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APPROXIMATION OF SOLUTIONS OF THE FORCED DUFFING EQUATION WITH m-POINT BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we discuss the existence and uniqueness of the solution of the forced Duffing equation with m-point boundary conditions. A monotone sequence of approximate solutions converging uniformly and quadratically to the unique solution of the problem is obtained by applying a generalized quasilinearization technique.

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1. INTRODUCTION

Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena such as periodic orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. Another important application of Duffing equation is in the field of the prediction of diseases. A careful measurement and analysis of a strongly chaotic voice has the potential to serve as an early warning system for more serious chaos and possible onset of disease. This chaos is stimulated with the help of Duffing equation. In fact, the success at analyzing and predicting the onset of chaos in speech and its simulation by equations such as the Duffing equation has enhanced the hope that we might be able to predict the onset of arrhythmia and heart attacks someday. However, such predictions are based on the numerical solutions of the Duffing equation.

The monotone iterative technique coupled with the method of upper and lower solutions [1–7] manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. In general, the convergence of the sequence of approximate solutions given by the monotone iterative technique is at most linear [8, 9]. To obtain a sequence of approximate solutions converging quadratically, we use the method of quasilinearization [10]. This method has been developed for a variety of problems

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[11–20]. In view of its diverse applications, this approach is quite an elegant and easier for application algorithms.

The subject of multi-point nonlocal boundary value problems, initiated by Ilin and Moiseev [21,22], has been addressed by many authors, for instance, [23–30]. The multi-point boundary conditions appear in certain problems of thermodynamics, elasticity and wave propagation, see [31] and the references therein. The multi-point boundary conditions may be understood in the sense that the controllers at the end points dissipate or add energy according to censors located at intermediate positions. To the best of our knowledge, the method of quasilinearization has not been developed for Duffing equation with multi-point boundary conditions.

In this paper, we apply a quasilinearization technique to obtain the analytic approximation of the solution of the forced Duffing equation with m-point boundary conditions. In fact, a sequence of approximate solutions converging monotonically and quadratically to the unique solution of the problem is presented. The results obtained in this paper offer an algorithm to study the various practical phenomena such as prediction of the possible onset of vascular diseases, onset of chaos in speech, etc.

2. PRELIMINARIES

Consider the following boundary value problem

$$\begin{cases} u''(t) + \sigma u'(t) + f(t,u) = 0, \ 0 < t < 1, \ \sigma \in \mathbb{R} - \{0\}, \\ u(0) - \mu_1 u'(0) = \sum_{i=1}^{m-2} p_i u(\eta_i), \ u(1) + \mu_2 u'(1) = \sum_{i=1}^{m-2} q_i u(\eta_i), \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, p_i, q_i $(i = 1, 2, \ldots, m-2)$ are nonnegative real constants such that $\sum_{i=1}^{m-2} p_i < 1$, $\sum_{i=1}^{m-2} q_i < 1$, $\eta_i \in (0,1)$, and μ_1, μ_2 are nonnegative constants. It can easily be verified that the homogeneous problem associated with (1.1) has only the trivial solution. Therefore, by Green's function method, the solution of (1.1) can be written as

$$\begin{aligned} u(t) &= \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big(\sum_{i=1}^{m-2} p_i u(\eta_i)\Big) \\ &+ \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big(\sum_{i=1}^{m-2} q_i u(\eta_i)\Big) + \int_0^1 G(t,s) f(s,u(s))ds, \end{aligned}$$

where

$$G(t,s) = \Lambda \begin{cases} [(1-\sigma\mu_2) - e^{\sigma(1-s)}][(1+\sigma\mu_1) - e^{-\sigma t}], & 0 \le t \le s, \\ [(1-\sigma\mu_2) - e^{\sigma(1-t)}][(1+\sigma\mu_1) - e^{-\sigma s}], & s \le t \le 1, \end{cases}$$

$$\Lambda = \frac{e^{\sigma s}}{\sigma[(1-\sigma\mu_2) - (1+\sigma\mu_1)e^{\sigma}]}.$$

We note that $G(t,s) > 0$ on $(0,1) \times (0,1)$.

Definition 2.1. A function $\alpha \in C^2[0,1]$ is a lower solution of (1.1) if

$$\begin{cases} \alpha''(t) + \sigma\alpha'(t) + f(t,\alpha(t)) \geq 0, \ 0 < t < 1, \\ \alpha(0) - \mu_1\alpha'(0) \leq \sum_{i=1}^{m-2} p_i\alpha(\eta_i), \qquad \alpha(1) + \mu_2\alpha'(1) \leq \sum_{i=1}^{m-2} q_i\alpha(\eta_i). \end{cases}$$

Similarly, $\beta \in C^2[0,1]$ is an upper solution of (1.1) if the inequalities in the definition of lower solution are reversed.

Theorem 2.1. Let α and β be lower and upper solutions of the boundary value problem (1.1) respectively. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be such that $f_u(t,u) < 0$. Then $\alpha(t) \leq \beta(t)$, $t \in [0,1]$.

Proof. Set $x(t) = \alpha(t) - \beta(t)$, $t \in [0,1]$ so that

$$\begin{cases} x(0) - \mu_1 x'(0) \leq \sum_{i=1}^{m-2} p_i x(\eta_i) \\ x(1) + \mu_2 x'(1) \leq \sum_{i=1}^{m-2} q_i x(\eta_i). \end{cases} \tag{2.2}$$

For the sake of contradiction, suppose that $x(t) > 0$ for every $t \in [0,1]$. Then $x(t)$ has a positive maximum at some $t_0 \in [0,1]$. If $t_0 \in (0,1)$, then $x(t_0) > 0$, $x'(t_0) = 0$ and $x''(t_0) \leq 0$. In view of the decreasing property of the function $f(t,u)$ in u, it follows that

$$x''(t_0)+\sigma x'(t_0) = \alpha''(t_0)+\sigma\alpha'(t_0)-(\beta''(t_0)+\sigma\beta'(t_0)) \geq -f(t_0,\alpha(t_0))+f(t_0,\beta(t_0)) > 0,$$

which is a contradiction. If $t_0 = 0$, then $x(0) > 0$, $x'(0) = 0$ and from (2.2), we obtain

$$x(0) = x(0) - \mu_1 x'(0) \leq \sum_{i=1}^{m-2} p_i x(\eta_i) \leq \sum_{i=1}^{m-2} p_i x(0),$$

that is, $\left(1 - \sum_{i=1}^{m-2} p_i\right) x(0) \leq 0$ which contradicts the assumption that $\sum_{i=1}^{m-2} p_i < 1$. We have a similar contradiction at $t_0 = 1$. Hence our claim that $x(t) > 0$ for every $t \in [0,1]$ is false. Thus, we can find $t_1 \in [0,1]$ such that $x(t_1) \leq 0$ with $t_0 < t_1$ (the case $t_0 > t_1$ is similar) and there exists $t_2 \in (t_0,t_1)$ such that $x(t_2) = 0$ and $x(t) > 0$ for every $t \in [t_0,t_2)$. Then, using the assumption that $f(t,u)$ is strictly decreasing in u, we find that

$$x''(t) + \sigma x'(t) \geq -f(t,\alpha(t)) + f(t,\beta(t)) > 0,$$

which can alternatively be written as $(x'(t)e^{\sigma t})' > 0$. Integrating from t_0 to t, and using $x'(t_0) = 0$, we obtain $x'(t) > 0$ for every $t \in [t_0,t_2)$ which together with $x'(t_0) = 0$ implies that $x'(t) \geq 0$ for every $t \in [t_0,t_2)$. Thus, $x(t)$ is nondecreasing on $[t_0,t_2)$ which is a contradiction as $x(t)$ has a positive maximum value at $t = t_0$. Similar contradiction occurs at $t_0 = 0, 1$. Thus, we conclude that $\alpha(t) \leq \beta(t)$, $t \in [0,1]$.

Theorem 2.2. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with $f_u(t,u) < 0$, and α, β are respectively lower and upper solutions of the boundary value problem (1.1) such that $\alpha(t) \le \beta(t)$. Then there exists a solution $u(t)$ of (1.1) such that $\alpha(t) \le u(t) \le \beta(t),\ t \in [0,1]$.

Proof. Let us define

$$F(t,u) = \begin{cases} f(t,\beta(t)) - \frac{u-\beta(t)}{1+|u-\beta|}, & \text{if } u > \beta, \\ f(t,u), & \text{if } \alpha \le u \le \beta, \\ f(t,\alpha(t)) - \frac{u-\alpha(t)}{1+|u-\alpha|}, & \text{if } u < \alpha. \end{cases}$$

Since $F(t,u)$ is continuous and bounded, it follows that there exists a solution $u(t)$ of the problem

$$\begin{cases} u''(t) + \sigma u'(t) + F(t,u) = 0,\ 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \sum_{i=1}^{m-2} p_i u(\eta_i), \quad u(1) + \mu_2 u'(1) = \sum_{i=1}^{m-2} q_i u(\eta_i). \end{cases} \tag{2.3}$$

In relation to (2.3), we have

$$\begin{cases} \alpha''(t) + \sigma\alpha'(t) + F(t,\alpha(t)) = \alpha''(t) + \sigma\alpha'(t) + f(t,\alpha(t)) \ge 0,\ 0 < t < 1, \\ \alpha(0) - \mu_1\alpha'(0) = \sum_{i=1}^{m-2} p_i\alpha(\eta_i), \quad \alpha(1) + \mu_2\alpha'(1) = \sum_{i=1}^{m-2} q_i\alpha(\eta_i) \end{cases}$$

and

$$\begin{cases} \beta''(t) + \sigma\beta'(t) + F(t,\beta(t)) = \beta''(t) + \sigma\beta'(t) + f(t,\beta(t)) \le 0, \quad 0 < t < 1, \\ \beta(0) - \mu_1\beta'(0) = \sum_{i=1}^{m-2} p_i\beta(\eta_i), \quad \beta(1) + \mu_2\beta'(1) = \sum_{i=1}^{m-2} q_i\beta(\eta_i), \end{cases}$$

which imply that α and β are lower and upper solutions of (2.3) respectively. By definition of $F(t,u)$, it follows that any solution $u \in [\alpha,\beta]$ of (2.3) is indeed a solution of (1.1). Thus, we just need to show that any solution $u(t)$ of (2.3) satisfies $\alpha(t) \le u(t) \le \beta(t)$, $t \in [0,1]$. Let us assume that $\alpha(t) > u(t)$ on $[0,1]$. Then the function $y(t) = \alpha(t) - u(t)$ has a positive maximum at some $t = t_0 \in [0,1]$. If $t_0 \in (0,1)$, then $y(t_0) > 0$, $y'(t_0) = 0$, $y''(t_0) \le 0$. On the other hand,

$$\begin{aligned} y''(t_0) + \sigma y'(t_0) &= \alpha''(t_0) + \sigma\alpha'(t_0) - [u''(t_0) + \sigma u'(t_0)] \\ &\ge -F(t_0,\alpha(t_0)) + F(t_0,u(t_0)) \\ &= -f(t_0,\alpha(t_0)) + f(t_0,\alpha(t_0)) - \frac{u-\alpha(t_0)}{1+|u-\alpha_0|} > 0, \end{aligned}$$

which contradicts our assumption. If $t_0 = 0$, then $y(0) > 0$, $y'(0) = 0$ and

$$y(0) = y(0) - \mu_1 y'(0) \le \sum_{i=1}^{m-2} p_i y(\eta_i) \le \sum_{i=1}^{m-2} p_i y(0),$$

that is, $\left(1 - \sum_{i=1}^{m-2} p_i\right) y(0) \le 0$ which contradicts the assumption that $\sum_{i=1}^{m-2} p_i < 1$. Similarly, $t_0 = 1$ yields a contradiction. As in the proof of Theorem 2.1, we also obtain the contradiction in the neighbourhood of the point $t_0 \in [0,1]$. Thus, $\alpha(t) \leq u(t)$, $t \in [0,1]$. In a similar manner, it can be shown that $u(t) \leq \beta(t)$, $t \in [0,1]$. Hence we conclude that $\alpha(t) \leq u(t) \leq \beta(t),\ t \in [0,1]$.

Corollary 2.3. Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous with $f_u(t,u) < 0$ on $[0,1] \times \mathbb{R}$. Then the solution of boundary value problem (1.1) is unique.

3. MAIN RESULT

Theorem 3.1. Assume that

($\mathbf{A_1}$) $\alpha, \beta \in C^2[0,1]$ are respectively lower and upper solutions of (1.1) such that $\alpha(t) \leq \beta(t)$, $t \in [0,1]$;

($\mathbf{A_2}$) $f \in C^2([0,1] \times \mathbb{R})$ be such that $f_u(t,u) < 0$ and $(f_{uu}(t,u) + \phi_{uu}(t,u)) \geq 0$, where $\phi_{uu}(t,u)) \geq 0$ for some continuous function $\phi(t,u)$ on $[0,1] \times \mathbb{R}$.

Then, there exists a sequence $\{\alpha_n\}$ of approximate solutions converging monotonically and quadratically to the unique solution of the problem (1.1).

Proof. Let $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(t,u) = f(t,u) + \phi(t,u)$ so that $F_{uu}(t,u) \geq 0$. Using the generalized mean value theorem together with (A_2), we obtain

$$f(t,u) \geq f(t,v) + F_u(t,v)(u-v) + \phi(t,v) - \phi(t,u). \tag{3.1}$$

Setting

$$g(t,u,v) = f(t,v) + F_u(t,v)(u-v) + \phi(t,v) - \phi(t,u), \tag{3.2}$$

we note that $g_u(t,u,v) = [F_u(t,v) - \phi_u(t,u)] \leq [F_u(t,u) - \phi_u(t,u)] = f_u(t,u) < 0$ and

$$\begin{cases} f(t,u) \geq g(t,u,v), \\ f(t,u) = g(t,u,u). \end{cases} \tag{3.3}$$

Now, we fix $\alpha_0 = \alpha$ and consider the problem

$$\begin{cases} u''(t) + \sigma u'(t) + g(t,u,\alpha_0) = 0,\ 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \sum_{i=1}^{m-2} p_i u(\eta_i), \quad u(1) + \mu_2 u'(1) = \sum_{i=1}^{m-2} q_i u(\eta_i). \end{cases} \tag{3.4}$$

Using (A_1) and (3.3), we obtain

$$\begin{cases} \alpha_0''(t) + \sigma\alpha_0'(t) + g(t,\alpha_0,\alpha_0) = \alpha_0''(t) + \sigma\alpha_0'(t) + f(t,\alpha_0) \geq 0, \quad 0 < t < 1, \\ \alpha(0) - \mu_1\alpha'(0) \leq \sum_{i=1}^{m-2} p_i\alpha(\eta_i), \quad \alpha(1) + \mu_2\alpha'(1) \leq \sum_{i=1}^{m-2} q_i\alpha(\eta_i) \end{cases}$$

and

$$\begin{cases} \beta''(t) + \sigma\beta'(t) + g(t,\beta,\alpha_0) \leq \beta''(t) + \sigma\beta'(t) + f(t,\beta) \leq 0, \quad 0 < t < 1, \\ \beta(0) - \mu_1\beta'(0) = \sum_{i=1}^{m-2} p_i\beta(\eta_i), \quad \beta(1) + \mu_2\beta'(1) = \sum_{i=1}^{m-2} q_i\beta(\eta_i), \end{cases}$$

which imply that α_0 and β are respectively lower and upper solutions of (3.4). It follows by Theorems 2.1 and 2.2 that there exists a unique solution α_1 of (3.4) such that

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta(t), \quad t \in [0,1].$$

Next, we consider

$$\begin{cases} u''(t) + \sigma u'(t) + g(t,u,\alpha_1) = 0, \quad 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \sum_{i=1}^{m-2} p_i u(\eta_i), \quad u(1) + \mu_2 u'(1) = \sum_{i=1}^{m-2} q_i u(\eta_i). \end{cases} \tag{3.5}$$

Using the earlier arguments, it can be shown that α_1 and β are lower and upper solutions of (3.5) respectively and hence by Theorems 2.1 and 2.2, there exists a unique solution α_2 of (3.5) such that $\alpha_1(t) \leq \alpha_2(t) \leq \beta(t)$, $t \in [0,1]$.

Continuing this process successively yields a sequence $\{\alpha_n\}$ of solutions satisfying

$$\alpha_0(t) \leq \alpha_1(t) \leq \alpha_2(t) \leq \cdots \leq \alpha_n \leq \beta(t), \quad t \in [0,1],$$

where the element α_n of the sequence $\{\alpha_n\}$ is a solution of the problem

$$\begin{cases} u''(t) + \sigma u'(t) + g(t,u,\alpha_{n-1}) = 0, \ 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \sum_{i=1}^{m-2} p_i u(\eta_i), \quad u(1) + \mu_2 u'(1) = \sum_{i=1}^{m-2} q_i u(\eta_i) \end{cases}$$

and is given by

$$\begin{aligned} \alpha_n(t) = & \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big(\sum_{i=1}^{m-2} p_i \alpha_n(\eta_i)\Big) \\ & + \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \left(\sum_{i=1}^{m-2} q_i \alpha_n(\eta_i)\right) \\ & + \int_0^1 G(t,s) g(s, \alpha_n(s), \alpha_{n-1}(s)) ds. \end{aligned} \tag{3.6}$$

Using the fact that $[0,1]$ is compact and the monotone convergence of the sequence $\{\alpha_n\}$ is pointwise, it follows by the standard arguments (Arzela Ascoli convergence criterion, Dini's theorem [19, 29]) that the convergence of the sequence is uniform. If $u(t)$ is the limit point of the sequence, taking the limit $n \to \infty$ in (3.6), we obtain

$$\begin{aligned} u(t) &= \frac{-(1-\sigma\mu_2)e^{-\sigma} + e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big(\sum_{i=1}^{m-2} p_i u(\eta_i)\Big) \\ &+ \frac{(1+\sigma\mu_1) - e^{-\sigma t}}{(1+\sigma\mu_1) - (1-\sigma\mu_2)e^{-\sigma}} \Big(\sum_{i=1}^{m-2} q_i u(\eta_i)\Big) + \int_0^1 G(t,s) f(s,u(s)) ds. \end{aligned}$$

Thus, $u(t)$ is a solution of (1.1). Now, we show that the convergence of the sequence is quadratic. For that we set $e_n(t) = (u(t) - \alpha_n(t)) \geq 0$, $t \in [0,1]$ so that

$$e_n(0) - \mu_1 e_n'(0) = \sum_{i=1}^{m-2} p_i e_n(\eta_i), \quad e_n(1) + \mu_2 e_n'(1) = \sum_{i=1}^{m-2} q_i e_n(\eta_i).$$

In view of (A_2) and (3.2), it follows by Taylor's theorem that

$$\begin{aligned}
e_n''(t)+\sigma e_n'(t) &= u''+\sigma u'-(\alpha_n''+\sigma\alpha_n') = -f(t,u)+g(t,\alpha_n,\alpha_{n-1})\\
&= -f(t,u)+f(t,\alpha_{n-1})+F_u(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})+\phi(t,\alpha_{n-1})-\phi(t,\alpha_n)\\
&= -f_u(t,c_1)(u-\alpha_{n-1})-F_u(t,\alpha_{n-1})(u-\alpha_n)+F_u(t,\alpha_{n-1})(u-\alpha_{n-1})\\
&\quad -\phi_u(t,c_2)(\alpha_n-\alpha_{n-1})\\
&= [-f_u(t,c_1)+F_u(t,\alpha_{n-1})-\phi_u(t,c_2)]e_{n-1}+[-F_u(t,\alpha_{n-1})+\phi_u(t,c_2)]e_n\\
&= [-F_u(t,c_1)+F_u(t,\alpha_{n-1})+\phi_u(t,c_1)-\phi_u(t,c_2)]e_{n-1}\\
&\quad +[-F_u(t,\alpha_{n-1})+\phi_u(t,c_2)]e_n\\
&\geq [-F_u(t,u)+F_u(t,\alpha_{n-1})+\phi_u(t,\alpha_{n-1})-\phi_u(t,\alpha_n)]e_{n-1}\\
&\quad +[-F_u(t,\alpha_{n-1})+\phi_u(t,\alpha_{n-1})]e_n\\
&= [-F_{uu}(t,c_3)-\phi_{uu}(t,c_4)]e_{n-1}^2-f_u(t,\alpha_{n-1})e_n\\
&\geq -[A+B]e_{n-1}^2\\
&= -M\|e_{n-1}\|^2,
\end{aligned}$$

where $\alpha_{n-1}\leq c_1,c_3\leq u$, $\alpha_{n-1}\leq c_2,c_4\leq\alpha_n$, A is a bound on $\|F_{uu}\|$, B is a bound on $\|\phi_{uu}\|$ for $t\in(0,1)$ and $M=A+B$. Thus, we have

$$\begin{aligned}
e_n(t) &= \frac{-(1-\sigma\mu_2)e^{-\sigma}+e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\left(\sum_{i=1}^{m-2}p_ie_n(\eta_i)\right)\\
&\quad+\frac{(1+\sigma\mu_1)-e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\left(\sum_{i=1}^{m-2}q_ie_n(\eta_i)\right)\\
&\quad+\int_0^1 G(t,s)[f(s,u(s))-g(t,\alpha_n,\alpha_{n-1})]ds\\
&\leq \frac{-(1-\sigma\mu_2)e^{-\sigma}+e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\left(\sum_{i=1}^{m-2}p_ie_n(\eta_i)\right)\\
&\quad+\frac{(1+\sigma\mu_1)-e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\left(\sum_{i=1}^{m-2}q_ie_n(\eta_i)\right)\\
&\quad-\int_0^1 G(t,s)[e_n''(s)+\sigma e_n'(s)]ds\\
&\leq \lambda\left(\sum_{i=1}^{m-2}p_i+\sum_{i=1}^{m-2}q_i\right)\|e_n\|+M_1\|e_{n-1}\|^2,
\end{aligned}\tag{3.7}$$

where M_1 provides a bound on $M\int_0^1 G(t,s)$, $\lambda=\max\{\lambda_1,\lambda_2\}$, and

$$\left|\frac{-(1-\sigma\mu_2)e^{-\sigma}+e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\right|\leq\lambda_1,\ \left|\frac{(1+\sigma\mu_1)-e^{-\sigma t}}{(1+\sigma\mu_1)-(1-\sigma\mu_2)e^{-\sigma}}\right|\leq\lambda_2$$

on $[0,1]$. Taking the maximum over the interval $[0,1]$ and solving (3.7) algebraically, we obtain

$$\|e_n\| \leq M_2 \|e_{n-1}\|^2,$$

where

$$M_2 = \left[1 - \lambda\left(\sum_{i=1}^{m-2} p_i + \sum_{i=1}^{m-2} q_i\right)\right]^{-1},$$

and $\|u\| = \{|u(t)| : t \in [0,1]\}$. This establishes the quadratic convergence of the sequence of iterates.

Example. Consider the boundary value problem

$$\begin{cases} u''(t) + \sigma u'(t) - te^{u(t)-1} - 2(u(t)-1) = 0, \ \sigma < 0, \ 0 < t < 1, \\ u(0) - \mu_1 u'(0) = \frac{1}{7}u(\frac{3}{4}) + \frac{1}{9}u(\frac{4}{5}), \qquad u(1) + \mu_2 u'(1) = \frac{1}{3}u(\frac{3}{4}), \end{cases} \tag{3.8}$$

where $0 \leq \mu_1 \leq 11/20$, $\mu_2 \geq 0$. Let $\alpha(t) = 0$ and $\beta(t) = 1 + t$ be respectively lower and upper solutions of (3.8). Clearly $\alpha(t)$ and $\beta(t)$ are not the solutions of (3.8) and $\alpha(t) < \beta(t)$, $t \in [0,1]$. Moreover, the assumption (A_2) of Theorem 3.1 is satisfied by choosing $\phi(t,u) = Mu^2$, $M \geq e/2$. Thus, the conclusion of Theorem 3.1 applies to the problem (3.8).

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