CONTINUATION THEOREMS FOR ACYCLIC MAPS IN TOPOLOGICAL SPACES

DONAL O’REGAN

1Department of Mathematics, National University of Ireland, Galway Ireland
E-mail: donal.oregan@nuigalway.ie

ABSTRACT. This paper discusses acyclic maps between topological spaces and we present a definition of an essential map in this setting. In addition we show that if $F$ is essential and $F \cong G$ then $G$ is essential.

AMS (MOS) Subject Classification. 47H10.

1. INTRODUCTION

The notion of an essential map in a Banach (or Fréchet) space setting introduced by Granas in [2] is more general than the notion of degree. In [2] he showed for single valued maps that if $F$ is essential and $F \cong G$ then $G$ is essential. In this paper we extend this notion to acyclic maps between topological spaces. Also in this general setting we show that if $F$ is essential and $F \cong G$ then $G$ is essential. In particular we note that the result holds for maps between Hausdorff topological spaces (i.e. the spaces need not be vector spaces).

Let $X$ and $Z$ be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of $Z$. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Z)$ is acyclic if $F$ is upper semicontinuous with acyclic values.

2. CONTINUATION THEORY

Throughout this section $Y$ will be a completely regular topological space and $U$ will be an open subset of $Y$.

Definition 2.1. We say $F \in AC(\overline{U}, Y)$ if $F : \overline{U} \to K(Y)$ is an acyclic compact map; here $\overline{U}$ denotes the closure of $U$ in $Y$.

Definition 2.2. We say $F \in AC_{\partial U}(\overline{U}, Y)$ if $F \in AC(\overline{U}, Y)$ with $x \notin F(x)$ for $x \in \partial U$; here $\partial U$ denotes the boundary of $U$ in $Y$. 
**Definition 2.3.** Let $F, G \in AC_{\partial U}(\overline{U}, Y)$. We say $F \cong G$ in $AC_{\partial U}(\overline{U}, Y)$ if there exists a upper semicontinuous compact map $\Psi : \overline{U} \times [0, 1] \to K(Y)$ with $\Psi_t \in AC_{\partial U}(\overline{U}, Y)$ for each $t \in [0, 1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(t, x)$).

Notice $\cong$ is an equivalence relation in $AC_{\partial U}(\overline{U}, Y)$.

**Definition 2.4.** We say a map $F \in AC_{\partial U}(\overline{U}, Y)$ is essential in $AC_{\partial U}(\overline{U}, Y)$ if every map $G \in AC_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and with $G \cong F$ in $AC_{\partial U}(\overline{U}, Y)$ has a fixed point in $U$. Otherwise $F$ is inessential in $AC_{\partial U}(\overline{U}, Y)$ i.e. there exists a fixed point free map $G \in AC_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and $G \cong F$ in $AC_{\partial U}(\overline{U}, Y)$.

**Theorem 2.5.** Let $Y$ be a completely regular topological space, $U$ an open subset of $Y$, and let $F \in AC_{\partial U}(\overline{U}, Y)$. Then the following are equivalent:

(i) $F$ is inessential in $AC_{\partial U}(\overline{U}, Y)$;
(ii) there exists a fixed point free map $G \in AC_{\partial U}(\overline{U}, Y)$ with $G \cong F$ in $AC_{\partial U}(\overline{U}, Y)$.

**Proof.** (i) implies (ii) is immediate. Now we prove (ii) implies (i). Let $H : \overline{U} \times [0, 1] \to K(Y)$ be a upper semicontinuous compact map with $H_t \in AC_{\partial U}(\overline{U}, Y)$ for each $t \in [0, 1]$ and with $H_0 = F$ and $H_1 = G$. Let

$$B = \{x \in \overline{U} : x \in H_t(x) \text{ for some } t \in [0, 1]\}.$$

If $B = \emptyset$ then in particular $x \notin H(x, 0) = F(x)$ for $x \in \overline{U}$ so $F$ is inessential in $AC_{\partial U}(\overline{U}, Y)$. Now suppose $B \neq \emptyset$. Clearly $B$ is closed (note $H$ is upper semicontinuous) and in fact compact (note $H$ is compact). Also note $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0, 1]$ so $B \cap \partial U = \emptyset$. Now since $Y$ is completely regular there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(B) = 1$ and $\mu(\partial U) = 0$. Define a map $R : \overline{U} \to K(Y)$ by $R(x) = H(x, \mu(x))$. Clearly $R \in AC_{\partial U}(\overline{U}, Y)$ since $R|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$. Also $x \notin R(x)$ for $x \in \overline{U}$ since if $x \in R(x)$ for some $x \in \overline{U}$ then $x \in B$ so $\mu(x) = 1$ i.e. $x \in H(x, 1) = G(x)$, a contradiction. We claim

$$R \cong F \text{ in } AC_{\partial U}(\overline{U}, Y). \quad (2.1)$$

If (2.1) is true then (i) holds, so it remains to check (2.1). Let $Q : \overline{U} \times [0, 1] \to K(Y)$ be given by

$$Q(x, t) = H(x, t \mu(x)).$$

Now $Q_0 = H_0 = F$ and $Q_1(x) = H(x, \mu(x)) = R(x)$ and clearly $Q : \overline{U} \times [0, 1] \to K(Y)$ is an upper semicontinuous compact map with $Q_t \in AC(\overline{U}, Y)$ for each $t \in [0, 1]$. Also $Q_t$ is fixed point free on $\partial U$ for each $t \in [0, 1]$ since if there exists $t \in [0, 1]$ and $x \in \partial U$ with $x \in Q_t(x)$ then $x \in H(x, t \mu(x))$ so $x \in B$ and as a result $\mu(x) = 1$ i.e. $x \in H(x, t)$, a contradiction. Thus (2.1) holds. \qed

Now Theorem 2.5 immediately guarantees the following continuation theorem.
Theorem 2.6. Let $Y$ be a completely regular topological space and $U$ an open subset of $Y$. Suppose $F$ and $G$ are two maps in $AC_0(U,Y)$ with $F \cong G$ in $AC_0(U,Y)$. Then $F$ is essential in $AC_0(U,Y)$ iff $G$ is essential in $AC_0(U,Y)$.

Proof. $F$ is inessential in $AC_0(U,Y)$ iff there exists a fixed point free map $\Phi \in AC_0(U,Y)$ with $F \cong \Phi$ in $AC_0(U,Y)$ iff (since $\cong$ is an equivalence relation in $AC_0(U,Y)$) there exists a fixed point free map $\Phi \in AC_0(U,Y)$ with $G \cong \Phi$ in $AC_0(U,Y)$ iff $G$ is inessential in $AC_0(U,Y)$. $\square$

In particular we mention two special cases of Theorem 2.6. We say $F \in C(\overline{U},Y)$ if $F: \overline{U} \to Y$ is a continuous single valued compact map. We can also write the analogue of $C_0(U,Y)$, essential in $C_0(U,Y)$ and $\cong$ in $C_0(U,Y)$.

Theorem 2.7. Let $Y$ be a completely regular topological space and $U$ an open subset of $Y$. Suppose $F$ and $G$ are two maps in $C_0(U,Y)$ with $F \cong G$ in $C_0(U,Y)$. Then $F$ is essential in $C_0(U,Y)$ iff $G$ is essential in $C_0(U,Y)$.

Next we suppose $Y$ is a convex subset of a locally convex linear topological space (so in particular $Y$ is completely regular). We say $F \in K(U,Y)$ if $F: U \to CK(Y)$ is a upper continuous compact map; here $CK(Y)$ denotes the family of nonempty, convex, compact subsets of $Y$. We can also write the analogue of $K_0(U,Y)$, essential in $K_0(U,Y)$ and $\cong$ in $K_0(U,Y)$.

Theorem 2.8. Let $Y$ be a a convex subset of a locally convex linear topological space and $U$ an open subset of $Y$. Suppose $F$ and $G$ are two maps in $K_0(U,Y)$ with $F \cong G$ in $K_0(U,Y)$. Then $F$ is essential in $K_0(U,Y)$ iff $G$ is essential in $K_0(U,Y)$.

An obvious question is if the condition $F \cong G$ in $AC_0(U,Y)$ automatically satisfied in Definition 2.4 i.e. if $F$ and $G$ are in $AC_0(U,Y)$ with $G|_{\partial U} = F|_{\partial U}$ is $F \cong G$ in $AC_0(U,Y)$? If the maps are in $K_0(U,Y)$ and $Y$ is a convex subset of locally convex linear topological space then it is easy to see that

$$\Psi(x,t) = t F(x) + (1-t) G(x)$$

guarantees that $F \cong G$ in $K_0(U,Y)$. However the acyclic map case seems to be much more difficult. Let $Y$ be a infinite dimensional normed linear space and $U$ an open convex subset of $Y$ with $0 \in U$. Let $F, G$ be in $AC_0(U,Y)$ with $G|_{\partial U} = F|_{\partial U}$. We know [1] there exists a continuous retraction $r: U \to \partial U$. Let the map $F^*$ be given by $F^*(x) = F(r(x))$ for $x \in \overline{U}$. Of course $F^*(x) = G(r(x))$ for $x \in \overline{U}$ since $G|_{\partial U} = F|_{\partial U}$. With

$$H(x, \lambda) = G(2 \lambda r(x) + (1 - 2 \lambda) x) = G \circ j(x, \lambda) \text{ for } (x, \lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$
Definition 2.10. Let $\Psi : \overline{U} \times [0, \frac{1}{2}] \to \overline{U}$ be a upper semicontinuous compact map $\Psi : \overline{U} \times [0, \frac{1}{2}] \to \overline{U}$ is given by $j(x, \lambda) = 2\lambda r(x) + (1 - 2\lambda)x$ it is easy to see that

$$G \cong F^* \quad \text{in} \quad AC_{\partial U}(\overline{U}, E);$$

notice if there exists $x \in \partial U$ and $\lambda \in [0, \frac{1}{2}]$ with $x \in H_\lambda(x)$ then since $r(x) = x$ we have $x \in G(2\lambda x + (1 - 2\lambda)x) = G(x)$, a contradiction. Similarly with

$$Q(x, \lambda) = F((2 - 2\lambda)r(x) + (2\lambda - 1)x) \quad \text{for} \quad (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

it is easy to see that

$$F^* \cong F \quad \text{in} \quad AC_{\partial U}(\overline{U}, E).$$

Combining (2.2) and (2.3) gives $G \cong F$ in $AC_{\partial U}(\overline{U}, E)$.

It is possible to we generalize the above by considering a subclass of the $U_c^k$ maps of Park. Let $X$ and $Y$ be Hausdorff topological spaces. Given a class $X$ of maps, $X(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of $Y$) belonging to $X$, and $X_c$ the set of finite compositions of maps in $X$. We let

$$F(X) = \{Z : Fix F \neq \emptyset \quad \text{for all} \quad F \in X(Z, Z)\}$$

where $Fix, F$ denotes the set of fixed points of $F$.

The class $U$ of maps is defined by the following properties:

(i). $A$ contains the class $C$ of single valued continuous functions;

(ii). each $F \in A_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in F(A_c)$ for all $n \in \{1, 2, \ldots\}$; here $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

We say $F \in U_c^k(X, Y)$ if for any compact subset $K$ of $X$ there is a $G \in U_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall $U_c^k$ is closed under compositions. Finally we consider a subclass $A$ of the $U_c^k$ maps. The following condition will be assumed:

$$\begin{cases} 
\text{for Hausdorff topological spaces } X_1, X_2 \text{ and } X_3, \\
\text{if } F \in A(X_1, X_3) \text{ and } f \in C(X_2, X_1), \\
\text{then } F \circ f \in A(X_2, X_3).
\end{cases}$$

Definition 2.9. We say $F \in A(\overline{U}, Y)$ if $F \in A(\overline{U}, Y)$ is a upper semicontinuous compact map.

Definition 2.10. We say $F \in A_{\partial U}(\overline{U}, Y)$ if $F \in A(\overline{U}, Y)$ with $x \notin F(x)$ for $x \in \partial U$.

Definition 2.11. Let $F, G \in A_{\partial U}(\overline{U}, Y)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y)$ if there exists a upper semicontinuous compact map $\Psi : \overline{U} \times [0, 1] \to 2^Y$ with $\Psi_t \in A_{\partial U}(\overline{U}, Y)$ for each $t \in [0, 1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(t,x)$).
**Definition 2.12.** We say a map $F \in A_{\partial U}(\overline{U}, Y)$ is essential in $A_{\partial U}(\overline{U}, Y)$ if every map $G \in A_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and with $G \cong F$ in $A_{\partial U}(\overline{U}, Y)$ has a fixed point in $U$. Otherwise $F$ is inessential in $A_{\partial U}(\overline{U}, Y)$ i.e. there exists a fixed point free map $G \in A_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y)$.

The following condition will be assumed:

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y). \quad (2.5)$$

Essentially the same reasoning as in Theorem 2.5 and Theorem 2.6 yield the following result.

**Theorem 2.13.** Suppose (2.4) and (2.5) hold. Let $Y$ be a completely regular topological space and $U$ an open subset of $Y$. Suppose $F$ and $G$ are two maps in $A_{\partial U}(\overline{U}, Y)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, Y)$. Then $F$ is essential in $A_{\partial U}(\overline{U}, Y)$ iff $G$ is essential in $A_{\partial U}(\overline{U}, Y)$.

We now discuss essential maps in a little more detail.

**Definition 2.14.** We say $F \in AC(Y, Y)$ if $F : Y \to K(Y)$ is an acyclic compact map.

**Definition 2.15.** If $F \in AC(Y, Y)$ and $p \in Y$ then we say $F \cong \{p\}$ in $AC(Y, Y)$ if there exists an upper semicontinuous compact map $R : Y \times [0,1] \to K(Y)$ with $R_t \in AC(Y, Y)$ for each $t \in [0,1]$, $R_1 = F$ and $R_0 = \{p\}$ (here $R_t(x) = R(x,t)$).

**Theorem 2.16.** Let $Y$ be a completely regular topological space, $U$ an open subset of $Y$ and $u_0 \in U$. Let $F(x) = \{u_0\}$ for each $x \in U$. Assume the following property holds:

$$\left\{ \begin{array}{l}
\text{for any } \Phi \in AC(Y, Y) \text{ and any } p \in Y \text{ with } \Phi \cong \{p\} \\
\text{in } AC(Y, Y) \text{ we have that } \Phi \text{ has a fixed point in } Y.
\end{array} \right. \quad (2.6)$$

Then $F$ is essential in $AC_{\partial U}(\overline{U}, Y)$.

**Proof.** Take any $G \in AC_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and $G \cong \{u_0\}$ in $AC_{\partial U}(\overline{U}, Y)$. To show $F$ is essential in $AC_{\partial U}(\overline{U}, Y)$ we must show $G$ has a fixed point in $U$.

We know there exists a upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(Y)$ with $\Lambda_t \in AC_{\partial U}(\overline{U}, Y)$ for each $t \in [0,1]$, $\Lambda_0 = \{u_0\}$ and $\Lambda_1 = G$. Now let

$$D = \left\{ x \in \overline{U} : x \in \Lambda_t(x) \text{ for some } t \in [0,1] \right\}.$$ 

Notice $D \neq \emptyset$ (since $u_0 \in U$) is closed and compact and $D \cap (Y\setminus U) = \emptyset$. Thus there exists a continuous map $\sigma : Y \to [0,1]$ with $\sigma(D) = 1$ and $\sigma(Y\setminus U) = 0$. Define $\Psi : Y \times [0,1] \to K(Y)$ by

$$\Psi(x,t) = \left\{ \begin{array}{ll}
\Lambda(x, t\sigma(x)), & x \in U \\
\{u_0\}, & x \in Y\setminus U.
\end{array} \right.$$
Clearly $\Psi : Y \times [0, 1] \to K(Y)$ is an upper semicontinuous compact map with $\Psi_t \in AC(Y,Y)$ for each $t \in [0, 1]$ and as a result $\Psi_1 \cong \{u_0\}$ in $AC(Y,Y)$. Now (2.6) guarantees that there exists $x \in Y$ with $x \in \Psi_1(x)$. If $x \in Y \setminus \overline{U}$ then $x \in \{u_0\}$ which is a contradiction since $u_0 \in U$. Thus $x \in U$ so $x \in \Lambda(x, \sigma(x))$ and as a result $x \in D$ which implies $\sigma(x) = 1$ and so $x \in \Lambda(x, 1) = G(x)$. 

\begin{proof}

Remark 2.17. Condition (2.6) was discussed in [3] and we refer the reader to that paper.

Combining Theorem 2.6 and Theorem 2.16 yields the following result.

Theorem 2.18. Let $Y$ be a completely regular topological space, $U$ an open subset of $Y$ and $u_0 \in U$. Let $F(x) = \{u_0\}$ for each $x \in \overline{U}$ and assume (2.6) holds. In addition suppose exists a upper semicontinuous compact map $H : \overline{U} \times [0, 1] \to K(Y)$ with $H_t \in AC(\overline{U},Y)$ for each $t \in [0, 1]$, $H_0 = F$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0, 1]$. Then $H_1$ is essential in $AC_{\partial U}(\overline{U}, Y)$ (in particular $H_1$ has a fixed point in $U$).

Let $X$ be a completely regular topological vector space, $Y$ a topological vector space, and $U$ an open subset of $X$. Also let $L : \text{dom } L \subseteq X \to Y$ be a linear (not necessarily continuous) single valued map; here $\text{dom } L$ is a vector subspace of $X$. Finally $T : X \to Y$ will be a linear, continuous single valued map with $L + T : \text{dom } L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(X,Y)$.

A map $F : \overline{U} \to 2^Y$ is said to be $(L,T)$ acyclic if $(L + T)^{-1}F : \overline{U} \to K(X)$ is an upper semicontinuous map (i.e. an acyclic map). Also $F : \overline{U} \to 2^Y$ is said to be $(L,T)$ compact if $(L + T)^{-1}F : \overline{U} \to 2^X$ is a compact map.

Definition 2.19. We let $F \in AC(\overline{U}, Y; L, T)$ if $(L + T)^{-1}F \in AC(\overline{U}, X)$.

Definition 2.20. We say $F \in AC_{\partial U}(\overline{U}, Y; L, T)$ if $F \in AC(\overline{U}, Y; L, T)$ with $L x \notin F(x)$ for $x \in \partial U \cap \text{dom } L$.

Definition 2.21. Two maps $F,G \in AC_{\partial U}(\overline{U}, Y; L, T)$ are homotopic in $AC_{\partial U}(\overline{U}, Y; L, T)$, written $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$, if there exists a $(L,T)$ upper semicontinuous, $(L,T)$ compact mapping $N : \overline{U} \times [0, 1] \to 2^Y$ such that $N_t(u) = N(u,t) : \overline{U} \to 2^Y$ belongs to $AC_{\partial U}(\overline{U}, Y; L, T)$ for each $t \in [0, 1]$ and $N_0 = F$ with $N_1 = G$.

Definition 2.22. A map $F \in AC_{\partial U}(\overline{U}, Y; L, T)$ is said to be L-essential in $AC_{\partial U}(\overline{U}, Y; L, T)$ if for every map $G \in AC_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ and with $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$ we have that there exists $x \in \overline{U} \cap \text{dom } L$ with $L x \in G(x)$. Otherwise $F$ is L-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$ i.e. there exists $G \in AC_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ and with $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$ and $L x \notin G(x)$ for $x \in \overline{U} \cap \text{dom } L$.

Theorem 2.23. Let $X, Y, U, L$ and $T$ be as above, and let $F \in AC_{\partial U}(\overline{U}, Y; L, T)$. Then the following conditions are equivalent:

1. $F$ is L-essential in $AC_{\partial U}(\overline{U}, Y; L, T)$.
2. $F$ is L-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$.
3. $F$ is an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(X,Y)$.
4. $F$ is an acyclic map.
5. $F$ is a compact map.
CONTINUATION THEOREMS

(i). $F$ is $L$-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$;

(ii). there exists a map $G \in AC_{\partial U}(\overline{U}, Y; L, T)$ with $L x \notin G(x)$ for $x \in \overline{U} \cap dom L$ and $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$.

Proof. We just need to show (ii) implies (i). Let $N : \overline{U} \times [0,1] \to 2^Y$ be a $(L, T)$ upper semicontinuous, $(L, T)$ compact map with $N_t \in AC_{\partial U}(\overline{U}, Y; L, T)$ for each $t \in [0,1]$ and with $N_0 = F$ and $N_1 = G$ [In particular $L x \notin N_t(x)$ for $x \in \partial U \cap dom L$ and for $t \in [0,1]$]. Let

$$B = \{ x \in \overline{U} : x \in (L + T)^{-1} (N_t + T) (x) \quad \text{for some} \quad t \in [0,1] \}.$$ 

Of course, it is immediate that

$$B = \{ x \in \overline{U} : x \in (L + T)^{-1} (N_t + T) (x) \quad \text{for some} \quad t \in [0,1] \}.$$ 

If $B = \emptyset$ then $F$ is L-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$. So it remains to consider the case when $B \neq \emptyset$. Now $B$ is closed and $\partial U \cap B = \emptyset$ so there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $J$ by $J(x) = N(x, \mu(x)) = N \circ j(x)$ where $j : \overline{U} \to \overline{U} \times [0,1]$ is given by $j(x) = (x, \mu(x))$. Clearly $J$ is a $(L, T)$ upper semicontinuous, $(L, T)$ compact map. Also $(L+T)^{-1}J \in AC(\overline{U}, X)$, $J|_{\partial U} = F|_{\partial U}$, $L x \notin J(x)$ for $x \in \overline{U} \cap dom L$ (since if $L x \in J(x)$ for $x \in \overline{U} \cap dom L$ then $x \in B$ and so $\mu(x) = 1$ i.e. $L x \in G(x)$, a contradiction) and $J \cong F$ in $AC_{\partial U}(\overline{U}, Y; L, T)$. \hfill \Box

Theorem 2.24. Let $X$, $Y$, $U$, $L$ and $T$ be as above and suppose $F$ and $G$ are two maps in $AC_{\partial U}(\overline{U}, Y; L, T)$ with $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$. Then $F$ is $L$-essential in $AC_{\partial U}(\overline{U}, Y; L, T)$ if and only if $G$ is $L$-essential in $AC_{\partial U}(\overline{U}, Y; L, T)$.

Remark 2.25. One could also easily replace $AC_{\partial U}(\overline{U}, Y; L, T)$ with the class $A_{\partial U}(\overline{U}, Y; L, T)$ (note $F \in A(\overline{U}, Y; L, T)$ if $(L + T)^{-1} F \in A(\overline{U}, X)$) in Theorem 2.24.

REFERENCES

