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STABILITY IN TERMS OF TWO MEASURES FOR PERTURBED DYNAMIC INTEGRO-DIFFERENTIAL EQUATIONS ON TIME SCALES

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ABSTRACT. In this paper, we will give a new comparison theorem that connects the solutions of perturbed and unperturbed dynamic systems in terms of two measures on time scales.

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1. Introduction

It is well known that the most used techniques in the study of the effect of perturbations of dynamic systems are the Lyapunov method and the nonlinear variation of parameters. In this paper, using the calculus on time scales and employing variation of Lyapunov's method and a family of perturbing Lyapunov functions, we prove a new comparison theorem that connects the solutions of perturbed and unperturbed dynamic systems on time scales in a manner useful to the theory of perturbations. Different from the proof of theorems in [1], here we verify the stability of trivial solution in terms of two measures, which unify varieties of stability notions and offer a general framework for investigation. This comparison result provides a flexible mechanism for preserving the nature of perturbation and shows the advantage of employing a family of Lyapunov functions than a single one in the study of stability properties. In section 2, we introduce the notions of strict stability in terms of two measures as well as strict boundedness concepts, which firstly introduced in [4]. Also, we develop the method of variation of parameters and Lyapunov-like functions. In section 3, we establish several stability criteria in terms of two measures for perturbed system which employs the interplay of the solution of unperturbed system and comparison system.

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2. Preliminaries

Let us list the following definitions and classes of functions for convenience.

$$K = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(u) \text{ is strictly increasing and } \sigma(0) = 0\}.$$
$$CK = \{\sigma \in C[\mathbb{R}_+^2, \mathbb{R}_+] : \sigma(t, u) \in K \text{ for each } t \in \mathbb{R}_+\}.$$
$$\Gamma = \{h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+] : \inf_{x \in \mathbb{R}^n} h(t, x) = 0 \text{ for each } t \in \mathbb{R}_+\}.$$

Definition 2.1. Let h_0, $h \in \Gamma$. Then we say that h_0 is finer than h if there exists a $\rho > 0$ and a function $\phi \in K$ such that

$$h_0(t, x) < \rho \text{ implies } h(t, x) \leq \phi(h_0(t, x)).$$

Definition 2.2. Let $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ and $h \in \Gamma$. Then V is said to be (i) h-positive definite if there exists a $\rho > 0$ and a function $b \in K$ such that $h(t, x) < \rho$ implies $b(h(t, x)) \leq V(t, x)$; (ii) h-decrescent if there exists a $\rho_0 > 0$ and a function $a \in K$ such that $h_0(t, x) < \rho_0$ implies $V(t, x) \leq a(h_0(t, x))$; (iii) weakly h-decrescent if there exists a $\rho_0 > 0$ and a function $a \in CK$ such that $h(t, x) < \rho_0$ implies $V(t, x) \leq a(t, h_0(t, x))$.

As the introduction in [2], let $\mathbb{T}$ be a time scale (an arbitrary closed set of $\mathbb{R}$) with $t_0 \geq 0$ as a minimal element. If a time scale has a maximal element which is also left-scattered, it is called a degenerate point. Let $\mathbb{T}^k$ represent the set of all non-degenerate points of $\mathbb{T}$.

Definition 2.3. The mapping $f : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be right-dense (rd) continuous and is denoted by $f \in C_{rd}[\mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ if
(i) it is continuous at each (t, x, y) with right-dense or maximal t and
(ii) the limits $f(t^-, x, y) = \lim_{(s,u,v) \to (t^-, x, y)} f(s, u, v)$ and $\lim_{(u,v) \to (x,y)} f(t, u, v)$ exists at each (t, x, y) with left-dense t.

Consider the two dynamic systems

$$y^\Delta = f(t, y, L_1 y), \qquad y(t_0) = x_0, \tag{2.1}$$

and

$$x^\Delta = F(t, x, L_2 x), \qquad x(t_0) = x_0, \tag{2.2}$$

where $f, F \in C_{rd}[\mathbb{T}^k \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $L_i x = \int_{t_0}^t K_i(t, s, x(s)) \Delta s$, $K_i : \mathbb{T}^k \times \mathbb{T}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is such that $K_i \in C_{rd}[\mathbb{T}^k \times \mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}^n]$ is continuous at each (t, s, x) with right-dense or maximal t and $K_i(t^-, s^-, x) = \lim_{(p,q,u) \to (t^-, s^-, x)} f(p, q, u)$ and $\lim_{u \to x} K_i(t, s, u)$ exists at each (t, s, x) with left-dense t and s, and $f(t, 0) \equiv 0$, $F(t, 0) \equiv 0$, $K_i(t, s, 0) \equiv 0$. Specially, when $F(t, x, L_1 x) = f(t, x, L_1 x) + R(t, x, Lx)$, $R(t, x, Lx)$ being the perturbation term. Relative to the system (2.1), let us assume that the following assumption (H) holds:

(H) The solutions $y(t,t_0,x_0)$ of (2.1) exists for all $t \geq t_0$, unique and rd-continuous with respect to the initial data, and $\|y(t,t_0,x_0)\|$ is locally Lipschitzian in x_0.

For any $V \in C_{rd}[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}_+]$ and any fixed $t \in \mathbb{T}$. Let $\mu^*(t)$ be defined as in [2], then we define

$$D_{-}V^{\Delta}(s, y(t,s,x))$$
$$\equiv \liminf_{\mu^*(s)\to 0} \frac{V(s,y(t,s,x)) - V(s-\mu^*(s), y(t,s-\mu^*(s), x-\mu^*(s)F(s,x,L_2x)))}{\mu^*(s)}$$
$$D^{+}V^{\Delta}(s, y(t,s,x))$$
$$\equiv \limsup_{\mu^*(s)\to 0} \frac{V(s+\mu^*(s), y(t,s+\mu^*(s), x+\mu^*(s)F(s,x,L_2x))) - V(s,y(t,s,x))}{\mu^*(s)}$$

for $t_0 < s \leq t$ and $x \in \mathbb{R}^n$.

The following comparison result in [1] which relates the solutions of (2.2) to the solutions of (2.1) is an important tool in the subsequent discussion. We state it here.

Theorem 2.1 ([1]). *Assume that the assumption (H) holds, and*

(i) $V \in C_{rd}[\mathbb{T}^k \times \mathbb{R}^n, \mathbb{R}_+], V(t,s)$ *is locally Lipschitzian in x and for $t_0 < s \leq t, x \in \mathbb{R}^n$,*

$$D_{-}V^{\Delta}(s, y(t,s,x)) \leq g(s, V(s, y(t,s,x)));$$

(ii) $g \in C_{rd}[\mathbb{T}^k \times \mathbb{R}_+, \mathbb{R}], g(t,u)\mu^*(t)$ *is nondecreasing in u for each $t \in \mathbb{T}$, and the maximal solution $r(t,t_0,u_0)$ of*

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \geq 0,$$

exists for $t \in \mathbb{T}^k$.

Then, if $x(t) = x(t,t_0,x_0)$ is any solution of (2.2) *we have*

$$V(t, x(t,t_0,x_0)) \leq r(t,t_0,u_0), \qquad t \in \mathbb{T}^k,$$

provided $V(t_0, y(t,t_0,x_0)) \leq u_0$.

Definition 2.4. The system (2.1) is said to be (h_0,h)-equistable, if given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists a $\delta = \delta(t_0,\varepsilon)$ that is continuous in t_0 for each ε such that

$$h_0(t_0,x_0) < \delta \text{ implies } h(t,y(t)) < \varepsilon, t \geq t_0.$$

Definition 2.5. Let $h_0, h \in \Gamma$. Then system (1) is said to be

(1) strictly (h_0,h)-bounded if for any $\alpha_1, \alpha_2 > 0$, $\alpha_1 > \alpha_2$, $t_0 \in \mathbb{R}_+$, there exist positive functions $\beta_1 = \beta_1(t_0,\alpha_1)$ and $\beta_2 = \beta_2(t_0,\alpha_2)$ which are continuous in (t_0,α_1) and (t_0,α_2), respectively, such that

$$\alpha_2 \leq h_0(t_0,x_0) < \alpha_1 \text{ implies } \beta_2 \leq h(t,y(t)) < \beta_1, t \geq t_0,$$

where $y(t) = y(t,t_0,x_0)$ is any solution of system (2.1);

(2) uniformly strictly (h_0, h)-bounded if β_1 and β_2 in (1) are independent of t_0;

(3) strictly (h_0, h)-stable if in (1), $\lim_{\alpha_1 \to 0} \beta_1(t_0, \alpha_1) = 0$ and $\lim_{\alpha_2 \to 0} \beta_2(t_0, \alpha_2) = 0$;

(4) uniformly strictly (h_0, h)-stable if in (2), $\lim_{\alpha_1 \to 0} \beta_1(\alpha_1) = 0$ and $\lim_{\alpha_2 \to 0} \beta_2(\alpha_2) = 0$.

3. **Main Results**

We begin by proving a result on nonuniform stability in terms of two measures under weaker assumptions.

Theorem 3.1. *Assume that (H) holds, and*

(A_0) *h_0, $h \in \Gamma$ and h_0 is finer than h;*

(A_1) *$V_1 \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$, $V_1(t, y)$ is locally Lipschitzian in y for each $t \in \mathbb{T}$, weakly h_0-decrescent, and*

$$D_- V_1^{\Delta}(s, y(t, s, x)) \leq g_1(s, V(s, y(t, s, x))), \qquad (t, x) \in \mathbb{T} \times \mathbb{R}^n,$$

$$V_1(s, y(t, s, x)) \leq a(h(s, y(t, s, x))),$$

where $g_1 \in C_{rd}[\mathbb{T}^k \times \mathbb{R}_+, \mathbb{R}], g_1(t, u)\mu^*(t)$ is nondecreasing in u for each $t \in \mathbb{T}^k$, and $g_1(t, 0) \equiv 0$;

(A_2) *for every $\eta > 0$, there exists a $V_{2\eta} \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+], V_{2\eta}$ is locally Lipschitzian in y,*

$$\begin{aligned} &D_- V_1^{\Delta}(s, y(t, s, x)) + D_- V_{2\eta}^{\Delta}(s, y(t, s, x)) \\ &\quad \leq g_2(s, V_1(s, y(t, s, x))) + V_{2\eta}(s, V(s, y(t, s, x))), \end{aligned}$$

$$b(h(s, y(t, s, x))) \leq V_{2\eta}(s, y(t, s, x) \leq a(h_0(s, y(t, s, x)));$$

where $g_2 \in C_{rd}[\mathbb{T}^k \times \mathbb{R}_+, \mathbb{R}], g_2(t, u)\mu^*(t)$ is nondecreasing in u for each $t \in \mathbb{T}^k$, and $g_2(t, 0) \equiv 0$;

(A_3) *suppose that the trivial solution of (2.1) is (h_0, h)-equistable, the trivial solution is equistable relative to the differential equation*

$$u^{\Delta} = g_1(t, u), \qquad u(t_0) = u_0, \tag{3.1}$$

and uniformly stable with respect to the differential equation

$$w^{\Delta} = g_2(t, w), \qquad w(t_0) = w_0, \tag{3.2}$$

then the differential system (2.2) is (h_0, h)-equistable.

Proof. Since V_1 is weakly h_0-decrescent, there exists a $0 < \rho_1 \leq \rho$ and a $\phi_0 \in CK$ such that

$$V_1(s, y(t, s, x)) \leq \phi_0(s, h_0(s, y(t, s, x))) \qquad \text{if } h_0(s, y(t, s, x)) < \rho_1, \tag{3.3}$$

Also, h_0 is finer than h implies that there exists a $0 < \rho_0 \leq \rho_1$ and a $\phi_1 \in K$ such that

$$h(s, y(t, s, x)) \leq \phi_1(h_0(s, y(t, s, x))) \qquad \text{provided } h_0(s, y(t, s, x)) < \rho_0, \tag{3.4}$$

where ρ_0 is such that $\phi_1(\rho_0) < \rho_1$.

Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{T}$ be given. Since the trivial solution of (3.2) is uniformly stable, given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta_0 = \delta_0(\varepsilon)$ such that

$$w(t, t_0, w_0) < b(\varepsilon), \quad t \geq t_0, \qquad \text{if } w_0 < \delta_0, \tag{3.5}$$

where $w(t, t_0, w_0)$ is any solution of (3.2). Since $a, \phi_1 \in K$, we can find a $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$a(\delta_1) < \frac{\delta_0}{2} \qquad \text{and } \phi_1(\delta_1) < \varepsilon, \tag{3.6}$$

the equistability of the trivial solution of (3.1) implies that given $\frac{\delta_0}{2} > 0$ and $t_0 \in \mathbb{T}$, there exists a t_0, such that

$$u_0 \leq \delta^* \qquad \text{implies } u(t, t_0, u_0) < \frac{\delta_0}{2}, \qquad t \geq t_0, \tag{3.7}$$

where $u(t, t_0, u_0)$ is any solution of (3.1).

Let $\delta_2 = a^{-1}(\delta^*)$, since $y = 0$ of (2.1) is (h_0, h)-equistable, given $\delta_2 > 0, t_0 \in \mathbb{T}$, there exists a $\delta^0 = \delta^0(t_0, \varepsilon)$ such that

$$h(t, y(t, t_0, x_0)) < \delta_2, \qquad t \geq t_0, \qquad \text{if } h_0(t_0, x_0) < \delta^0$$

Choose $u_0 = V_1(t_0, y(t, t_0, x_0))$. Since $\phi_0 \in CK$ and (3.3) holds, there exists a $\delta_3 = \delta_3(t_0, \varepsilon) > 0$ such that $\delta_3 \in (0, \min(\delta_1, \rho_1))$, and

$$h_0(t_0, x_0) < \delta_3 \qquad \text{implies } V_1(t_0, y(t, t_0, x_0)) \leq \phi_0(t_0, h_0(t_0, y(t, t_0, x_0))) < \delta^* \quad (3.8),$$

We set $\delta = \min(\delta_1, \delta_3, \delta^0)$ and suppose that $h_0(t_0, x_0) < \delta$,

$$h_0(t_0, x_0) \leq \phi_1(h_0(t_0, x_0)) \leq \phi_1(\delta) \leq \phi_1(\delta_1) < \varepsilon, \tag{3.9}$$

We claim that $h_0(t_0, x_0) < \delta$ implies $h(t, x(t, t_0, x_0)) < \varepsilon, t \geq t_0$. If this is not true, because of (3.9), there exists a solution $x(t)$ of (2.2) with $h_0(t_0, x_0) < \delta$ and $t_2 > t_1 > t_0$ such that

$$h(t_1, x(t_1, t_0, x_0)) = \delta_1, \qquad h(t_2, x(t_2, t_0, x_0)) = \varepsilon. \tag{3.10}$$

Setting $\eta = \delta_1(\varepsilon)$, we see by ($A_2$), there exists a $V_{2\eta}$ and hence letting $m(s) = V_1(s, y(t, s, x)) + V_{2\eta}(s, y(t, s, x)), t \in [t_1, t_2]$, we obtain the differential inequality

$$D^+ m(s) \leq g_2(s, m(s)),$$

Hence by the comparison Theorem 2.1, we have

$$m(s) \leq r_2(s, t_1, m(t_1)), \qquad t \in [t_1, t_2] \tag{3.11}$$

where $r_2(s, t_1, m(t_1))$ is the maximal solution of (3.2). Also, we can obtain similarly the estimate

$$V_1(t, x(t, t_0, x_0)) \le r_1(t, t_0.u_0), \qquad t \in [t_1, t_2]$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (3.1). Hence by (3.7) and (3.8), we have

$$V_1(t_1, x(t_1, t_0, x_0)) \le r_1(t_1, t_0, V_1(t_0, y(t, t_0, x_0))) \le r_1(t_1, t_0, a(h(t_0, y(t, t_0, x_0))))$$
$$\le r_1(t_1, t_0, a(\delta_2)) = r_1(t_1, t_0, \delta^*) < \tfrac{\delta_0}{2}.$$

Also by (A_2) and (3.6), we get

$$V_{2\eta}(t_1, x(t_1, t_0, x_0)) \le a(h_0(t_1, x(t_1, t_0, x_0))) = a(\delta_1) < \frac{\delta_0}{2}.$$

Since

$$m(t) = V_1(t, y(t, t, x)) + V_{2\eta}(t, y(t, t, x)) = V_1(t, x(t, t_0, x_0)) + V_{2\eta}(t, x(t, t_0, x_0))$$

so

$$m(t_1) = V_1(t_1, x(t_1, t_0, x_0)) + V_{2\eta}(t_1, x(t_1, t_0, x_0)) < \frac{\delta_0}{2},$$

Therefore (3.5) and (3.11) imply that

$$m(t_2) \le r_2(t_2, t_1, m(t_1)) < b(\varepsilon).$$

But

$$m(t_2) \ge V_{2\eta}(t_2, x(t_2, t_0, x_0)) \ge b(h(t_2, x(t_2, t_0, x_0))) = b(\varepsilon),$$

which leads to a contradiction. Hence the proof is complete.

Theorem 3.2. *Assume that*

(i) $h_0, h \in \Gamma$ *and* h_0 *is finer than* h;

(ii) $V_1 \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+], V_1(t, y)$ *is locally Lipschitzian in* y *for each* $t \in \mathbb{T}$ *and weakly* h_0*-decrescent and*

$$V_1(s, y(t, s, x)) \le a(h(s, y(t, s, x)));$$

(iii) for every $\eta > 0$, *there exists a* $V_{2\eta} \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+], V_{2\eta}$ *is locally Lipschitzian in* y,

$$D_-V_1^{\Delta}(s, y(t, s, x)) + D_-V_{2\eta}^{\Delta}(s, y(t, s, x))$$
$$\le g(s, V_1(s, y(t, s, x))) + V_{2\eta}(s, V(s, y(t, s, x))),$$

$$b(h(s, y(t, s, x))) \le V_{2\eta}(s, y(t, s, x) \le a(h_0(s, y(t, s, x))),$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+, \mathbb{R}]$ *with* $g(t, 0) \equiv 0$;

(iv) the trivial solution of $u^{\Delta} = g(t, u), u(t_0) = u_0 \ge 0$ *is uniformly stable;*

(v) there exist two functions $V_3, V_4 \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+]$ *such that* $V_1 = V_3 + V_4$, *where* V_3 *is* h*-positive definite and*

$$D_-V_1^{\Delta}(s, y(t, s, x)) \le -\lambda(t)C(V_3(s, y(t, s, x))),$$

where $C \in K$ and $\lambda \in C[\mathbb{T}, \mathbb{R}^n]$ is integrally positive, that is, $\int_I \lambda(s)ds = \infty$, whenever $I = \cup_{i=1}^{\infty}[\alpha_i, \beta_i], \alpha_i < \beta_i < \alpha_{i+1}$, and $\beta_i - \alpha_i \geq \delta > 0$;

(vi) for every function $y \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$, the function $\int_0^t [D_-V_4^{\triangle}(s, y(t,s,x))]_{\pm} ds$ is uniformly continuous on $\mathbb{T}$, where $[\cdot]_{\pm}$ means that either the positive or negative part is considered for all $s \in \mathbb{T}$.

Then the trivial solution of (2.1) *is (h_0, h)-equistable implies the trivial solution of* (2.2) *is (h_0, h)-equi-asymptotically stable.*

Proof. Since (v) implies that $D_-V_1(t, x) \leq 0$, the assumptions (i) to (iv) yield by Theorem 3.1 that the differential system (2.2) is (h_0, h)-equistable. Choose $\varepsilon = \rho$ and designating by $\delta_0 = \delta_0(t_0, \rho)$, it is clear that we have

$$h_0(t_0, x_0) < \delta_0 \text{ implies } h(t, x(t, t_0, x_0)) < \rho, \qquad t \geq t_0 \tag{3.12}$$

Let $x(t)$ be any solution of (2.2) satisfying (3.12). Define the functions $m_1(s) = V_1(s, y(t,s,x))$, $m_3(s) = V_3(s, y(t,s,x))$, $m_4(s) = V_4(s, y(t,s,x))$, so that

$$\begin{aligned} m_1(t) &= V_1(t, y(t,t,x)) = V_1(t, x(t, t_0, x_0)), \\ m_3(t) &= V_3(t, x(t, t_0, x_0)), \\ m_4(t) &= V_4(t, x(t, t_0, x_0)), \end{aligned}$$

Since $m_1(s) = m_3(s) + m_4(s)$, we have $m_1(t) = m_3(t) + m_4(t)$. The same as the proof of Theorem 3.2 in [3], we get $\lim_{t\to\infty} m_3(t) = 0$, which means $\lim_{s\to\infty} m_3(s) = 0$. That is $\lim_{s\to\infty} V_3(s, y(t,s,x)) = 0$.

Since V_3 is h-positive, i.e.

$$h_0(t_0, x_0) < \delta_0 \text{ implies } b(h(s, y(t,s,x))) \leq V_3(s, y(t,s,x))$$

We get in turn

$$\lim_{s\to 0} h(s, y(t,s,x)) = 0,$$

hence

$$\lim_{t\to 0} h(t, x(t, t_0, x_0)) = 0.$$

The proof is complete.

Remark. From this theorem, we can see the advantage of employing a family of Lyapunov functions even clearly. We need only the stability of unperturbed system to proof the asymptotical stability of the perturbed system.

The next result is on uniform asymptotic stability for (2.2). We also need only the strict stability of (2.1) but a single Lyapunov function to do that. So we can see the advantage of introducing the concept of strict (h_0, h) stability.

Theorem 3.3. *Assume that*

(i) $h_0, h_1, h_2 \in \Gamma$ *and* h_2 *is uniformly finer than* $h_1, V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}_+], h_1(t,x)$ *is nondecreasing in* t *for each fixed* x *and there exists a function* $b \in K$ *such that*

$$V(t,x) \geq b(h_2(t,x)), (t,x) \in \mathbb{T} \times \mathbb{R}^n;$$

(ii) h_0 *is locally Lipschitzian in* x *and*

$$D_y^+ h_0^\Delta(t,y) \leq 0;$$

(iii) h_0 *is uniformly finer than* h_2 *and*

$$V(t,x) \leq a_1(h_1(t,x)) + a_0(h_0(t,x)), (t,x) \in \mathbb{T} \times \mathbb{R}^n,$$

where $a_1, a_0 \in K$*;*

(iv)

$$D_- V^\Delta(s, y(t,s,x)) \leq g(s, V(s, y(t,s,x))), t_0 \leq s \leq t, (t,x) \in \mathbb{T} \times \mathbb{R}^n,$$

(v) system (2.1) *is strictly uniformly* (h_0, h_1)*-stable.*

Then the uniformly asymptotical stability of differential equation

$$u^\Delta = g(t,u), \qquad u(t_0) = u_0, \tag{3.13}$$

implies the uniformly asymptotically (h_0, h_2)*-stability of* (2.2).

Proof. Since h_0 is uniformly finer than h_2, there exist $\varphi \in K$ and $\delta_0 > 0$ such that

$$h_2(t,x) \leq \varphi(t, h_0(t,x)), \qquad \text{if } h_0(t,x) < \delta_0, \tag{3.14}$$

Let $\varepsilon > 0$ and $t_0 \in \mathbb{T}$. There exist $\eta > 0$ and $\delta_1 > 0$ such that

$$a_1(t_0, \eta) + a_0(t_0, \delta_1) < \delta^*, \tag{3.15}$$

and

$$\varphi(t_0, \delta_1) < \varepsilon. \tag{3.16}$$

Since system (2.1) is uniformly (h_0, h_1)-stable, it follows that there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$h_0(t_0, x_0) < \delta_2 \qquad \text{implies } h_1(t, y(t, t_0, x_0)) < \eta, t \geq t_0, \tag{3.17}$$

where $y(t) = y(t, t_0, x_0)$ is any solution of system (2.1).

Now choose $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ and let $x(t) = x(t, t_0, x_0)$ be a solution of system (2.2) with $h_0(t_0, x_0) < \delta$. By the choice of δ and (3.14), (3.16), we see that $h_2(t_0, x_0(t)) < \varepsilon$. We claim that

$$h_2(t, x(t)) < \varepsilon, \quad t \geq t_0,$$

If this is not true, then there exists a $t_1 > t_0$ such that

$$h_2(t_1, x(t_1)) = \varepsilon \qquad \text{and} \qquad h_2(t, x(t)) < \varepsilon, t \in [t_0, t_1).$$

For $t \in [t_0, t_1]$, define $m(s) = V(s, y(t, s, x(s))), s \in [t_0, t]$. Then we get

$$D^+ m(s) \le g(s, m(s)), \quad t_0 \le s \le t \le t_1,$$

Let $m(t_0) = u_0$, by Theorem 2.1, we get

$$m(s) \le r(s, t_0, m(t_0)), \quad t_0 \le s \le t \le t_1, \tag{3.18}$$

where $r(s, t_0, m(t_0))$ is the maximal solution of (3.13).

Since (3.13) is uniformly asymptotically stable, give $b(\varepsilon) > 0$ and $t_0 \in \mathbb{T}$, for δ^*, we have

$$u_0 < \delta^* \quad \text{implies} \quad u(t, t_0, u_0) < b(\varepsilon), t \ge t_0, \tag{3.19}$$

By condition (iii) we have

$$\begin{aligned} u_0 &= m(t_0) = V(t_0, y(t, t_0, x_0)) \\ &\le a_1(t_0, h_1(t_0, y(t, t_0, x_0))) + a_0(t_0, h_0(t_0, y(t, t_0, x_0))) \end{aligned} \tag{3.20}$$

In view of condition (i) and (ii), we obtain

$$h_1(t_0, y(t, t_0, x_0)) \le h_1(t, y(t, t_0, x_0)), \tag{3.21}$$

and

$$h_0(t_0, y(t, t_0, x_0)) \le h_0(t_0, x_0), \tag{3.22}$$

Then from (3.15), (3.17) and (3.20)-(3.22),we obtain

$$m(t_0) < \delta^*$$

Then by (3.18), (3.19)

$$V(t_1, x(t_1)) \le r(t_1, t_0, m(t_0)) < b(\varepsilon)$$

But in view of condition (i)

$$V(t_1, x(t_1)) \ge b(h_2(t_1, x(t_1))) = b(\varepsilon)$$

which is a contradiction, hence system (2.2) is uniformly (h_0, h_2) stable.

Next we will prove the attractivity of (2.2). Since system (2.1) is strictly uniformly (h_0, h_1)-stable, there exist $\delta_3 > 0$ and $\eta_1 > 0$ such that

$$\delta^0 \le h_0(t_0, x_0) < \delta_3 \quad \text{implies} \quad \eta_1 \le h_1(t, y(t)) < \eta, t \ge t_0,$$

Choose $\sigma = \min\{\delta, \delta_3\}$. For any $t_0 \in \mathbb{T}$, let $x(t) = x(t, t_0, x_0)$ be any of system (2.2) with $h_0(t_0, x_0) < \sigma$, we are going to prove

$$h_2(t, x(t)) < \varepsilon, t \ge t_0 + T.$$

By uniform (h_0, h)-stability, it is sufficient to show that there exists a $t^* \in [t_0, t_0 + T]$ such that

$$h_0(t^*, x(t^*)) < \delta^0.$$

If this is not true, then there exists a solution $x(t) = x(t, t_0, x_0)$ of (2.2) with $h_0(t_0, x_0) < \sigma$ such that

$$h_0(t, x(t)) \geq \delta^0, t \in [t_0, t_0 + T],$$

Since h_2 is uniformly finer than h_1, there exists a $\varphi_2 \in K$ such that

$$h_1(t, x) \leq \varphi(h_2(t, x)) \tag{3.23}$$

For given $\varepsilon > 0$,we let

$$\varphi(t + T, \eta_1) < \varepsilon. \tag{3.24}$$

So in view of condition (i) and (3.23), (3.24)

$$\begin{aligned} V(t_0 + T, x(t_0 + T)) &\geq b(h_2(t_0 + T, x(t_0 + T))) \\ &\geq b(\varphi(t_0 + T, h_1(t_0 + T, x(t_0 + T)))) \\ &\geq b(\varphi(t_0 + T, \eta_1)) \geq b(\varepsilon), \end{aligned}$$

But

$$V(t_0 + T, x(t_0 + T)) = m(t_0 + T) \leq r(s, t_0, m(t_0)) < b(\varepsilon).$$

This is a contradiction and the theorem is proved.

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