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OSCILLATION CRITERIA OF HILLE AND NEHARI TYPES FOR THIRD-ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. The objective of this paper is to systematically study oscillation and asymptotic behavior of the third order-nonlinear delay differential equation

$$((x''(t))^{\gamma})' + q(t)x^{\gamma}(\tau(t)) = 0, \quad t \geq t_0, \tag{$*$}$$

where $q(t)$ is a positive function, $\gamma > 0$ is a quotient of odd positive integers and the delay function $\tau(t) \leq t$ satisfies $\lim_{t\to\infty} \tau(t) = \infty$. We establish some sufficient conditions of Hille and Nehari types, which ensure that $(*)$ is oscillatory or the solutions converge to zero. Our results in the nondelay case extend and improve some known results in the literature and in the delay case the results can be applied to new classes of equations which are not covered by the known criteria. Some examples are considered to illustrate the main results.

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1. INTRODUCTION

In recent years, the qualitative theory and asymptotic behavior of differential equations and their applications have been and still are receiving intensive attention. As far as oscillation theory is concerned, most texts in differential equations, both elementary and advanced, deal with second-order equations. In fact, in the last few years several monographs and hundreds of research papers concerning oscillation theory have been written, see for example the monographs [1, 7, 9, 12, 16, 27]. Although differential equations of second-order have been studied extensively, the study of qualitative behavior of third-order differential equations has received considerably less attention in the literature, especially the third-order delay differential equations, we refer the reader to [4, 5, 6, 8, 10, 14, 15, 17, 20, 23, 24, 25, 26], however certain results for third-order differential equations have been known for a long time along with their applications in mathematical modeling in biology and physics.

Birkhoff in 1908 (see [3]), applied methods of projective geometry and started the study of separation and comparison theorems for third-order equations. While Birkhoff's paper is a necessary reference for any paper on third-order equations, his results or methods are seldom quoted. In 1961 Hanan [10] studied the oscillation

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and nonoscillation of two different types of third order differential equations and gave definitions of two types of solutions. The paper was the starting point for many investigations to the asymptotic behavior of third-order equations. In 1970 Barrett [2] made a self-contained inductive developments from equations of one order to the next. Most of the discussion dealt with equations of second, third and fourth orders, but most of the results for third-order equations dealt with the canonical forms, asymptotic behavior of the fundamental nonoscillatory solutions and the disconjuagcy.

In this paper, we are concerned with oscillation of third-order delay differential equations of the form

$$\left((x''(t))^{\gamma}\right)' + q(t)x^{\gamma}(\tau(t)) = 0, \quad t \geq t_0. \tag{E$_0$}$$

With regard of (E_0) we will assume that the following condition is satisfied:

(h_1). $q(t)$ is a positive real-valued continuous function, $\tau(t) \leq t$ and satisfies $\lim_{t\to\infty} \tau(t) = \infty$ and γ is a quotient of odd positive integers.

Let $T_0 = \min\{\tau(t) : t \geq 0\}$ and $\tau^{-1}(t) = \sup\{s \geq 0 : \tau(s) \leq t\}$ for $t \geq T_0$. Clearly $\tau^{-1}(t) \geq t$ for $t \geq T_0$, $\tau^{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau(t)$ when the latter exists. By a solution of (E_0) we mean a nontrivial real-valued function $x(t)$ which has the properties $x'(t) \in C^1[\tau^{-1}(t_0), \infty)$, and $(x''(t))^{\gamma} \in C^1[\tau^{-1}(t_0), \infty)$. Our attention is restricted to those solutions of (E_0) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geq t_x$. We make a standing hypothesis that (E_0) does possess such solutions. A solution of (E_0) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is nonoscillatory. Equation (E_0) is said to be oscillatory in case there exists at least one oscillatory solution. We recall that equation (E_0) is disconjugate on an interval $I = [t_0, \infty)$ in case no nontrivial solution has more than two zeros on $I = [t_0, \infty)$, counting multiplicity.

Our motivation for considering the third-order differential equation is that it is an appropriate equation for some applications in extrema, biology and physics. For completeness, we present some of the applications of third-order differential equations.

In the early years of eighteenth century a number of problems led to differential equations of second and third orders. In 1701 James Bernoulli published the solution to the Isoperimetric Problem- a problem in which it is required to make one integral a maximum or minimum, while keeping constant the integral of a second given function, thus resulting in a differential equation of the third-order, see [18].

In the early of 1950's Alan Lloyd Hodgkin and Andrew Huxley developed a mathematical model for the propagation of electrical pulses in the nerve of a squid. The original model described the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. The Hodgkin-Huxley model, is a set of non-linear ordinary differential equations, that approximates the electrical characteristics of excitable cells such as neurons and cardiac myocytes. The model has played a seminal role in biophysics and neuronal modeling. Alan Lloyd Hodgkin and Andrew Huxley were awarded A Nobel prize in 1963 for this work. A reduced version of the Hodgkin-Huxley model was proposed by Nagumo, he suggested a relatively third-order differential equation of the form

$$y'''(x) - cy''(x) + f'(y)y'(x) - (b/c)y(x) = 0,\ y' = \frac{dy}{dx}0,\ f'(y) = \frac{df}{dy},$$

as a model exhibiting many of the features of the Hodgkin-Huxley equations, where the function f is cubic function. For more details of Nagumo's equation, we refer the reader to the paper by Mckean [19] who gave some of the background of these equations and summarized some of the numerical results of this model.

For application in physics, Vreeke and Sandquist [28] proposed the systems of differential equations (which is equivalent to third-order differential equation),

$$\left.\begin{array}{l} \frac{dx_1(t)}{dt} = x_1(\gamma_1(1-x_2) + \gamma_2(1-x_3)), \\ \frac{dx_2(t)}{dt} = \gamma_3(x_1 - x_2), \\ \frac{dx_2(t)}{dt} = \gamma_4(x_1 - x_3), \end{array}\right\}$$

to describe the two temperature feedback nuclear reactor problem, where x_1 is normalized neutron density, x_2 and x_3 are normalized temperatures, x_2 being associated with fuel and x_3, with the moderator or coolant, γ_3 and γ_4 are positive heat transfer coefficients, γ_1 and γ_2 are normalized effective neutron lifetime parameters associated with the temperature feedbacks. The expression $\rho = \gamma_1(1-x_2) + \gamma_2(1-x_3)$ in the first equation is called the reactivity and is a measure of multiplication factor of the neutrons in the fission reactor.

Also in physics the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + f(u) = 0,$$

arises in a wide variety fascinating physical phenomena. For instance, we recall that the Kuramoto-Sivashinsky equation is introduced to describe pattern formulation in reaction diffusion systems, and to model the instability of flame front propagation, see Kuramoto and Yamada [13] and Michelson [21]. To find the travelling wave solutions of this partial differential equation, we may use the substitution of the form $u(x, ct) = u(x - ct)$ with speed c and one has to solve the nonlinear-third order differential equation of the form

$$\lambda u'''(x) + u'(x) + f(u) = 0,$$

where λ is a parameter and f is an even function, (for example, $f(u) = u^2$ or $f(u) = u^2 - 1$. It would be interesting to study the behavior of the nonlinear third-order differential equation of Emden-Fowler type with forced term taking the form

$$u'''(x) + a(x)u'(x) + b(x)u^\alpha = c(x), \qquad \alpha \geq 2,\ c(x) \geq 0.$$

For oscillation of second order equations, Hille [11] considered the equation

$$x''(t) + q(t)x(t) = 0, \quad t \geq t_0, \tag{1.1}$$

and established a sharp sufficient condition for oscillation. He proved that every solution of (1.1) oscillates if

$$\liminf_{t\to\infty} t \int_t^\infty q(s)ds > \frac{1}{4}. \tag{1.2}$$

Nehari [22] by a different approach proved that if

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^t s^2 q(s)ds > \frac{1}{4}, \tag{1.3}$$

then every solution of (1.1) oscillates. Hille [11] also extended Kneser's theorem and proved that if

$$q^* = \lim_{t\to\infty} \sup t^2 q(t) > \frac{1}{4},$$

then (1.1) is oscillatory and nonoscillatory if

$$q_* = \lim_{t\to\infty} \inf t^2 q(t) < \frac{1}{4}.$$

The equation can be either oscillatory or nonoscillatory if either q_* or $q^* = \frac{1}{4}$. Note that the oscillation constant of (1.1) is $\frac{1}{4}$.

For oscillation of third-order differential equations related to (E_0), Hanan [10] considered the equation

$$x'''(t) + q(t)x(t) = 0, \quad t \in (0, \infty), \tag{E_1}$$

and established some sufficient conditions for oscillation and nonoscillation. He established a sufficient conditions for oscillation and proved that if (1.1) is nonoscillatory and

$$\int_{t_0}^\infty tq(t)dt = \infty, \tag{1.4}$$

then $(E_1)$is oscillatory. For nonoscillation he proved that if (1.1) is nonoscillatory and

$$\int_{t_0}^\infty t^2 q(t)dt < \infty, \tag{1.5}$$

then (1.1) is nonoscillatory. He also used a comparison theorem and extended the Hille-Kneser condition and established a sharp sufficient condition for oscillation and nonoscilation. He proved that (E_1) is nonoscillatory if

$$\limsup_{t\to\infty} t^3 q(t) < \frac{2}{3\sqrt{3}}, \tag{1.6}$$

and (E_1) is oscillatory if

$$\liminf_{t\to\infty} t^3 q(t) > \frac{2}{3\sqrt{3}}. \tag{1.7}$$

Note that the condition (1.7) cannot be weakened. Indeed, let $q(t) = \frac{2}{3\sqrt{3}t^3}$ for $t \geq 1$. Then, we have

$$\liminf_{t\to\infty} t^3 q(t) = \frac{2}{3\sqrt{3}},$$

and (E_1) becomes the third-order Euler differential equation

$$x'''(t) + \frac{2}{3\sqrt{3}}\frac{1}{t^3}x(t) = 0, \quad t \geq 1,$$

which has only nonoscillatory solutions. Note that the roots of the characteristic equation are given by $m = -0.153\,06$, $m = 1.523\,3$, and $m = 1.629\,8$. In other words the oscillation constant $\frac{2}{3\sqrt{3}}$ is the lower bound for oscillation for all solutions of (E_1).

Mehri [20] considered the third-order linear differential equation (E_1) and established sufficient condition for oscillation and proved that (E_1) is oscillatory if and only if

$$\int_{t_0}^{\infty} q(t)dt = \infty. \tag{1.8}$$

Lazer [17] considered a third order differential equation in a general form and as a corollary of his results he proved that (E_1) is oscillatory in case

$$\int_{t_0}^{\infty} t^{1+\delta} q(t)dt = \infty, \quad \text{for some} \quad 0 < \delta < 1, \tag{1.9}$$

which improves the condition (1.8). But one can easily see that the conditions (1.4), (1.8) and (1.9) can not be applied to the cases when $q(t) = \frac{\beta}{t^2}$ and $q(t) = \frac{\beta}{t^3}$ for some $\beta > 0$.

For oscillation of third-order delay differential equations of type (E_0), Ladas et. al. [15] considered the equation

$$x'''(t) + x(t-\tau) = 0,$$

and proved that all solutions are oscillatory if and only if

$$\tau e > 3. \tag{1.10}$$

The natural question now is: *Can the oscillation conditions (1.2) and (1.3) that has been established by Hille* [11] *and Nehari* [22] *for second-order differential equations and the condition (1.7) that has been established by Hanan* [10] *for third-order differential equation without delay can be extended to the third-order nonlinear delay differential equation (E_0)?*

The purpose of this paper is to give an affirmative answer to this question. We will establish some new oscillation criteria for (E_0) which guarantee that every solution oscillates or converges to zero. Our results improve the oscillation conditions (1.4), (1.8), (1.9) and (1.10) that has been established by Hanan [10], Mehri [20] and Lazar [17] and Ladas et. al. [15] and extend the condition (1.7) that has been established by Hanan [10]. The approach that we will use to extend the condition of Hanan is different from the technique that has been applied in [10]. Some examples which dwell upon the importance of our main results are given.

2. MAIN RESULTS

In this Section, we establish some new sufficient conditions which ensure that the solution $x(t)$ of (E_0) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. First, we state and prove some useful lemmas, which we will use in the proof of our main results. We note that if $x(t)$ is a solution of (E_0) then $z = -x$ is also solution of (E_0). Thus, concerning nonoscillatory solutions of (E_0) we can restrict our attention only to the positive ones.

Lemma 2.1. *Assume that* (h_1) *holds and let* $x(t)$ *be an eventually positive solution of* (E_0)*. Then there are only the following two cases for* $t \geqslant t_1$ *sufficiently large:*

Case (I). $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$,

Case (II). $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$.

Proof. Assume that $x(t)$ is a positive solution of (E_0) on $[t_0, \infty)$. Pick $t_1 \in [t_0, \infty)$ so that $t_1 > t_0$ and $x(\tau(t)) > 0$ on $[t_1, \infty)$. From (E_0) and (h_1), we have

$$\left((x''(t))^\gamma\right)' = -q(t)x^\gamma(\tau(t)) < 0 \text{ for } t \geqslant t_1.$$

Thus $(x''(t))^\gamma$ is nonincreasing and of one sign and this implies that $x''(t)$ is of one sign. If we admit that $x''(t) \leq 0$ then $x'(t)$ is decreasing and there exists a negative constant d and $t_2 \geqslant t_1$ such that

$$(x''(t))^\gamma \leq d \quad \text{for} \quad t \geqslant t_2.$$

Integrating from t_2 to t, we obtain

$$x'(t) \leq x'(t_2) + d^{\frac{1}{\gamma}} \int_{t_2}^{t} ds. \tag{2.1}$$

Letting $t \to \infty$, then $x'(t) \to -\infty$. Thus, there is an integer $t_3 \geqslant t_2$ such that for $t \geqslant t_3$, $x'(t) \leq x'(t_3) < 0$. Integrating from t_3 to t, we obtain

$$x(t) - x(t_3) \leq a(t_3)x'(t_3) \int_{t_3}^{t} ds, \tag{2.2}$$

which implies that $x(t) \to -\infty$ as $t \to \infty$, a contradiction with the fact that $x(t) > 0$. Hence $x''(t) > 0$ and $x'(t)$ is increasing and of one sign. The proof is complete. □

Lemma 2.2. *Assume that* (h_1) *holds and let* $x(t)$ *be a solution of* (E_0) *which satisfies Case* (II) *of Lemma 2.1. If*

$$\int_{t_0}^{\infty} \int_{z}^{\infty} \left[\int_{u}^{\infty} q(s)ds\right]^{\frac{1}{\gamma}} du\, dz = \infty, \tag{2.3}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that Case (II) in Lemma 2.1 holds for $t \geq t_1$. In this case since $x(t)$ is positive and decreasing, it follows that

$$\lim_{t\to\infty} x(t) = l \quad \text{exists} \quad \text{(finite)}.$$

We prove $l = 0$. Assume not, i.e., $l > 0$. Hence $x(\tau(t)) \geq x(t) > l$, for $t \geq t_2 > t_1$ sufficiently large. Integrating (E_0) from t to ∞, and using $x^\gamma(\tau(t)) \geq l^\gamma$ we get

$$x''(t) \geq l \left[\int_t^\infty q(s)ds\right]^{\frac{1}{\gamma}}.$$

Integrating again from t to ∞, we have

$$-x'(t) \geq l \int_t^\infty \left[\int_u^\infty q(s)ds\right]^{\frac{1}{\gamma}} du.$$

Integrating from t_2 to ∞, we obtain

$$x(t_2) \geq l \int_{t_2}^\infty \int_z^\infty \left[\int_u^\infty q(s)ds\right]^{\frac{1}{\gamma}} du\, dz.$$

This is a contradiction with (2.3). Then $l = 0$ and the proof is complete. □

Now, we state and prove the main results. For simplicity, we introduce the following notations:

$$P_* := \liminf_{t\to\infty} t^\gamma \int_t^\infty P(s)ds \text{ and } Q_* := \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^t s^{\gamma+1}P(s)ds, \tag{2.4}$$

where $P(s) = q(s)\left(\frac{\tau(s)}{s}\right)^\gamma \left(\frac{\tau(s)-T}{2}\right)^\gamma$, and we assume that

$$\int_T^\infty q(s)\tau^\gamma(s)(s-T)^\gamma ds = \infty, \text{ for } T \geq t_0. \tag{2.5}$$

Theorem 2.3. *Assume that (h_1) and (2.5) hold. Let $x(t)$ be a nonoscillatory solution of (E_0) such that $x(t)$ and $x(\tau(t)) > 0$ and suppose that Case (I) of Lemma 2.1 holds for $t \geq T > t_0$. Define $w(t) := \left(\frac{x''(t)}{x'(t)}\right)^\gamma$, and*

$$r := \liminf_{t\to\infty} t^\gamma w(t), \quad \textit{and} \quad R := \limsup_{t\to\infty} t^\gamma w(t). \tag{2.6}$$

Then

$$P_* \leq r - r^{1+\frac{1}{\gamma}}, \tag{2.7}$$

and

$$Q_* \leq \gamma R - \gamma R^{1+\frac{1}{\gamma}}. \tag{2.8}$$

Proof. From Lemma 2.1 when the *Case (I)* is satisfied, we can easily see that there exists $T \geq t_0$ such that $x(t)$ satisfies

$$x(\tau(t)) > 0, \quad x'(t) > 0, \quad x''(t) > 0, \quad x''' \leq 0, \quad t \in [T,\ \infty).$$

From the definition of $w(t)$ and (E_0), we see that $w(t)$ is positive and satisfies

$$\begin{aligned} w'(t) &= \frac{(x'(t))^{\gamma}\left((x''(t))^{\gamma}\right)' - \gamma\left(x'(t)\right)^{\gamma-1}\left(x''(t)\right)^{\gamma+1}}{(x'(t))^{2\gamma}} \\ &= \frac{\left((x''(t))^{\gamma}\right)'}{(x(\tau(t)))^{\gamma}}\frac{(x(\tau(t)))^{\gamma}}{(x'(t))^{\gamma}} - \gamma w^{\frac{\gamma+1}{\gamma}}(t) \\ &= -q(t)\frac{(x(\tau(t)))^{\gamma}}{(x'(t))^{\gamma}} - \gamma w^{\frac{\gamma+1}{\gamma}}(t). \end{aligned} \tag{2.9}$$

Let $h_1(t,T) = (t-T)$, $h_2(t,T) = \frac{(t-T)^2}{2}$ and define

$$X(t) := (t-T)x(t) - h_2(t,T)x'(t).$$

Then $X(T) = 0$, and

$$\begin{aligned} X'(t) &= (t-T)x'(t) + x(t) - h_2(t,T)x''(t) - (t-T)x'(t) \\ &= x(t) - h_2(t,T)x''(t) \\ &= x(t) - \left(\int_T^t (u-T)du\right)x''(t). \end{aligned}$$

By Taylor's Theorem, since $x''(t)$ is nonincreasing, we have

$$\begin{aligned} x(t) &= x(T) + h_1(t,T)x'(T) + \int_T^t h_1(t,u)x''(u)du \\ &\geq x(T) + h_1(t,T)x'(T) + x''(t)\int_T^t h_1(t,u)du. \end{aligned}$$

Hence $X'(t) > 0$ on $[T,\infty)$. Since $X(T) = 0$, we get that $X(t) > 0$ on (T,∞). This implies that

$$\frac{x(t)}{x'(t)} > \frac{h_2(t,T)}{(t-T)} = \frac{(t-T)}{2}, \quad t \in (T,\infty). \tag{2.10}$$

Next let

$$U(t) := x'(t) - tx''(t).$$

Since $U'(t) = -tx'''(t) > 0$ for $t \in [T,\,\infty)$, we have that $U(t)$ is strictly increasing on $[T,\infty)$. We claim that there is a $t_1 \in [T,\infty)$ such that $U(t) > 0$ on $[t_1,\infty)$. Assume not. Then $U(t) < 0$ on $[t_1,\,\infty)$. Therefore,

$$\left(\frac{x'(t)}{t}\right)' = \frac{tx''(t) - x'(t)}{t^2} = -\frac{U(t)}{t^2} > 0, \quad t \in [t_1,\infty),$$

which implies that $x'(t)/t$ is strictly increasing on $[t_1,\infty)$. Pick $t_2 \in [t_1,\,\infty)$ so that $\tau(t) \geq \tau(t_2)$, for $t \geq t_2$. Then, since $x'(t)/t$ is strictly increasing, we have

$$x'(\tau(t))/\tau(t) \geq x'(\tau(t_2))/\tau(t_2) =: d > 0,$$

so that $x'(\tau(t)) \geq d\tau(t)$ for $t \geq t_2$. This implies that, since $x(t) \geq \frac{(t-T)}{2}x'(t)$,

$$x^{\gamma}(\tau(t)) \geq d^{\gamma}\tau^{\gamma}(t)\frac{(t-T)^{\gamma}}{2}.$$

Now by integrating both sides of (E_0) from t_2 to t, we have

$$(x''(t))^\gamma - (x''(t_2))^\gamma + \frac{d^\gamma}{2^\gamma}\int_{t_2}^{t} q(s)\tau^\gamma(s)(s-T)^\gamma ds \le 0.$$

This implies that

$$(x''(t_2))^\gamma \ge \frac{d^\gamma}{2^\gamma}\int_{t_2}^{t} q(s)\tau^\gamma(s)(s-T)^\gamma ds,$$

which contradicts (2.5). Hence there is a $t_1 \in [T, \infty)$ such that $U(t) > 0$ on $[t_1, \infty)$. Consequently,

$$\left(\frac{x'(t)}{t}\right)' = \frac{tx''(t) - x'(t)}{t^2} = -\frac{U(t)}{t^2} < 0, \quad t \in [t_1, \infty).$$

Then $x'(t) > tx''(t)$ and $\frac{x'(t)}{t}$ is strictly decreasing on $[t_1, \infty)$. This implies that

$$\frac{x'(\tau(t))}{\tau(t)} \ge \frac{x'(t)}{t}.$$

Substituting into (2.9), we have

$$w'(t) \le -q(t)\left(\frac{\tau(t)}{t}\right)^\gamma \frac{(x(\tau(t)))^\gamma}{(x'(\tau(t))^\gamma} - \gamma w^{\frac{\gamma+1}{\gamma}}(t).$$

Using (2.10) and the fact that $x(t) \ge \frac{(t-T)}{2}x'(t)$, we have

$$w'(t) + P(t) + \gamma w^{\frac{\gamma+1}{\gamma}}(t) \le 0. \tag{2.11}$$

Since $P(t) > 0$ and $w(t) > 0$ for $t \ge t_1$, we have from (2.11) that $w'(t) \le 0$, and

$$-\left(w'(t)/\gamma w^{\frac{\gamma+1}{\gamma}}(t)\right) > 1, \quad \text{for } t \ge t_1. \tag{2.12}$$

This implies that

$$\left(1/w^{\frac{1}{\gamma}}(t)\right)' > 1. \tag{2.13}$$

Integrating the last inequality from t_1 to t, we obtain

$$(t - t_1)^\gamma w(t) < 1. \tag{2.14}$$

This implies that

$$\lim_{t\to\infty} w(t) = 0, \quad \lim_{t\to\infty} t^\mu w(t) = 0, \text{ for } \mu < \gamma, \ \lim_{t\to\infty}\frac{1}{t}\int_{t_1}^{t} s^{\gamma-1}w(s)ds = 0. \tag{2.15}$$

From (2.6) and (2.14), we have

$$0 < r < 1 \quad \text{and} \quad 0 < R < 1, \tag{2.16}$$

since if $r = 0$ and $R = 0$, there is nothing to prove. Now, we prove that (2.7) holds. Integrating (2.11) from t to ∞ ($t \ge t_1$) and using (2.15), we have

$$w(t) \ge \int_t^\infty P(s)ds + \gamma\int_t^\infty w^{\frac{\gamma+1}{\gamma}}(s)ds \ \text{ for } t \ge t_1. \tag{2.17}$$

This implies that

$$r = \lim_{t\to\infty} \inf t^{\gamma} w(t) \geq P_*.$$

It follows from (2.6) that for any arbitrary $\epsilon > 0$ and sufficiently small there exists $t_2 \geq t_1$ such that

$$r - \epsilon < t^{\gamma} w(t) < r + \epsilon,\ t^{\gamma} \int_t^{\infty} P(s)ds \geq P_* - \epsilon,\ t \geq t_2. \tag{2.18}$$

Again from (2.17), we have

$$\begin{aligned} t^{\gamma} w(t) &\geq t^{\gamma} \int_t^{\infty} P(s)ds + \gamma t^{\gamma} \int_t^{\infty} w^{\frac{\gamma+1}{\gamma}}(s)ds \\ &\geq t^{\gamma} \int_t^{\infty} P(s)ds + \gamma t^{\gamma} \int_t^{\infty} \frac{s^{\gamma+1} w^{\frac{\gamma+1}{\gamma}}}{s^{\gamma+1}} ds \\ &\geq t^{\gamma} \int_t^{\infty} P(s)ds + t^{\gamma} \int_t^{\infty} (r-\epsilon)^{1+\frac{1}{\gamma}} \frac{\gamma}{s^{\gamma+1}} ds \\ &\geq t^{\gamma} \int_t^{\infty} P(s)ds + (r-\epsilon)^{1+\frac{1}{\gamma}}\, t^{\gamma} \int_t^{\infty} \left(\frac{-1}{s^{\gamma}}\right)' ds \\ &= t^{\gamma} \int_t^{\infty} P(s)ds + (r-\epsilon)^{1+\frac{1}{\gamma}}. \end{aligned} \tag{2.19}$$

Then from (2.18) and (2.19), we have

$$r \geq P_* - \epsilon + (r-\epsilon)^{1+\frac{1}{\gamma}}.$$

Then (since ϵ is arbitrary small), we have

$$P_* \leq r - r^{1+\frac{1}{\gamma}},$$

and this proves (2.7). Now, we prove (2.8). Form (2.11), we see that

$$w'(t) + P(t) + \gamma\,(w(t))^{\lambda} \leq 0, \quad \text{for} \ \ t \geq t_1, \tag{2.20}$$

where $\lambda = 1 + \frac{1}{\gamma}$. Multiplying (2.20) by $s^{\gamma+1}$, and integrating from t_1 to t $(t \geq t_1)$ and using integration by parts, we obtain

$$\begin{aligned} \int_{t_1}^{t} s^{\gamma+1} P(s)ds &\leq -\int_{t_1}^{t} s^{\gamma+1} w'(s)ds - \gamma \int_{t_1}^{t} s^{\gamma+1}\,(w(s))^{\lambda}\, ds \\ &= \left[-s^{\gamma+1} w\right]_{t_1}^{t} + \int_{t_1}^{t} (s^{\gamma+1})' w(s)ds - \gamma \int_{t_1}^{t} s^{\gamma+1}\,(w(s))^{\lambda}\, ds \\ &\leq -t^{\gamma+1} w(t) + t_1^{\gamma+1} w(t_1) + \int_{t_1}^{t} (\gamma+1) s^{\gamma} w(s)ds \\ &\quad -\gamma \int_{t_1}^{t} s^{\gamma+1} (w(s))^{\lambda} ds \\ &= -t^{\gamma+1} w(t) + t_1^{\gamma+1} w(t_1) \\ &\quad + \int_{t_1}^{t} (\gamma+1) s^{\gamma} w(s)ds - \int_{t_1}^{t} \gamma s^{\gamma+1} (w(s))^{\lambda} ds \end{aligned}$$

Thus

$$\int_{t_1}^{t} s^{\gamma+1}P(s)ds \leq -t^{\gamma+1}w(t)+t_1^{\gamma+1}w(t_1)+\int_{t_1}^{t}(\gamma+1)s^{\gamma}w(s)ds - \int_{t_1}^{t}\gamma s^{\gamma+1}(w(s))^{\lambda}ds.$$

It follows that

$$t^{\gamma+1}w(t) \leq (t_1)^{\gamma+1}w(t_1)-\int_{t_1}^{t}s^{\gamma+1}P(s)ds + \int_{t_1}^{t}\left((w(s))^{\frac{1}{\gamma}}s\right)^{\gamma}\left[\gamma+1-\gamma s^{\gamma+1}(w(s))^{\frac{1}{\gamma}}\right]ds.$$

Then, we have

$$t^{\gamma}w(t) \leq \frac{(t_1)^{\gamma+1}w(t_1)}{t}-\frac{1}{t}\int_{t_1}^{t}s^{\gamma+1}P(s)ds + \frac{1}{t}\int_{t_1}^{t}\left((w(s))^{\frac{1}{\gamma}}s\right)^{\gamma}\left[\gamma+1-\gamma s^{\gamma+1}(w(s))^{\frac{1}{\gamma}}\right]ds. \tag{2.21}$$

Using the fact that $Bu-Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}$, where A and B are constants and $A>0$, we see that

$$\left((w)^{\frac{1}{\gamma}}s\right)^{\gamma}\left[\gamma+1-\gamma s^{\gamma+1}(w)^{\frac{1}{\gamma}}\right] \leq 1.$$

This and (2.21) imply that

$$t^{\gamma}w(t) \leq \frac{(t_1)^{\gamma+1}w(t_1)}{t}-\frac{1}{t}\int_{t_1}^{t}s^{\gamma+1}P(s)ds+1-\frac{t_1}{t}. \tag{2.22}$$

Thus

$$\limsup_{t\to\infty} t^{\gamma}w(t) \leq 1-\liminf_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}s^{\gamma+1}P(s)ds.$$

This implies that

$$R=\limsup_{t\to\infty} t^{\gamma}w(t) \leq 1-Q_*.$$

It follows from (2.4) and (2.6) that for any arbitrary $\epsilon>0$ and sufficiently small there exists $t_2 \geq t_1$ such that

$$R-\epsilon < t^{\gamma}w(t) < R+\epsilon, \quad \frac{1}{t}\int_{t_0}^{t}s^{\gamma+1}P(s)ds > Q_*-\epsilon, \ t \geq t_2. \tag{2.23}$$

Then from (2.21) and (2.23), we have

$$R \leq -Q_*+\epsilon+(R+\epsilon)\left((\gamma+1)-\gamma(R+\epsilon)^{\frac{1}{\gamma}}\right). \tag{2.24}$$

Then (since ϵ is arbitrary small), we obtain

$$Q_* \leq \gamma R-\gamma R^{1+\frac{1}{\gamma}}.$$

which proves (2.8). This completes the proof. $\square$

From Theorem 2.3 we have the following new oscillation criteria of (E_0), which is of Nehari [22] type.

Theorem 2.4. *Assume that (h_1), (2.3) and (2.5) hold. Let $x(t)$ be a solution of (E_0). If*

$$Q_* = \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^{\gamma+1} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}(\tau(s)-T)^{\gamma}\,ds > \frac{2^{\gamma}\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}}, \tag{2.25}$$

then $x(t)$ is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (E_0) with $x(\tau(t)) > 0$ on $[t_1,\infty)$. Assume that the part (I) of Lemma 2.1 holds. Let $w(t)$ be as defined in Theorem 2.3 and $R = \limsup_{t\to\infty} t^{\gamma} w(t)$. Then from Theorem 2.3, we see that R satisfies the inequality

$$Q_* \le \gamma R - \gamma R^{1+\frac{1}{\gamma}}.$$

Using the fact that

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \text{ for } A > 0,$$

we have

$$Q_* \le \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}},$$

which contradicts (2.25). If part (II) of Lemma 2.1 holds, then by (2.3) and Lemma 2.2, we can easily prove that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

From Theorem 2.3, we have the following oscillation criteria of (E_0) which of Hille [11] type.

Theorem 2.5. *Assume that (h_1), (2.3) and (2.5) holds and let $x(t)$ be a solution of (E_0). If*

$$P_* = \liminf_{t\to\infty} t^{\gamma}\int_{t}^{\infty} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma}(\tau(s)-T)^{\gamma}\,ds > \frac{2^{\gamma}\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{2.26}$$

then $x(t)$ oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (E_0) with $x(\tau(t)) > 0$ on $[t_1,\infty)$. Assume that the part (I) of Lemma 2.1 holds. Let $w(t)$ be as defined in Theorem 2.3 and $r = \liminf_{t\to\infty} t^{\gamma} w(t)$. Then from Theorem 2.3, we see that r satisfies the inequality

$$P_* \le r - r^{1+\frac{1}{\gamma}}.$$

As in the proof of Theorem 2.3, we see that

$$P_* \le \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}},$$

which contradicts (2.26). Now if part (II) of Lemma 2.1 holds, then by (2.3) and Lemma 2.2, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

From Theorems 2.4 and 2.5 as a special case when $\gamma = 1$ and $\tau(t) = t$, we have the following new oscillation result for equation (E_1).

Corollary 2.6. *Assume that $q(t)$ is a positive function such that*

$$\int_{t_0}^{\infty} q(s)\tau(s)(s-T)ds = \infty, \ \text{and} \int_{t_0}^{\infty}\int_{z}^{\infty}\int_{u}^{\infty} q(s)ds\,du\,dz = \infty.$$

Let $x(t)$ be a solution of (E_1). If for $T \geq t_0$

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 q(s)\,(s-T)\,ds > \frac{1}{2}, \tag{2.27}$$

or

$$\liminf_{t\to\infty} t\int_{t}^{\infty} q(s)\,(s-T)\,ds > \frac{1}{2}, \tag{2.28}$$

then $x(t)$ is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Remark 2.7. From Corollary 2.6, we note that the oscillation constant that we obtained, which is $\frac{1}{2}$, for the third order differential equation (E_1) is different from the oscillation constant $\frac{1}{4}$ for second-order differential equation that has been established by Hille and Nehari. One may ask that why the two conditions are different. To give the answer to this question, we consider the equation of the form

$$x'''(t) + \frac{3}{10t^3}x(t) = 0, \quad t \geq 1.$$

In this case the roots of the characteristic equation

$$m(m-1)(m-2) + \frac{3}{10} = 0,$$

are given by $m = -0.125\,42$, $m = 1.338\,9$ and $m = 1.786\,5$. One can easily see that there is no any oscillatory solutions. Note that $\frac{1}{4} < \frac{3}{10} < \frac{1}{2}$. But for the third order differential equation

$$x'''(t) + \frac{6}{10t^3}x(t) = 0, \quad t \geq 1.$$

The roots of the characteristic equation of this equation are given by $m = 1.610\,6 - 0.344\,23i$, $m = 1.610\,6 + 0.344\,23i$ and $m = -0.221\,20$. Then the solutions are oscillatory or converge to zero. Note that $\frac{6}{10} > \frac{1}{2}$.

Open problems. (i). What are the sufficient conditions for oscillation of (E_1) when (2.25) and (2.26) are not satisfied.

(ii). If the conditions in Theorems 2.4 and 2.5 can be improved to fill the gap between $1/2$ and $\frac{2}{3\sqrt{3}}$ when $\gamma = 1$.

Example 2.8. Consider the Euler third-order linear differential equation

$$x'''(t) + \frac{6}{t^3}x(t) = 0, \quad t \geq 1. \tag{2.29}$$

It is clear that the conditions (2.3) and (2.5) hold. To apply Theorem 2.5 it remains to prove that (2.26) is satisfied. In our case the condition reads

$$\begin{aligned} & \liminf_{t\to\infty} t^{\gamma} \int_t^{\infty} P(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} (\tau(s)-T)^{\gamma}\, ds \\ = & \liminf_{t\to\infty} t \int_t^{\infty} \frac{6}{s^3}(s-1)\, ds = \liminf_{t\to\infty} t \int_t^{\infty} \left[\frac{6}{s^2} - \frac{6}{s^3}\right] ds \\ = & \liminf_{t\to\infty} t \left[\frac{6}{t} - \frac{3}{t^2}\right] = 6 > 1/2. \end{aligned}$$

Then any solution of (2.29) oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$. One can easily see that the basis of solution space of (2.29) is given by

$$\{t^{-1},\ t^2\cos\sqrt{2}\log t,\ t^2\sin\sqrt{2}\log t\}.$$

Example 2.9. Consider the linear differential equation

$$\left((x''(t))^3\right)' + \frac{3}{t^7}x^3(t) = 0,\ t \geq 1. \tag{2.30}$$

It is clear that the (2.3) and (2.5) hold. To apply Theorem 2.5 it remains to prove that (2.26) is satisfied. In this case (2.26) reads

$$\liminf_{t\to\infty} t^{\gamma} \int_t^{\infty} P(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} (\tau(s)-T)^{\gamma}\, ds = \liminf_{t\to\infty} t^3 \int_t^{\infty} \frac{3}{s^7}(s-1)^3\, ds = 1 > \frac{27}{32}.$$

Then the solutions of (2.30) oscillates or satisfy $\lim_{t\to\infty} x(t) = 0$. Note the results by [5, 6, 10, 17, 15, 20] cannot be applied to the equations (2.29) and (2.30). So our results improve the results in [5, 6, 10, 17, 15, 20].

In the following, we consider the equation

$$x'''(t) + q(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{E$_2$}$$

and prove that the equation is oscillatory or the solutions tend to zero.

Theorem 2.10. *Assume that* (h_1) *holds. Let* $x(t)$ *be a solution of* (E_2)*. If*

$$\int_{t_0}^{\infty} \left[\tau^2(s)q(s) - \frac{2}{3\sqrt{3}s}\right] ds = \infty, \tag{2.31}$$

then $x(t)$ *is oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (E_2) on $[t_0, \infty)$. Without loss of generality we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > t_1$. It follows by Lemma 2.1 that here exists $t_2 \geq t_1$ such that either:

(I) $x(t)x'(t) > 0$, $x''(t) > 0$ and $x'''(t) < 0$ for $t \geq t_2$, or

(II). $x(t)x'(t) < 0$, $x''(t) > 0$ and $x'''(t) < 0$ for $t \geq t_2$.

First, we consider the Case (I) and define

$$u(t) := \frac{tx(t)}{x'(t)} > 0. \tag{2.32}$$

Using (E_2), we see that $u(t)$ satisfies the second-order equation

$$((tu)' + \frac{3}{2}u^2 - 4u)' + \frac{1}{t}\left(u^3 - 3u^2 + 2u + t^3 q(t)\frac{x(\tau(t))}{x(t)}\right) = 0. \tag{2.33}$$

Define $P(u) := u^3 - 3u^2 + 2u + t^3 q\frac{x(\tau(t))}{x(t)}$. One can easily see that $P(u)$ attends its minimum at $u = \frac{1}{3}\sqrt{3} + 1$, and

$$P(u) \geq t^3 q(t)\frac{x(\tau(t))}{x(t)} - \frac{2}{3\sqrt{3}}. \tag{2.34}$$

Substituting the estimate (2.34) into (2.33), we have

$$((tu)' + \frac{3}{2}u^2 - 4u)' \leq -\frac{1}{t}\left(t^3 q(t)\frac{x(\tau(t))}{x(t)} - \frac{2}{3\sqrt{3}}\right). \tag{2.35}$$

From Kiguradze Lemma [12], which show that if a function $x(t)$ satisfies $x^{(i)} > 0$, $i = 0, 1, 2, \ldots, n$ and $x^{(n+1)} < 0$, then

$$\frac{x}{t^n/ni} \geq \frac{x'}{t^{n-1}/(n-1)i}.$$

Then, from (I), we see that $n = 2$, and then from the last inequality we have

$$\frac{x'(t)}{x(t)} \leq \frac{2}{t}, \quad \text{for } t \geq t_2.$$

Integrating the last inequality from $\tau(t)$ to t, we obtain

$$x(\tau(t)) \geq \frac{\tau^2(t)}{t^2}x(t). \tag{2.36}$$

Substituting (2.36) into (2.35), we have

$$((tu)' + \frac{3}{2}u^2 - 4u)' \leq -\left(q(t)\tau^2(t) - \frac{2}{3\sqrt{3}t})^{\frac{3}{2}}\right).$$

Integrating the above inequality from t_2 to t, we have

$$(tu)' + \frac{3}{2}u^2 - 4u \leq K_0 - (\frac{3}{2}u^2 - 4u) - \int_{t_2}^{t}\left(q(s)\tau^2(s) - \frac{2}{3\sqrt{3}s})^{\frac{3}{2}}\right) ds,$$

where K_0 is a constant. Since $\frac{3}{2}u^2 - 4u \geq \frac{-8}{3}$, then

$$(tu)' \leq K_1 - \int_{t_2}^{t}\left(q(s)\tau^2(s) - \frac{2}{3\sqrt{3}s})^{\frac{3}{2}}\right) ds,$$

where $K_1 = K_0 + \frac{8}{3}$. An integration the above inequality again form t_2 to t, yields that

$$tu \leq K_2 + K_1 t - \int_{t_2}^{t}\int_{t_2}^{s}\left(q(v)\tau^2(v) - \frac{2}{3\sqrt{3}v})^{\frac{3}{2}}\right) dvds,$$

where K_2 is a constant. So it follows from (2.31) that $u(t) < 0$ for sufficiently large t, which contradicts the positivity of $u(t)$. Next, we consider the Case (II) and suppose that $x'(t) < 0$ for $t \geq t_2$. Hence $\lim_{t\to\infty} x(t) = L \geq 0$ exists. Let $L > 0$. Then $x(\tau(t)) \geq x(t) \geq L$ for $t \in [t_2, \infty)$. From (2.31), it follows that, since $\tau(t) \leq t$,

$$\int_{t_0}^{\infty} s^2 q(s)ds \geq \int_{t_0}^{\infty} \tau^2(s)q(s)ds = \infty.$$

Multiplying (E_2) by t^2 and integrating from t_2 to t, we have

$$t^2x''(t) - 2tx'(t) + \frac{9}{4}x(t) \leq K - L\int_{t_2}^{t} s^2q(s)ds,$$

where K is some constant. From the last inequality, we see that $x''(t) < 0$ for large t, which is a contradiction. Hence $L = 0$ and then $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

From Theorem 2.10, we have the following result which of Hille-Kneser type and can be considered as the extension of the condition (1.7) of Hanan [10].

Corollary 2.11. *Assume that (h_1) holds. Let $x(t)$ be a solution of (E_2). If*

$$\liminf_{t\to\infty} \tau^2(s)sq(s) > \frac{2}{3\sqrt{3}}, \tag{2.37}$$

then $x(t)$ is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

The following example illustrates the main results in Theorem 2.10.

Example 2.12. Consider the third order linear delay differential equation

$$x'''(t) + \frac{2}{3\sqrt{3}\tau^3(t)}x(\tau(t)) = 0, t \geq 1, \tag{2.38}$$

where $\tau(t) = t/2$. Now the condition (2.31) reads

$$\begin{aligned}
\int_{t_0}^{\infty}\left[\tau^2(s)q(s) - \frac{2}{3\sqrt{3}s}\right]ds &= \int_{1}^{\infty}\left[\tau^2(s)\frac{2}{3\sqrt{3}\tau^3(s)} - \frac{2}{3\sqrt{3}s}\right]ds \\
&= \int_{1}^{\infty}\left[\frac{4}{3\sqrt{3}s} - \frac{2}{3\sqrt{3}s}\right]ds \\
&= \int_{t_0}^{\infty}\frac{2}{3\sqrt{3}s}ds = \infty.
\end{aligned}$$

Thus (2.31) is satisfied. Then by Theorem 2.10, if $x(t)$ is a solution of (2.38), then $x(t)$ is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. In fact the corresponding characteristic equation which is given by

$$2^{m-4}m(m-1)(m-2) + \frac{1}{3\sqrt{3}} = 0,$$

has only a negative real root given by -1.0232, which implies that the corresponding solution tends to zero. Note that when $\tau(t) = t$, equation (2.38) becomes the third-order Euler differential equation

$$x'''(t) + \frac{2}{3\sqrt{3}t^3}x(t) = 0, t \geq 1,$$

which is disconjugate and has a negative root and two equal positive roots given by $1+\frac{1}{\sqrt{3}}$. This means that this equation is not oscillatory and the delay in the equation has a large effect on the asymptotic behavior of the solutions.

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