITY OF SOME QUADRATIC FUNCTIONAL-INTEGRAL EQUATION IN BANACH ALGEBRA

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ABSTRACT. Using the technique of a suitable measure of noncompactness in Banach algebra, we prove an existence theorem for some functional-integral equations which contains as particular cases a lot of integral and functional-integral equations that arise in many branches of nonlinear analysis and its applications. Also, the famous Chandrasekhar's integral equation is considered as a special case.

AMS (MOS) Subject Classification. 45G10, 45M99, 47H09.

1. INTRODUCTION

In this paper we study the quadratic functional-integral equation with singular kernel, namely

$$x(t) = f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right) \cdot g\left(t, x(\gamma(t)), x(t) \int_0^1 v(t,s,x(s))\, ds\right),\ t \in \mathrm{I} = [0,1],\ 0 < \alpha \le 1. \tag{1.1}$$

The equations of such kind contain as spacial case many integral and functional equations that arise in nonlinear analysis and its applications. Also, Eq.(1.1) contains, as spacial case, the integral equation of Chandrasekhar which arises in radiative transfer, neutron transport and the kinetic theory of gases, [1, 5, 7, 8, 10, 11, 12, 15, 16, 17, 18].

Using the technique of a suitable measure of noncompactness in Banach algebra, we prove an existence theorem for Eq.(1.1). In fact, our results in this paper are motivated by extensions and generalization of the results in [2] and [7] based on the regular measure of noncompactness in Banach algebra and fixed point theorem due to Darbo.

2. AUXILIARY FACTS AND RESULTS

This section is devoted to collect some definitions and results which will be needed further on. Assume that $(\mathrm{E}, \|.\|)$ is a real Banach space with zero element θ. Denote

Received July 23, 2007

by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a subset of E, then $\bar{\mathrm{X}}$ and $ConvX$ denote the closure and convex closure of X, respectively. We denote the standard algebraic operations on sets by the symbols λX and X + Y. Moreover, we denote by $\mathcal{M}_{\mathrm{E}}$ the family of all nonempty and bounded subsets of E and $\mathcal{N}_{\mathrm{E}}$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a regular measure of noncompactness [6]:

Definition 2.1. A mapping $\mu : \mathcal{M}_{\mathrm{E}} \to [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) $\mu(\mathrm{X}) = 0 \Leftrightarrow \mathrm{X} \in \mathcal{N}_{\mathrm{E}}$.
2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3) $\mu(\bar{X}) = \mu(ConvX) = \mu(X)$.
4) $\mu(\lambda X) = |\lambda| \, \mu(X)$, for $\lambda \in \mathbb{R}$.
5) $\mu(\mathrm{X} + \mathrm{Y}) \leq \mu(X) + \mu(Y))$.
6) $\mu(\mathrm{X} \cup \mathrm{Y}) = \max\{\mu(X), \mu(Y)\}$.

Remark 2.2. Notice that the condition

7) If $\{\mathrm{X}_n\}$ is a sequence of nonempty, bounded, closed subsets of E such that $\mathrm{X}_{n+1} \subset \mathrm{X}_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \mu(\mathrm{X}_n) = 0$ then the set $\mathrm{X}_\infty = \bigcap_{n=1}^{\infty} \mathrm{X}_n$ is nonempty.

follows immediately from definition 2.1 above. To see this, let us consider an arbitrary sequence $\{x_n\}$, where $x_n \in \mathrm{X}_n$ for $n = 1,\ 2,\ 3,\ 3,\ \ldots$. Further, fix an arbitrary natural number k, $k \geq 2$. Then, using $1) - 6)$, we get

$$\begin{aligned} \mu(\{x_n\}) &= \mu(\{x_1,\ x_2,\ \ldots\} \cup \{x_k,\ x_{k+1},\ \ldots\}) \\ &= \max\{\mu(\{x_1,\ x_2,\ \ldots\}), \mu(\{x_k,\ x_{k+1},\ \ldots\})\} \\ &= \mu(\{x_k,\ x_{k+1},\ \ldots\}) \\ &\leq \mu(\mathrm{X}_k). \end{aligned}$$

Since k was chosen arbitrary, the above estimate implies that $\mu(\{x_n\}) = 0$. Thus the sequence $\{x_n\}$ is relatively compact, so it has an accumlation point x. In view of the closeness of X_k we infer that $x \in \mathrm{X}_k$ for any $k = 1,\ 2,\ 3,\ \ldots$ (since x is an accumlation point of any subsequence $\{x_k,\ x_{k+1},\ \ldots\}$). Hence we deduce that

$$x \in \bigcap_{n=1}^{\infty} \mathrm{X}_n.$$

This means that the set $\bigcap_{n=1}^{\infty} \mathrm{X}_n$ is nonempty. This completes the proof of 7). Some authors define a measure of noncompactness as satisfying 1-7, although we see that it is not necessary to include 7).

In what follows we will work in the Banach space C(I) consisting of all real functions defined and continuous on I. The space $C(\mathrm{I})$ is equipped with the standard norm

$$\| x \| = \max\{|x(t)| : t \in \mathrm{I}\}$$

Obviously, the space $C(\mathrm{I})$ has also the structure of Banach algebra. Now, we recollect the construction of a special measure of noncompactness in $C(\mathrm{I})$ which will be used in the next section, see [6].

Let us fix a nonempty and bounded subset X of $C(\mathrm{I})$. For $x \in X$ and $\varepsilon > 0$ denoted by $\omega(x, \varepsilon)$ the modulus of continuity of the function x, i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t,\ s \in [0, 1],\ |t - s| \leq \varepsilon\}$$

Further, let us put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon).$$

It can be shown [4] that the function $\omega_0(X)$ is a regular measure of noncompactness in the space C(I). Finally, the fixed point theorem due to Darbo will be recalled [13]:

Theorem 2.3. *Let Q be a nonempty, bounded, closed and convex subset of the space* E *and let*

$$H : Q \to Q$$

be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e., there exists a constant $0 \leq k < 1$ such that $\mu(H\ X) \leq k\ \mu(X)$ for any nonempty subset X of Q.

Then H has a fixed point in the set Q

Moreover, the following theorem holds which is the main tool in carrying out our proof, [4].

Theorem 2.4. *Assume that Ω is nonempty, bounded, convex, and closed subset of $C(\mathrm{I})$ and the operators $\mathcal{F}$ and $\mathcal{G}$ transform continuously the set Ω into $C(\mathrm{I})$ in such a way that $\mathcal{F}(\Omega)$ and $\mathcal{G}(\Omega)$ are bounded. Moreover, assume that the operator $\mathcal{T} = \mathcal{F} \cdot \mathcal{G}$ transforms Ω into itself. If the operators $\mathcal{F}$ and $\mathcal{G}$ satisfy on the set Ω the Darbo condition, with respect to the measure of noncompactness ω_0, with constant k_1 and k_2, respectively. Then the operator $\mathcal{T}$ satisfies the Darbo condition on Ω with the constant*

$$\|\mathcal{F}(\Omega)\|\ k_2 + \|\mathcal{G}(\Omega)\|\ k_1.$$

In particular, if $\|\mathcal{F}(\Omega)\|\ k_2 + \|\mathcal{G}(\Omega)\|\ k_1 < 1$ then $\mathcal{T}$ is a contraction with respect to the measure of noncompactness ω_0 and so has at least on fixed point in Ω.

3. MAIN THEOREM

In this section, we will study Eq.(1.1) assuming that the following assumptions are satisfied:

a_1) $f,\ g : \mathrm{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist nonnegative constants $a_i,\ b_i;\ i = 1,\ 2$, such that

$$|f(t, x, 0)| \leq a_1 + b_1\ |x|$$
$$|g(t, x, 0)| \leq a_2 + b_2\ |x|,$$

for all $t \in \mathrm{I}$ and $x \in \mathbb{R}$.

a_2) The functions $f(t, x, y)$ and $g(t, x, y)$ satisfy the Lipschitz condition with respect to the variables x and y with constants $l_1,\ l_2 \geq 0$ respectively, i.e.,

$$|f(t, x_1, y) - f(t, x_2, y)| \leq l_1\ |x_1 - x_2|$$
$$|g(t, x_1, y) - g(t, x_2, y)| \leq l_1\ |x_1 - x_2|,$$

for all $t \in \mathrm{I}$ and $x_1,\ x_2,\ y \in \mathbb{R}$, and

$$|f(t, x, y_1) - f(t, x, y_2)| \leq l_2\ |x_1 - x_2|$$
$$|g(t, x, y_1) - g(t, x, y_2)| \leq l_2\ |x_1 - x_2|,$$

for all $t \in \mathrm{I}$ and $x,\ y_1,\ y_2 \in \mathbb{R}$.

a_3) $u,\ v : \mathrm{I} \times \mathrm{I} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist nonnegative constants $c_i,\ d_i;\ i = 1,\ 2$, such that

$$|u(t, s, x)| \leq c_1 + d_1\ |x|$$
$$|v(t, s, x)| \leq c_2 + d_2\ |x|,$$

for all $s,\ t \in \mathrm{I}$ and $x \in \mathbb{R}$.

a_4) $\beta,\ \gamma : \mathrm{I} \to \mathrm{I}$ are continuous and satisfy,

$$|\beta(t_1) - \beta(t_2)| \leq |t_1 - t_2|$$
$$|\gamma(t_1) - \gamma(t_2)| \leq |t_1 - t_2|,$$

for all $t_1,\ t_2 \in \mathrm{I}$.

a_5) The inequality

$$[l_2\ (c + d\ r) + (a + b\ r)\ \Gamma(\alpha + 1)] \cdot [l_2\ r\ (c + d\ r) + (a + b\ r)] \leq r\ \ \Gamma(\alpha + 1)$$

has a positive solution r_0, where $a = \max\{a_1, a_2\}$, $b = \max\{b_1, b_2\}$, $c = \max\{c_1, c_2\}$ and $d = \max\{d_1, d_2\}$.

a_6)

$$l_1\ [l_2\ (1 + r_0)\ (c + d\ r_0) + (a + b\ r_0)\ \Gamma(\alpha + 1)] < \Gamma(\alpha + 1).$$

Now, we are in a position to state and prove our main result in this paper

Theorem 3.1. *Let assumptions $a_1) - a_6)$ be satisfied. Then Eq.(1.1) has at least one solution x in the Banach algebra* C(I).

Proof. Define the operators $\mathcal{F}$ and $\mathcal{G}$ on the space C(I) in the following way

$$(\mathcal{F}x)(t) = f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right),$$

$$(\mathcal{G}x)(t) = g\left(t, x(\gamma(t)), x(t) \int_0^1 v(t,s,x(s))\, ds\right).$$

From Assumptions $a_1)$ and $a_3)$, it follows that the operators $\mathcal{F}$ and $\mathcal{G}$ transform the space C(I) into itself.

Now, let us define the operator $\mathcal{T}$ on C(I) by setting

$$\mathcal{T}x = (\mathcal{F}x) \cdot (\mathcal{G}x).$$

Obviously, $\mathcal{T}$ transforms C(I) into itself. Also, let us fix $x \in$ C(I). Then, using our assumptions for $t \in$ I, we get

$$\begin{aligned}
|(\mathcal{F}\, x)(t)| &\leq \left| f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right)\right| \\
&\leq \left| f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right) - f\left(t, x(\beta(t)), 0\right)\right| \\
&\quad + |f\left(t, x(\beta(t)), 0\right)| \\
&\leq \frac{l_2}{\Gamma(\alpha)} \int_0^t \frac{|u(t,s,x(s))|}{(t-s)^{1-\alpha}}\, ds + a_1 + b_1\, |x(\beta(t))| \\
&\leq \frac{l_2}{\Gamma(\alpha+1)}(c_1 + d_1\, \|x\|) + (a_1 + b_1\, \|x\|). \qquad (3.1)
\end{aligned}$$

On the other hand

$$\begin{aligned}
|(\mathcal{G}\, x)(t)| &\leq \left| g\left(t, x(\gamma(t)), x(t) \int_0^1 v(t,s,x(s))\, ds\right)\right| \\
&\leq \left| g\left(t, x(\gamma(t)), x(t) \int_0^1 v(t,s,x(s))\, ds\right) - g\left(t, x(\gamma(t)), 0\right)\right| \\
&\quad + |g\left(t, x(\gamma(t)), 0\right)| \\
&\leq l_2\, |x(t)| \int_0^1 |v(t,s,x(s))|\, ds + a_2 + b_2\, |x(\gamma(t))| \\
&\leq l_2\, \|x\|\, (c_2 + d_2\, \|x\|) + (a_2 + b_2\, \|x\|). \qquad (3.2)
\end{aligned}$$

By (3.1) and (3.2), we obtain

$$\begin{aligned}
|(\mathcal{T}\, x)(t)| &= |(\mathcal{F}x)(t)| \cdot |(\mathcal{G}x)(t)| \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[l_2\, (c_1 + d_1\, \|x\|) + (a_1 + b_1\, \|x\|)\, \Gamma(\alpha+1)\right] \\
&\quad \times [l_2\, \|x\|\, (c_2 + d_2\, \|x\|) + (a_2 + b_2\, \|x\|)].
\end{aligned}$$

Hence

$$\|\mathcal{T}x\| \leq [\frac{l_2}{\Gamma(\alpha+1)}(c + d\, \|x\|) + (a + b\, \|x\|)] \cdot [l_2\, \|x\|\, (c + d\, \|x\|) + (a + b\, \|x\|)].$$

We deduce that, by taking into account assumption a_5), the operator $\mathcal{T}$ maps the ball B_{r_0} into itself.

Next, we show that the operator $\mathcal{F}$ is continuous on B_{r_0}. To do this fix $\varepsilon > 0$ and take $x,\ y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then, for $t \in \mathrm{I}$ we get

$$\begin{aligned}
|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| &= \left| f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right)\right. \\
&\qquad \left. - f\left(t, y(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\, ds\right)\right| \\
&\leq \left| f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds\right)\right. \\
&\qquad \left. - f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\, ds\right)\right| \\
&\quad + \left| f\left(t, x(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\, ds\right)\right. \\
&\qquad \left. - f\left(t, y(\beta(t)), \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,y(s))}{(t-s)^{1-\alpha}}\, ds\right)\right| \\
&\leq \frac{l_2}{\Gamma(\alpha)} \int_0^t \frac{|u(t,s,x(s)) - u(t,s,y(s))|}{(t-s)^{1-\alpha}}\, ds \\
&\qquad + l_1\, |x(\beta(t)) - y(\beta(t))| \\
&\leq \frac{l_2}{\Gamma(\alpha+1)}\, \omega_r(u,\varepsilon) + l_1\, \|x - y\| \\
&\leq \frac{l_2}{\Gamma(\alpha+1)}\, \omega_r(u,\varepsilon) + \varepsilon\, l_1,
\end{aligned}$$

where

$$\omega_r(u,\varepsilon) = \mathrm{Sup}\left\{|u(t,s,x) - u(t,s,y)| : t,\ s \in \mathrm{I},\ x,\ y \in [-r_0, r_0],\ |x - y| \leq \varepsilon\right\}.$$

By uniform continuity of the functions u on the set $\mathrm{I} \times \mathrm{I} \times [-r_0, r_0]$, we infer that $\omega_r(u,\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, the above estimate shows that the operator $\mathcal{F}$ is continuous on the ball B_{r_0}. Similarly, we can show that the operator $\mathcal{G}$ is continuous on the ball B_{r_0} and consequently the operator $\mathcal{T}$ is continuous on the ball B_{r_0}.

Now, we show that the operators $\mathcal{F}$ and $\mathcal{G}$ satisfy the Darbo condition on the ball B_{r_0}. To do this take a nonempty subset X of B_{r_0}. Next, choose an arbitrary number $\varepsilon > 0$ and $t_1,\ t_2 \in \mathrm{I}$ such that $|t_2 - t_1| \leq \varepsilon$. Then we obtain

$$\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| &= \left| f\left(t_2, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds\right)\right. \\
&\qquad \left. - f\left(t_1, x(\beta(t_1)), \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right| \\
&\leq \left| f\left(t_2, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds\right)\right.
\end{aligned}$$

$$- f\left(t_2, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\Bigg|$$
$$+\left|f\left(t_2, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right.$$
$$\left. - f\left(t_1, x(\beta(t_1)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right|$$

$$\leq \frac{l_2}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right|$$
$$+\left|f\left(t_2, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right.$$
$$\left. - f\left(t_1, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right|$$
$$+\left|f\left(t_1, x(\beta(t_2)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right.$$
$$\left. - f\left(t_1, x(\beta(t_1)), \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right)\right|.$$

But

$$\left|\int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right|$$
$$\leq \left|\int_0^{t_2} \frac{u(t_2,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_2} \frac{u(t_1,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds\right|$$
$$+\left|\int_0^{t_2} \frac{u(t_1,s,x(s))}{(t_2-s)^{1-\alpha}}\, ds - \int_0^{t_2} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right|$$
$$+\left|\int_0^{t_2} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds - \int_0^{t_1} \frac{u(t_1,s,x(s))}{(t_1-s)^{1-\alpha}}\, ds\right|$$
$$\leq \int_0^{t_2} \frac{|u(t_2,s,x(s)) - u(t_1,s,x(s))|}{(t_2-s)^{1-\alpha}}\, ds$$
$$+\int_0^{t_2} |u(t_1,s,x(s))|[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]\, ds$$
$$+\int_{t_1}^{t_2} \frac{|u(t_1,s,x(s))|}{(t_1-s)^{1-\alpha}}\, ds$$
$$\leq \omega_u(\varepsilon,.,.)\int_0^{t_2} (t_2-s)^{\alpha-1}\, ds$$
$$+L\int_0^{t_2} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]\, ds$$
$$+L\int_{t_1}^{t_2} (t_1-s)^{\alpha-1}\, ds.$$

Then

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \leq \frac{l_2}{\Gamma(\alpha+1)}\{\omega_u(\varepsilon,.,.)\, {t_2}^{\alpha} + L\, ({t_2}^{\alpha} - {t_1}^{\alpha})\}$$

$$+\omega_f(\varepsilon,.,.) + l_1\ |x(\beta(t_2))x(\beta(t_1))|\,,$$

where

$$\omega_u(\varepsilon,.,.) = \text{Sup}\left\{|u(\tau,s,x) - u(t,s,x)| : t,\ \tau,\ s \in \text{I},\ x \in [-r_0, r_0],\ |\tau - t| \leq \varepsilon\right\},$$

$$L = \text{Sup}\left\{|u(t,s,x)| : t,\ s \in \text{I},\ x \in [-r_0, r_0]\right\}$$

and

$$\omega_f(\varepsilon,.,.) = \text{Sup}\left\{|f(\tau,x,y) - f(t,x,y)| : t,\ \tau \in \text{I},\ x \in [-r_0, r_0],\ y \in [-r_0\ L, r_0\ L],\ |\tau - t| \leq \varepsilon\right\}.$$

Thus from the last inequality, we get

$$|(\mathcal{F}x)(\tau) - (\mathcal{F}x)(t)| \ \leq\ \frac{l_2}{\Gamma(\alpha+1)}\left\{\omega_u(\varepsilon,.,.)\ + L\ \alpha\ \varepsilon\ \delta^{\alpha-1}\right\} +\omega_f(\varepsilon,.,.) + l_1\ \omega(x,\varepsilon).$$

or

$$\omega(\mathcal{F}x,\varepsilon) \leq \frac{l_2}{\Gamma(\alpha+1)}\left\{\omega_u(\varepsilon,.,.)\ + L\ \alpha\ \varepsilon\ \delta^{\alpha-1}\right\} + \omega_f(\varepsilon,.,.) + l_1\ \omega(x,\varepsilon),$$

where $\delta \in (t_1, t_2)$. Thus, taking the supremum in X, then the limit as $\varepsilon \to 0$, and taking into account the uniform continuity of the functions f and u on bounded sets, we can deduce that

$$\omega_0(\mathcal{F}\text{X}) \leq l_1\ \omega_0(\text{X}). \tag{3.3}$$

In the similar way, we can prove that

$$\omega_0(\mathcal{G}\text{X}) \leq l_1\ \omega_0(\text{X}). \tag{3.4}$$

Finally, liking $(3.1) - (3.4)$ and keeping in mind Theorem2.3, we deduce that the operator $\mathcal{T}$ satisfies on the ball B_{r_0} the Darbo condition with respect to the measure ω_0 with constant

$$k = \frac{l_1\ l_2}{\Gamma(\alpha+1)}(1+r_0)(c+d\ r_0) + l_1\ (a+b\ r_0).$$

Thus, the operator $\mathcal{T}$ is a contraction on B_{r_0} with respect to the measure ω_0, thanks to assumption a_6). Therefore, applying Darbo's theorem we get that the operator $\mathcal{T}$ has a fixed point in B_{r_0}. Consequently, Eq.(1.1) has at least one solution in B_{r_0}. This completes the proof. □

4. EXAMPLES

Example 4.1. If $f(t,x,y) = f_1(t,x)$ and $g(t,x,y) = 1$, then Eq.(1.1) is the well-known functional equation of the first order with delay

$$x(t) = f_1(t, x(\beta(t))),$$

see [7] and references therein.

Example 4.2. If $f(t,x,y) = a(t) + y$ and $g(t,x,y) = 1$, then Eq.(1.1) reduces to the Abel integral equation of the second kind

$$x(t) = a(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(s))}{(t-s)^{1-\alpha}}\, ds.$$

On the other hand, for $f(t,x,y) = 1$ and $g(t,x,y) = a(t) + y$, Eq.(1.1) reduces to the well-known quadratic integral equation of Urysohn type

$$x(t) = a(t) + x(t) \int_0^1 v(t,s,x(s))\, ds.$$

Example 4.3. If $g(t,x,y) = 1$, $\beta(t) = t$ and $\alpha = 1$, then Eq.(1.1) becomes a functional-integral equation

$$x(t) = f\left(t, x(t), \int_0^t u(t,s,x(s))\, ds\right). \tag{4.1}$$

The authors proved in [14] the existence of solutions to Eq.(4.1). These solutions are continuous and bounded on the interval $[0,\infty)$ and are globally attractive.

Example 4.4. In the case $f(t,x,y) = 1$, $g(t,x,y) = 1 + y$ and $v(t,s,x) = \frac{t}{t+s}\,\phi(s)\,x$, Eq.(1.1) has the form

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s}\, \phi(s)\, x(s)\, ds. \tag{4.2}$$

Eq.(4.2) is the famous quadratic integral equation of Chandrasekhar type considered in many papers and monographs (cf. [1, 5, 10, 15] for instance). Some Problems considered in the theory of radiative transfer, in the theory of neutron transport and in the kinetic theory of gases lead to Eq.(4.2) (cf. [3, 5, 7, 9, 10, 12, 15, 16]).

Remark 4.5. In order to apply our technique to Eq.(4.2) we have to impose an additional condition that the characteristic function ϕ is continuous and satisfies $\phi(0) = 0$. This condition will ensure that the kernel $v(t,s,x)$ defined by

$$v(t,s,x) = \begin{cases} 0, & s = 0,\ t \geq 0,\ x \in \mathbb{R} \\ \frac{t}{t+s}\phi(s)\, x, & s \neq 0,\ t \geq 0,\ x \in \mathbb{R} \end{cases}$$

is continuous on $\mathrm{I} \times \mathrm{I} \times \mathbb{R}$ in accordance with assumption a_3), see [7].

5. ACKNOWLEDGMENTS

This work was completed when the author was visiting the Abdus Salam ICTP, Trieste, Italy as Regular Associate. It is a pleasure for him to express gratitude for its financial support and the warm hospitality. The author is grateful to the referee and Professor J. Graef for their remarks, especially Remark 2.2.

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