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BOUNDARY VALUE PROBLEMS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

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ABSTRACT. In this paper we prove some existence results for boundary value problems for a functional differential equation of fractional order with both retarded and advanced arguments. The nonlinear alternative of Leray-Schauder type is the main tool in carrying out our proof.

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1. Introduction

Functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Kolmanovskii and Myshkis [16] and Hale and Verduyn Lunel [11], and the references therein. On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [9, 13, 15, 18, 19, 20, 22, 25] and references therein.

Differential equations of fractional order play a very important role in describing some real world problems. For example certain problems in physics, mechanics and other fields can be described with the help of fractional differential equations, see [5, 6, 8, 13, 22, 23, 24] and references therein. The theory of differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared which are devoted to fractional differential equations, for example see [1, 15, 17, 20, 26]. However, there are few papers in which the authors consider

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the Dirichlet-type problem for linear and nonlinear ordinary differential equations of fractional order, see for example [2, 7, 14, 27].

In this paper we study the existence of solutions to boundary value problems (BVP for short) of a fractional functional differential equation of mixed type

$$D^\beta x(t) = f(t, x^t),\ 0 \le t \le 1,\ 1 < \beta < 2, \tag{1}$$

$$x(t) = \phi(t),\ -r_1 \le t \le 0, \tag{2}$$

$$x(t) = \psi(t),\ 1 \le t \le 1 + r_2, \tag{3}$$

where D^β is the standard Riemman-Liouville fractional derivative. Here, $f : [0,1] \times C([-r_1, r_2], \mathbb{R}) \to \mathbb{R}$ is a given function, $\phi \in C([-r_1, 0], \mathbb{R})$ with $\phi(0) = 0$ and $\psi \in C([1, 1 + r_2], \mathbb{R})$ with $\psi(1) = 0$. For any function x defined on $[-r_1, 1 + r_2]$ and any $0 \le t \le b$, we denote by x^t the element of $C([-r_1, r_2], \mathbb{R})$ defined by $x^t(\theta) = x(t + \theta)$ for $-r_1 \le \theta \le r_2$, where $r_1,\ r_2 \ge 0$ are constants.

We remark that:

- If $r_1 = r_2 = 0$ then we have an ordinary differential equation of fractional order.
- If $r_1 > 0$ and $r_2 = 0$ then we have a retarded functional differential equation of fractional order.
- If $r_1 = 0$ and $r_2 > 0$ then we have an advanced differential equation of fractional order.
- If $r_1 > 0$ and $r_2 > 0$ then we have a mixed differential equation of fractional order. Moreover, in the limit case $\beta = 1$, the boundary value problem (1)–(3) represents a model for disease in epidemiology theory, where the physical state of the subject (the delayed argument), the treatment that should given (the advanced argument), the boundary condition (2) (the clinical observations from other subjects) and the boundary condition (3) (the expectations for the evolution of the disease). For more information, see [4, 21] and references therein.

In fact, our result is motivated by the extension and generalization of our result in [3] concerning the existence of solutions to the initial value problems of a mixed type functional differential equations of fractional order. It is worthwhile mentioning that no contributions exist, as far as we know, concerning the existence of solutions to problem (1)–(3).

2. **Auxiliary Facts and Results**

This section is devoted to collecting some definitions and results which will be needed throughout this paper.

By $C := C([-r_1, r_2], \mathbb{R})$ we denote the Banach space of all continuous functions from $[-r_1, r_2]$ into $\mathbb{R}$ equipped with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r_1 \le \theta \le r_2\}$$

and $C([0,1],\mathbb{R})$ is endowed with norm $\|x\|_0 = \sup\{|x(t)| : 0 \le t \le 1\}$. Also, let

$$\|x\|_{r_1,r_2} = \max\{\sup_{-r_1\le t\le 0}|x(t)|,\ \|x\|_0,\ \sup_{b\le t\le b+r_2}|x(t)|\}.$$

Now, we recall some definitions and facts about fractional derivatives and fractional integrals of arbitrary orders, see [15, 19, 20, 22].

Definition 2.1. The fractional primitive of order $\beta > 0$ of a function $g : (0,1] \to \mathbb{R}$ is defined by

$$I_0^\beta g(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{g(s)}{(t-s)^{1-\beta}}\,ds,$$

provided the right hand side is pointwise defined on $(0,1]$, where Γ is the gamma function.

Note that $I^\beta g$ exists for all $\beta > 0$ and $g \in C((0,1],\mathbb{R}) \cap L^1((0,1],\mathbb{R})$. Also, when $g \in C([0,1],\mathbb{R})$ then $I^\beta g \in C([0,1],\mathbb{R})$ and $I^\beta g(0) = 0$.

Definition 2.2. The fractional derivative of order $\beta > 0$ of a continuous function $g : (0,1] \to \mathbb{R}$ is defined by

$$\begin{aligned} D^\beta g(t) \equiv \frac{d^\beta}{dt^\beta}g(t) &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_a^t \frac{g(s)}{(t-s)^\beta}\,ds \\ &= \frac{d}{dt}I_a^{1-\beta}g(t). \end{aligned}$$

For our existence results we will need the following lemma.

Lemma 2.3. *Let $0 < \beta < 1$ and let $g : (0,1] \to \mathbb{R}$ be a continuous function with $\lim_{t\to 0^+} g(t) = g(0^+) \in \mathbb{R}$. Then x is a solution of the integral equation of fractional order*

$$x(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{g(s)}{(t-s)^{1-\beta}}\,ds$$

if and only if x is a solution of the initial value problem for the fractional differential equation

$$\begin{aligned} D^\beta x(t) &= g(t),\ t \in (0,1], \\ x(0) &= 0. \end{aligned}$$

Lemma 2.4. [2] *Given $h \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$D^\beta u(t) + g(t) = 0, \quad 0 < t < 1 \tag{4}$$

$$u(0) = u(1) = 0, \tag{5}$$

is

$$u(t) = \int_0^1 G(t,s)g(s)ds,$$

where

$$G(t,s)=\begin{cases}\frac{[t(1-s)]^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & \text{if } 0\le s\le t\le 1,\\ \\ \frac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & \text{if } 0\le t\le s\le 1.\end{cases} \tag{6}$$

Making use of (6) the unique solution u of (4)–(5) may be written as

$$u(t)=-\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\,g(s)\,ds+\frac{1}{\Gamma(\beta)}\int_0^1 t^{\beta-1}(1-s)^{\beta-1}\,g(s)\,ds,\quad t\in J.$$

3. Main result

By a solution of (1)–(3) we mean a function $x\in C([-r_1,1+r_2],\mathbb{R})$ that satisfies the equation $D^\beta x(t)=f(t,x^t)$ on $[0,1]$ and the conditions $x(t)=\phi(t)$, $\phi(0)=0$ on $[-r_1,0]$ and $x(t)=\psi(t)$, $\psi(1)=0$ on $[1,b+r_2]$.

Theorem 3.1. *Assume the following assumptions hold:*
(A_1) $f:[0,1]\times C([-r_1,r_2],\mathbb{R})\to\mathbb{R}$ *is a continuous function,*
(A_2) *for each* $r>0$ *there exists a function* $h_r\in L^1([0,1],\mathbb{R})$ *such that*

$$|f(t,u)|\le h_r(t)$$

for $t\in[0,1]$ *and each* $u\in C([-r_1,r_2],\mathbb{R})$ *with* $\|u\|\le r$.

Then the boundary value problem (1)–(3) *has at least one solution on the interval* $[-r_1,1+r_2]$.

Proof. Transform the problem (1)–(3) into a fixed point problem. Let us consider the operator $\mathcal{F}:C([-r_1,1+r_2],\mathbb{R})\to C([-r_1,1+r_2],\mathbb{R})$ defined by

$$(\mathcal{F}x)(t)=\begin{cases}\phi(t), & \text{if } t\in[-r_1,0],\\ \displaystyle\int_0^1 G(t,s)f(s,x^s)\,ds, & \text{if } t\in[0,1],\\ \psi(t), & \text{if } t\in[1,1+r_2].\end{cases}$$

Let $u:[-r_1,1+r_2]\to\mathbb{R}$ be a function defined by

$$u(t)=\begin{cases}\phi(t), & \text{if } t\in[-r_1,0],\\ 0, & \text{if } t\in[0,1],\\ \psi(t), & \text{if } t\in[1,1+r_2].\end{cases}$$

For each $y\in C([0,1],\mathbb{R})$ with $y(0)=0$ we denote by z the function defined by

$$z(t)=\begin{cases}0, & \text{if } t\in[-r_1,0],\\ y(t), & \text{if } t\in[0,1],\\ 0, & \text{if } t\in[1,1+r_2].\end{cases}$$

If the function x satisfies the integral

$$x(t)=\int_0^1 G(t,s)f(s,x^s)\,ds$$

we can decompose the function x as $x(t) = y(t) + u(t)$ for $0 \leq t \leq 1$. This implies $x^t = y^t + u^t$ for every $0 \leq t \leq 1$ and the function y satisfies

$$y(t) = \int_0^1 G(t,s) f(s, y^s + u^s)\, ds.$$

In what follows, let $B = \{y \in C([-r_1, 1 + r_2], \mathbb{R}) : y_0 = 0\}$ and let $\mathfrak{F} : B \to B$ defined by

$$(\mathfrak{F}y)(t) = \begin{cases} 0, & -r_1 \leq t \leq 0, \\ \displaystyle\int_0^1 G(t,s) f(s, y^s + u^s)\, ds, & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq 1 + r_2. \end{cases} \tag{7}$$

Then the operator $\mathcal{F}$ having a fixed point is equivalent to the operator $\mathfrak{F}$ having a fixed point, and so we turn to proving that $\mathfrak{F}$ has a fixed point which is a solution of the problem (1)–(3). We shall show that the operator $\mathfrak{F}$ is continuous and completely continuous.

Step 1: $\mathfrak{F}$ *is continuous.*

Let (y_n) be a sequence such that $y_n \to y$ in B. Then we have

$$\begin{aligned} |(\mathfrak{F}y_n)(t) - (\mathfrak{F}y)(t)| &\leq \int_0^1 G(t,s) |f(s, y^{ns} + u^s) - f(s, y^s + u^s)| ds \\ &\leq \|f(., y^{n(.)} + u^{(.)}) - f(., y^{(.)} + u^{(.)})\|_0 \int_0^1 G(t,s)\, ds. \end{aligned}$$

Since the function f is continuous, then we have

$$\|\mathfrak{F}y_n - \mathfrak{F}y\|_{[-r_1, 1+r_2]} \leq \|f(., y^{n(.)} + u^{(.)}) - f(., y^{(.)} + u^{(.)})\|_0 \int_0^1 G(t,s)\, ds \to 0 \text{ as } n \to \infty.$$

Step 2: $\mathfrak{F}$ *maps bounded sets into bounded sets in* B.

It is enough to show that, for any $\alpha > 0$, there exists a positive constant $\hat{L}$ such that, for each

$$y \in \Omega_\alpha = \{y \in B : \|y\|_{[-r_1, 1+r_2]} \leq \alpha\},$$

we have $\|\mathfrak{F}y\|_0 \leq \hat{L}$.

Let $y \in \Omega_\alpha$. Since f is continuous, we have for each $t \in [0,1]$

$$\begin{aligned} |(\mathfrak{F}y)(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} h_r(s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1} h_r(s)\, ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^1 h_r(s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^1 h_r(s)\, ds = \frac{2}{\Gamma(\beta)} \int_0^1 h_r(s)\, ds, \end{aligned}$$

and so

$$\|\mathfrak{F}y\|_0 \leq \frac{2}{\Gamma(\beta)} \int_0^1 h_r(s)\, ds := \hat{L}.$$

Consequently, $\mathfrak{F}$ maps bounded sets into bounded sets in B.

Step 3**:** $\mathfrak{F}$ *maps bounded sets into equicontinuous sets of* B.

Let $t_1,\ t_2 \in (0,b]$ with $t_1 < t_2$ and Ω_α be a bounded set of B as in Step 2. Let $y \in \Omega_\alpha$ then we have

$$\begin{aligned}
|(\mathfrak{F}y)(t_2) - (\mathfrak{F}y)(t_1)| \ &\leq \ \frac{t_2^{\beta-1} - t_1^{\beta-1}}{\Gamma(\beta)} \int_0^1 |f(s, y^s + u^s)|\, ds \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] |f(s, y^s + u^s)|\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} |f(s, y^s + u^s)|\, ds \right| \\
&\leq \ \frac{t_2^{\beta-1} - t_1^{\beta-1}}{\Gamma(\beta)} \int_0^1 h_r(s)\, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^{t_1} [(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}] h_r(s)\, ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} h_r(s)\, ds.
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the last inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

From the steps 1 to 3, by the aid of the Arzelá-Ascoli theorem, we conclude that the operator $\mathfrak{F}$ is continuous and completely continuous.

Step 4**:** *A priori bounds.*

We will show that there exists an open set $\Omega \subset B$ with $y \neq \lambda \mathfrak{F} y$ for $0 < \lambda < 1$ and $y \in \partial\Omega$.

Let $y \in B$ and $y = \lambda \mathfrak{F} y$ for some $0 < \lambda < 1$. Thus for each $t \in [0,1]$ we have

$$y(t) = \lambda \int_0^1 G(t,s) f(s, y^s + u^s)\ ds.$$

Therefore, by the aid of (A_2), we have

$$|y(t)| \leq \frac{2}{\Gamma(\beta)} \int_0^1 h_r(s)\, ds := R.$$

Finally, set

$$\Omega = \{x \in B : \|y\|_{r_1, r_2} < R + 1\}.$$

By our choice of Ω, there is no $y \in \partial\Omega$ such that $y = \lambda \mathfrak{F} y$ for some $0 < \lambda < 1$. As a consequence of the nonlinear alternative of Leray-Schauder type [10], we deduce that $\mathfrak{F}$ has a fixed point $y \in \bar{\Omega}$ which is a solution to problem (1)–(3). □

In the next theorem we assume that f satisfies a linear growth condition.

Theorem 3.2. *Assume that* (A1) *and the following condition hold.*

(A3) *There exist constants* $0 < c_1 < \frac{\Gamma(\beta+1)}{6}$, $c_2 > 0$ *such that*

$$|f(t,u)| \leq c_1\|u\| + c_2 \quad \text{for all} \quad t \in J \quad \text{and} \quad u \in C([-r_1, r_2], \mathbb{R}).$$

Then the boundary value problem (1)–(3) *has at least one solution on the interval* $[-r_1, 1+r_2]$.

Proof. Consider the operator $\mathcal{F}$ defined in Theorem 3.1, which as proved is completely continuous. Let

$$\Omega = \{x \in B : \|y\|_{r_1,r_2} < R\},$$

be open subset of B, where

$$R > \max\left\{1, \frac{6\ c_1}{\Gamma(\beta+1)}(\|\phi\| + \|\psi\|), \frac{6\ c_2}{\Gamma(\beta+1)}\right\}.$$

We suppose that there exists $z \in \partial\Omega$ and $\lambda \in (0,1)$ such that $y = \lambda\mathfrak{F}y$. So, for $y \in \partial\Omega$, we have

$$\begin{aligned}
|\mathfrak{F}y(t)| &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(c_1\|y^s + u^s\| + c_2)\,ds \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}(c_1\|y^s + u^s\| + c_2)\,ds \\
&\leq \frac{c_1}{\Gamma(\beta+1)}(R + \|\phi\| + \|\psi\|)t^\beta + \frac{c_2}{\Gamma(\beta+1)}t^\beta \\
&\quad + \frac{c_1}{\Gamma(\beta+1)}(R + \|\phi\| + \|\psi\|) + \frac{c_2}{\Gamma(\beta+1)} \\
&< \frac{2c_1}{\Gamma(\beta+1)}R + \frac{2c_1}{\Gamma(\beta+1)}(\|\phi\| + \|\psi\|) + \frac{2c_2}{\Gamma(\beta+1)} \\
&< \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R,
\end{aligned}$$

which implies that $\|\mathfrak{F}y\| \neq R = \|y\|$, a contradiction. By the Leray-Schauder nonlinear alternative we have the result. □

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