Communications in Applied Analysis **13** (2009), no. 1, 39–46

CONTINUATION THEOREMS FOR ACYCLIC MAPS IN TOPOLOGICAL SPACES

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ABSTRACT. This paper discusses acyclic maps between topological spaces and we present a definition of an essential map in this setting. In addition we show that if F is essential and $F \cong G$ then G is essential.

AMS (MOS) Subject Classification. 47H10.

1. INTRODUCTION

The notion of an essential map in a Banach (or Fréchet) space setting introduced by Granas in [2] is more general than the notion of degree. In [2] he showed for single valued maps that if F is essential and $F \cong G$ then G is essential. In this paper we extend this notion to acyclic maps between topological spaces. Also in this general setting we show that if F is essential and $F \cong G$ then G is essential. In particular we note that the result holds for maps between Hausdorff topological spaces (i.e. the spaces need not be vector spaces).

Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of Z. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Z)$ is acyclic if F is upper semicontinuous with acyclic values.

2. CONTINUATION THEORY

Throughout this section Y will be a completely regular topological space and U will be an open subset of Y.

Definition 2.1. We say $F \in \mathrm{AC}(\overline{\mathrm{U}}, \mathrm{Y})$ if $F : \overline{U} \to K(Y)$ is an acyclic compact map; here $\overline{U}$ denotes the closure of U in Y.

Definition 2.2. We say $F \in AC_{\partial U}(\overline{U}, Y)$ if $F \in AC(\overline{U}, Y)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in Y.

Received January 26, 2009

Definition 2.3. Let $F, G \in AC_{\partial U}(\overline{U}, Y)$. We say $F \cong G$ in $AC_{\partial U}(\overline{U}, Y)$ if there exists a upper semicontinuous compact map $\Psi : \overline{U} \times [0,1] \to K(Y)$ with $\Psi_t \in AC_{\partial U}(\overline{U}, Y)$ for each $t \in [0,1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(t,x)$).

Notice $\cong$ is an equivalence relation in $AC_{\partial U}(\overline{U}, Y)$.

Definition 2.4. We say a map $F \in AC_{\partial U}(\overline{U}, Y)$ is essential in $AC_{\partial U}(\overline{U}, Y)$ if every map $G \in AC_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and with $G \cong F$ in $AC_{\partial U}(\overline{U}, Y)$ has a fixed point in U. Otherwise F is inessential in $AC_{\partial U}(\overline{U}, Y)$ i.e. there exists a fixed point free map $G \in AC_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and $G \cong F$ in $AC_{\partial U}(\overline{U}, Y)$.

Theorem 2.5. *Let Y be a completely regular topological space, U an open subset of Y, and let $F \in AC_{\partial U}(\overline{U}, Y)$. Then the following are equivalent:*

(i). F is inessential in $AC_{\partial U}(\overline{U}, Y)$;

(ii). there exists a fixed point free map $G \in AC_{\partial U}(\overline{U}, Y)$ with $G \cong F$ in $AC_{\partial U}(\overline{U}, Y)$.

Proof. (i) implies (ii) is immediate. Now we prove (ii) implies (i). Let $H : \overline{U} \times [0,1] \to K(Y)$ be a upper semicontinuous compact map with $H_t \in AC_{\partial U}(\overline{U}, Y)$ for each $t \in [0,1]$ and with $H_0 = F$ and $H_1 = G$. Let

$$B = \left\{ x \in \overline{U} : \; x \in H_t(x) \; \text{ for some } \; t \in [0,1] \right\}.$$

If $B = \emptyset$ then in particular $x \notin H(x,0) = F(x)$ for $x \in \overline{U}$ so F is inessential in $AC_{\partial U}(\overline{U}, Y)$. Now suppose $B \neq \emptyset$. Clearly B is closed (note H is upper semicontinuous) and in fact compact (note H is compact). Also note $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ so $B \cap \partial U = \emptyset$. Now since Y is completely regular there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(B) = 1$ and $\mu(\partial U) = 0$. Define a map $R : \overline{U} \to K(Y)$ by $R(x) = H(x, \mu(x))$. Clearly $R \in AC_{\partial U}(\overline{U}, Y)$ since $R|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$. Also $x \notin R(x)$ for $x \in \overline{U}$ since if $x \in R(x)$ for some $x \in \overline{U}$ then $x \in B$ so $\mu(x) = 1$ i.e. $x \in H(x,1) = G(x)$, a contradiction. We claim

$$R \cong F \;\text{ in }\; AC_{\partial U}(\overline{U}, Y). \tag{2.1}$$

If (2.1) is true then (i) holds, so it remains to check (2.1). Let $Q : \overline{U} \times [0,1] \to K(Y)$ be given by

$$Q(x,t) = H(x, t\,\mu(x)).$$

Now $Q_0 = H_0 = F$ and $Q_1(x) = H(x, \mu(x)) = R(x)$ and clearly $Q : \overline{U} \times [0,1] \to K(Y)$ is an upper semicontinuous compact map with $Q_t \in AC(\overline{U}, Y)$ for each $t \in [0,1]$. Also Q_t is fixed point free on ∂U for each $t \in [0,1]$ since if there exists $t \in [0,1]$ and $x \in \partial U$ with $x \in Q_t(x)$ then $x \in H(x, t\mu(x))$ so $x \in B$ and as a result $\mu(x) = 1$ i.e. $x \in H(x,t)$, a contradiction. Thus (2.1) holds. □

Now Theorem 2.5 immediately guarantees the following continuation theorem.

Theorem 2.6. *Let Y be a completely regular topological space and U an open subset of Y. Suppose F and G are two maps in $AC_{\partial U}(\overline{U},Y)$ with $F \cong G$ in $AC_{\partial U}(\overline{U},Y)$. Then F is essential in $AC_{\partial U}(\overline{U},Y)$ iff G is essential in $AC_{\partial U}(\overline{U},Y)$.*

Proof. F is inessential in $AC_{\partial U}(\overline{U},Y)$ iff there exists a fixed point free map $\Phi \in AC_{\partial U}(\overline{U},Y)$ with $F \cong \Phi$ in $AC_{\partial U}(\overline{U},Y)$ iff (since $\cong$ is an equivalence relation in $AC_{\partial U}(\overline{U},Y)$) there exists a fixed point free map $\Phi \in AC_{\partial U}(\overline{U},Y)$ with $G \cong \Phi$ in $AC_{\partial U}(\overline{U},Y)$ iff G is inessential in $AC_{\partial U}(\overline{U},Y)$. □

In particular we mention two special cases of Theorem 2.6. We say $F \in C(\overline{U},Y)$ if $F : \overline{U} \to Y$ is a continuous single valued compact map. We can also write the analogue of $C_{\partial U}(\overline{U},Y)$, essential in $C_{\partial U}(\overline{U},Y)$ and $\cong$ in $C_{\partial U}(\overline{U},Y)$.

Theorem 2.7. *Let Y be a completely regular topological space and U an open subset of Y. Suppose F and G are two maps in $C_{\partial U}(\overline{U},Y)$ with $F \cong G$ in $C_{\partial U}(\overline{U},Y)$. Then F is essential in $C_{\partial U}(\overline{U},Y)$ iff G is essential in $C_{\partial U}(\overline{U},Y)$.*

Next we suppose Y is a convex subset of a locally convex linear topological space (so in particular Y is completely regular). We say $F \in K(\overline{U},Y)$ if $F : \overline{U} \to CK(Y)$ is a upper continuous compact map; here $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. We can also write the analogue of $K_{\partial U}(\overline{U},Y)$, essential in $K_{\partial U}(\overline{U},Y)$ and $\cong$ in $K_{\partial U}(\overline{U},Y)$.

Theorem 2.8. *Let Y be a a convex subset of a locally convex linear topological space and U an open subset of Y. Suppose F and G are two maps in $K_{\partial U}(\overline{U},Y)$ with $F \cong G$ in $K_{\partial U}(\overline{U},Y)$. Then F is essential in $K_{\partial U}(\overline{U},Y)$ iff G is essential in $K_{\partial U}(\overline{U},Y)$.*

An obvious question is if the condition $F \cong G$ in $AC_{\partial U}(\overline{U},Y)$ automatically satisfied in Definition 2.4 i.e. if F and G are in $AC_{\partial U}(\overline{U},Y)$ with $G|_{\partial U} = F|_{\partial U}$ is $F \cong G$ in $AC_{\partial U}(\overline{U},Y)$? If the maps are in $K_{\partial U}(\overline{U},Y)$ and Y is a convex subset of locally convex linear topological space then it is easy to see that

$$\Psi(x,t) = t\,F(x) + (1-t)\,G(x)$$

guarantees that $F \cong G$ in $K_{\partial U}(\overline{U},Y)$. However the acyclic map case seems to be much more difficult. Let Y be a infinite dimensional normed linear space and U an open convex subset of Y with $0 \in U$. Let F, G be in $AC_{\partial U}(\overline{U},Y)$ with $G|_{\partial U} = F|_{\partial U}$. We know [1] there exists a continuous retraction $r : \overline{U} \to \partial U$. Let the map $F^\star$ be given by $F^\star(x) = F(r(x))$ for $x \in \overline{U}$. Of course $F^\star(x) = G(r(x))$ for $x \in \overline{U}$ since $G|_{\partial U} = F|_{\partial U}$. With

$$H(x,\lambda) = G(2\,\lambda\, r(x) + (1-2\,\lambda)\,x) = G \circ j\,(x,\lambda) \quad \text{for} \quad (x,\lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$

(here $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ is given by $j(x, \lambda) = 2\lambda\, r(x) + (1 - 2\lambda)x)$ it is easy to see that

$$G \cong F^{\star} \quad \text{in} \quad AC_{\partial U}(\overline{U}, E); \tag{2.2}$$

notice if there exists $x \in \partial U$ and $\lambda \in \left[0, \frac{1}{2}\right]$ with $x \in H_{\lambda}(x)$ then since $r(x) = x$ we have $x \in G(2\lambda x + (1 - 2\lambda)x) = G(x)$, a contradiction. Similarly with

$$Q(x, \lambda) = F((2 - 2\,\lambda)\, r(x) + (2\,\lambda - 1)\, x) \quad \text{for} \quad (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

it is easy to see that

$$F^{\star} \cong F \quad \text{in} \quad AC_{\partial U}(\overline{U}, E). \tag{2.3}$$

Combining (2.2) and (2.3) gives $G \cong F$ in $AC_{\partial U}(\overline{U}, E)$.

It is possible to we generalize the above by considering a subclass of the $\mathbf{U}_c^k$ maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathbf{X}(Z, Z)\}$$

where Fix, F denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i). $\mathbf{A}$ contains the class $\mathbf{C}$ of single valued continuous functions;

(ii). each $F \in \mathbf{A}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathbf{F}(\mathbf{A}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

We say $F \in \mathbf{U}_c^k(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall $\mathbf{U}_c^k$ is closed under compositions. Finally we consider a subclass $\mathbf{A}$ of the $\mathbf{U}_c^k$ maps. The following condition will be assumed:

$$\begin{cases} \text{for Hausdorff topogical spaces } \ X_1,\ X_2 \ \text{ and } \ X_3, \\ \text{if } \ F \in \mathbf{A}(X_1, X_3) \ \text{ and } \ f \in \mathbf{C}(X_2, X_1), \\ \text{then } \ F \circ f \in \mathbf{A}(X_2, X_3). \end{cases} \tag{2.4}$$

Definition 2.9. We say $F \in A(\overline{U}, Y)$ if $F \in \mathbf{A}(\overline{U}, Y))$ is a upper semicontinuous compact map.

Definition 2.10. We say $F \in A_{\partial U}(\overline{U}, Y)$ if $F \in A(\overline{U}, Y)$ with $x \notin F\,(x)$ for $x \in \partial U$.

Definition 2.11. Let $F, G \in A_{\partial U}(\overline{U}, Y)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y)$ if there exists a upper semicontinuous compact map $\Psi : \overline{U} \times [0, 1] \to 2^Y$ with $\Psi_t \in A_{\partial U}(\overline{U}, Y)$ for each $t \in [0, 1]$, $\Psi_1 = F$ and $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(t, x)$).

Definition 2.12. We say a map $F \in A_{\partial U}(\overline{U}, Y)$ is essential in $A_{\partial U}(\overline{U}, Y)$ if every map $G \in A_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and with $G \cong F$ in $A_{\partial U}(\overline{U}, Y)$ has a fixed point in U. Otherwise F is inessential in $A_{\partial U}(\overline{U}, Y)$ i.e. there exists a fixed point free map $G \in A_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y)$.

The following condition will be assumed:

$$\cong \quad \text{is an equivalence relation in} \quad A_{\partial U}(\overline{U}, Y). \tag{2.5}$$

Essentially the same reasoning as in Theorem 2.5 and Theorem 2.6 yield the following result.

Theorem 2.13. *Suppose* (2.4) *and* (2.5) *hold. Let Y be a completely regular topological space and U an open subset of Y. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, Y)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, Y)$. Then F is essential in $A_{\partial U}(\overline{U}, Y)$ iff G is essential in $A_{\partial U}(\overline{U}, Y)$.*

We now discuss essential maps in a little more detail.

Definition 2.14. We say $F \in AC(Y, Y)$ if $F : Y \to K(Y)$ is an acyclic compact map.

Definition 2.15. If $F \in AC(Y, Y)$ and $p \in Y$ then we say $F \cong \{p\}$ in $AC(Y, Y)$ if there exists an upper semicontinuous compact map $R : Y \times [0, 1] \to K(Y)$ with $R_t \in AC(Y, Y)$ for each $t \in [0, 1]$, $R_1 = F$ and $R_0 = \{p\}$ (here $R_t(x) = R(x, t)$).

Theorem 2.16. *Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Let $F(x) = \{u_0\}$ for each $x \in \overline{U}$. Assume the following property holds:*

$$\begin{cases} \textit{for any} \;\; \Phi \in AC(Y, Y) \;\; \textit{and any} \;\; p \in Y \;\; \textit{with} \;\; \Phi \cong \{p\} \\ \textit{in} \;\; AC(Y, Y) \;\; \textit{we have that} \;\; \Phi \;\; \textit{has a fixed point in} \;\; Y. \end{cases} \tag{2.6}$$

Then F is essential in $AC_{\partial U}(\overline{U}, Y)$.

Proof. Take any $G \in AC_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$ and $G \cong \{u_0\}$ in $AC_{\partial U}(\overline{U}, Y)$. To show F is essential in $AC_{\partial U}(\overline{U}, Y)$ we must show G has a fixed point in U.

We know there exists a upper semicontinuous compact map $\Lambda : \overline{U} \times [0, 1] \to K(Y)$ with $\Lambda_t \in AC_{\partial U}(\overline{U}, Y)$ for each $t \in [0, 1]$, $\Lambda_0 = \{u_0\}$ and $\Lambda_1 = G$. Now let

$$D = \left\{ x \in \overline{U} : \; x \in \Lambda_t(x) \;\; \text{for some} \;\; t \in [0, 1] \right\}.$$

Notice $D \neq \emptyset$ (since $u_0 \in U$) is closed and compact and $D \cap (Y \backslash U) = \emptyset$. Thus there exists a continuous map $\sigma : Y \to [0, 1]$ with $\sigma(D) = 1$ and $\sigma(Y \backslash U) = 0$. Define $\Psi : Y \times [0, 1] \to K(Y)$ by

$$\Psi(x, t) = \begin{cases} \Lambda(x, t\, \sigma(x)), & x \in \overline{U} \\ \{u_0\}, & x \in Y \backslash U. \end{cases}$$

Clearly $\Psi : Y \times [0,1] \to K(Y)$ is an upper semicontinuous compact map with $\Psi_t \in AC(Y,Y)$ for each $t \in [0,1]$ and as a result $\Psi_1 \cong \{u_0\}$ in $AC(Y,Y)$. Now (2.6) guarantees that there exists $x \in Y$ with $x \in \Psi_1(x)$. If $x \in Y\backslash\overline{U}$ then $x \in \{u_0\}$ which is a contradiction since $u_0 \in U$. Thus $x \in U$ so $x \in \Lambda(x, \sigma(x))$ and as a result $x \in D$ which implies $\sigma(x) = 1$ and so $x \in \Lambda(x, 1) = G(x)$. □

Remark 2.17. Condition (2.6) was discussed in [3] and we refer the reader to that paper.

Combining Theorem 2.6 and Theorem 2.16 yields the following result.

Theorem 2.18. *Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Let $F(x) = \{u_0\}$ for each $x \in \overline{U}$ and assume* (2.6) *holds. In addition suppose exists a upper semicontinuous compact map $H : \overline{U} \times [0,1] \to K(Y)$ with $H_t \in AC(\overline{U}, Y)$ for each $t \in [0,1]$, $H_0 = F$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. Then H_1 is essential in $AC_{\partial U}(\overline{U}, Y)$ (in particular H_1 has a fixed point in U).*

Let X be a completely regular topological vector space, Y a topological vector space, and U an open subset of X. Also let $L : dom\, L \subseteq X \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of X. Finally $T : X \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(X, Y)$.

A map $F : \overline{U} \to 2^Y$ is said to be (L, T) acyclic if $(L+T)^{-1}F : \overline{U} \to K(X)$ is an upper semicontinuous map (i.e. an acyclic map). Also $F : \overline{U} \to 2^Y$ is said to be (L, T) compact if $(L+T)^{-1}F : \overline{U} \to 2^X$ is a compact map.

Definition 2.19. We let $F \in AC(\overline{U}, Y; L, T)$ if $(L+T)^{-1}\, F \in AC(\overline{U}, X)$.

Definition 2.20. We say $F \in AC_{\partial U}(\overline{U}, Y; L, T)$ if $F \in AC(\overline{U}, Y; L, T)$ with $L\, x \notin F(x)$ for $x \in \partial U \cap dom\, L$.

Definition 2.21. Two maps $F, G \in AC_{\partial U}(\overline{U}, Y; L, T)$ are homotopic in $AC_{\partial U}(\overline{U}, Y; L, T)$, written $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$, if there exists a (L, T) upper semicontinuous, (L, T) compact mapping $N : \overline{U} \times [0,1] \to 2^Y$ such that $N_t(u) = N(u,t) : \overline{U} \to 2^Y$ belongs to $AC_{\partial U}(\overline{U}, Y; L, T)$ for each $t \in [0,1]$ and $N_0 = F$ with $N_1 = G$.

Definition 2.22. A map $F \in AC_{\partial U}(\overline{U}, Y; L, T)$ is said to be L-essential in $AC_{\partial U}(\overline{U}, Y; L, T)$ if for every map $G \in AC_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ and with $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$ we have that there exists $x \in \overline{U} \cap dom\, L$ with $L\, x \in G(x)$. Otherwise F is L-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$ i.e. there exists $G \in AC_{\partial U}(\overline{U}, Y; L, T)$ with $G|_{\partial U} = F|_{\partial U}$ and with $F \cong G$ in $AC_{\partial U}(\overline{U}, Y; L, T)$ and $L\, x \notin G(x)$ for $x \in \overline{U} \cap dom\, L$.

Theorem 2.23. *Let X, Y, U, L and T be as above, and let $F \in AC_{\partial U}(\overline{U}, Y; L, T)$. Then the following conditions are equivalent:*

(i). F is L-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$;

(ii). there exists a map $G \in AC_{\partial U}(\overline{U}, Y; L, T)$ *with* $L\,x \notin G(x)$ *for* $x \in \overline{U} \cap dom\,L$ *and* $F \cong G$ *in* $AC_{\partial U}(\overline{U}, Y; L, T)$.

Proof. We just need to show (ii) implies (i). Let $N : \overline{U} \times [0,1] \to 2^Y$ be a (L,T) upper semicontinuous, (L,T) compact map with $N_t \in AC_{\partial U}(\overline{U}, Y; L, T)$ for each $t \in [0,1]$ and with $N_0 = F$ and $N_1 = G$ [In particular $L\,x \notin N_t(x)$ for $x \in \partial U \cap dom\,L$ and for $t \in [0,1]$]. Let

$$B = \left\{x \in \overline{U} \cap dom\,L : \; L\,x \in N(x,t) \quad \text{for some} \quad t \in [0,1]\right\}.$$

Of course, it is immediate that

$$B = \left\{x \in \overline{U} : \; x \in (L+T)^{-1}\,(N_t \,+\, T)\,(x) \quad \text{for some} \quad t \in [0,1]\right\}.$$

If $B = \emptyset$ then F is L-inessential in $AC_{\partial U}(\overline{U}, Y; L, T)$. So it remains to consider the case when $B \neq \emptyset$. Now B is closed and $\partial U \cap B = \emptyset$ so there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map J by $J(x) = N(x, \mu(x)) = N \circ j(x)$ where $j : \overline{U} \to \overline{U} \times [0,1]$ is given by $j(x) = (x, \mu(x))$. Clearly J is a (L,T) upper semicontinuous, (L,T) compact map. Also $(L+T)^{-1} J \in AC(\overline{U}, X)$, $J|_{\partial U} = F|_{\partial U}$, $L\,x \notin J(x)$ for $x \in \overline{U} \cap dom\,L$ (since if $L\,x \in J(x)$ for $x \in \overline{U} \cap dom\,L$ then $x \in B$ and so $\mu(x) = 1$ i.e. $L\,x \in G(x)$, a contradiction) and $J \cong F$ in $AC_{\partial U}(\overline{U}, Y; L, T)$. □

Theorem 2.24. *Let X, Y, U, L and T be as above and suppose F and G are two maps in* $AC_{\partial U}(\overline{U}, Y; L, T)$ *with* $F \cong G$ *in* $AC_{\partial U}(\overline{U}, Y; L, T)$. *Then F is L-essential in* $AC_{\partial U}(\overline{U}, Y; L, T)$ *if and only if G is L-essential in* $AC_{\partial U}(\overline{U}, Y; L, T)$.

Remark 2.25. One could also easily replace $AC_{\partial U}(\overline{U}, Y; L, T)$ with the class $A_{\partial U}(\overline{U}, Y; L, T)$ (note $F \in A(\overline{U}, Y; L, T)$ if $(L+T)^{-1}\,F \in A(\overline{U}, X)$) in Theorem 2.24.

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