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ON THE PERIODICITY OF LOZI'S EQUATION

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ABSTRACT. We investigate the periodic nature of the solutions of the second order nonlinear difference equation

$$y_{n+1} = \alpha|y_n| + \beta y_{n-1} + \gamma, \quad n = 0, 1, 2, \ldots$$

with real parameters, $\alpha,\ \beta,\ \gamma$, and real initial condition, $y_{-1},\ y_0$. Indeed, we show that if $\gamma \neq 0$, then all solutions are periodic of the same period p if and only if $\alpha = 0$ and $\beta = -1$, in which case, $p = 4$. Also, we identify all periodic solutions of minimal period 2 and 3.

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1. INTRODUCTION

The present work complements an earlier work done by the first author in [1] and is motivated by Open Problem 3.4.3 in [5, page 50]. This open problem is concerned with the global periodicity of the difference equation

$$y_{n+1} = \alpha|y_n| + \beta y_{n-1} + \gamma, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

with real parameters $\alpha,\ \beta,\ \gamma$ and real initial conditions $y_{-1},\ y_0$.

Eq. (1.1) is known in the literature by the name Lozi equation [4, page 40]. Its dynamic behavior is rich, and for some values of the parameters, it was proved by Misiurewicz [7] that it possesses a strange attractor (See also the paper by Liu et. al. [6]). The special case with $\alpha = 1,\ \beta = -1$ and $\gamma = 1$ is also called the Gingerbreadman Map and was investigated by Devaney in [3].

Our main objective in this paper is to investigate the periodic nature of the solutions of Eq. (1.1). In particular, we are concerned with the global periodicity, i.e., we would like to develop necessary and sufficient conditions on the parameters $\alpha,\ \beta,\ \gamma$ so that every solution of Eq. (1.1) is periodic of the same period p.

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The case $\gamma = 0$ was investigated in [1]. Therefore, we assume that $\gamma \neq 0$. Since periodicity is preserved under the transformation $y_n = |\gamma| z_n$, and z_n satisfies the difference equation

$$z_{n+1} = \frac{y_{n+1}}{|\gamma|} = \alpha \frac{|y_n|}{|\gamma|} + \beta \frac{y_{n-1}}{|\gamma|} + \frac{\gamma}{|\gamma|} = \alpha |z_n| + \beta z_{n-1} + \delta; \quad \delta = \frac{\gamma}{|\gamma|} = \pm 1,$$

it is enough to consider $\gamma = \pm 1$. Furthermore, the transformation $w_n = -z_n$ also preserves periodicity, and w_n satisfies the difference equation

$$w_{n+1} = -z_{n+1} = -\alpha |z_n| - \beta z_{n-1} - \delta = -\alpha |w_n| + \beta w_{n-1} - \delta.$$

Therefore, it is sufficient to consider the case $\delta = 1$. Finally, in view of Theorem 3.1 in [2], $f(x, y) = \alpha |x| + \beta y + 1$ is self inverse in y if and only if $\beta = -1$. Hence, we focus our attention to the difference equation

$$y_{n+1} = \alpha |y_n| - y_{n-1} + 1; \quad \alpha \in \mathbb{R}. \tag{1.2}$$

In the next section, we develop necessary and sufficient conditions for the existence of a periodic solution of a given period p. These conditions are used to characterize all solution of Eq. (1.2) of minimal period $p = 2, 3$. In Section 3, we further utilize these conditions and develop necessary and sufficient conditions for the existence of a minimal period p that works for all solutions of Eq. (1.2), that is, every solution of Eq. (1.2) is periodic of the same period p.

2. CONDITIONS FOR THE EXISTENCE OF A PERIODIC SOLUTION

In this section, we give necessary and sufficient conditions for a solution of Eq. (1.2) to be periodic of period p. These conditions will prove to be useful in the sequel. However, we first establish the following lemma.

Lemma 2.1. *If $\{y_n\}_{n=-1}^{\infty}$ is a solution of Eq.* (1.2), *then*

$$y_n = y_0 + n\left(y_0 - y_{-1}\right) + \frac{n(n+1)}{2} - \sum_{j=0}^{n-1} (n-j)\left(2y_j - \alpha |y_j|\right); \quad n = -1, 0, 1, 2, \ldots$$

Proof. Observe that Eq. (1.2) can be written as

$$\Delta^2 y_{n-1} = y_{n+1} - 2y_n + y_{n-1} = -\left(2y_n - \alpha |y_n|\right) + 1; \quad n = 0, 1, 2, \ldots$$

Thus, by the telescoping sum,

$$\Delta y_{n-1} - \Delta y_{-1} = \sum_{j=0}^{n-1} \Delta^2 y_{j-1} = -\sum_{j=0}^{n-1} \left(2y_j - \alpha |y_j|\right) + n.$$

Again, by the telescoping sum,

$$\begin{aligned} y_n - y_0 - n\Delta y_{-1} &= -\sum_{k=1}^{n}\sum_{j=0}^{k-1}(2y_j - \alpha|y_j|) + \sum_{k=1}^{n} k \\ &= -\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}(2y_j - \alpha|y_j|) + \frac{n(n+1)}{2} \\ &= -\sum_{j=0}^{n-1}(n-j)\left(2y_j - \alpha|y_j|\right) + \frac{n(n+1)}{2}. \end{aligned}$$

Rearranging the terms, the result follows for $n = 1, 2, \ldots$. Since an empty sum is equal to 0, the result holds for $n = -1, 0, 1, 2, \ldots$ as required. □

Using Lemma (2.1), we obtain the following necessary and sufficient conditions for a solution to be periodic of period p.

Theorem 2.2. *Suppose $\{y_n\}_{n=-1}^{\infty}$ is a solution of Eq.* (1.2). *Then y_n is periodic of period p if and only if*

$$\sum_{j=0}^{p-1}(2y_j - \alpha|y_j|) = p \tag{2.1}$$

and

$$\sum_{j=0}^{p-1}(p-j)\left(2y_j - \alpha|y_j|\right) = p\left(y_0 - y_{-1}\right) + \frac{p(p+1)}{2} \tag{2.2}$$

or, equivalently,

$$\sum_{j=0}^{p-1} j\left(2y_j - \alpha|y_j|\right) = \frac{p(p-1)}{2} - p\left(y_0 - y_{-1}\right). \tag{2.3}$$

Proof. A solution y_n of Eq. (1.2) is periodic of period p if and only if $y_p = y_0$ and $y_{p-1} = y_{-1}$. Therefore, using Lemma 2.1, y_n is periodic of period p if and only if

$$y_0 = y_p = y_0 + p\left(y_0 - y_{-1}\right) + \frac{p(p+1)}{2} - \sum_{j=0}^{p-1}(p-j)\left(2y_j - \alpha|y_j|\right)$$

which gives Condition (2.2), and

$$\begin{aligned}
y_{-1} = y_{p-1} &= y_0 + (p-1)\,(y_0 - y_{-1}) + \frac{(p-1)p}{2} - \sum_{j=0}^{p-2}(p-1-j)\,(2y_j - \alpha|y_j|) \\
&= y_0 + p\,(y_0 - y_{-1}) + (y_{-1} - y_0) + \frac{p(p+1)}{2} - p \\
&\quad - \sum_{j=0}^{p-1}(p-1-j)\,(2y_j - \alpha|y_j|) \\
&= y_0 + p\,(y_0 - y_{-1}) + (y_{-1} - y_0) + \frac{p(p+1)}{2} - p - \sum_{j=0}^{p-1}(p-j)\,(2y_j - \alpha|y_j|) \\
&\quad + \sum_{j=0}^{p-1}(2y_j - \alpha|y_j|) \\
&= y_{-1} - p + \sum_{j=0}^{p-1}(2y_j - \alpha|y_j|)
\end{aligned}$$

which gives Condition (2.1). □

The following results illustrate the applicability of Theorem 2.2.

Lemma 2.3. *Eq. (1.2) has a periodic solution of minimal period 2 if and only if $\alpha \le -2$. In fact, the periodic solutions of minimal period 2 are the ones with*

$$(y_{-1}, y_0) \in \left\{(y_{-1}, y_0) \in \mathbb{R}_+^2 :\ y_{-1} \neq y_0 \quad \textit{and} \quad y_0 + y_{-1} = \frac{1}{2}\right\} \quad \textit{if} \quad \alpha = -2$$

and the ones with

$$(y_{-1}, y_0) \in \left\{\left(\frac{2-\alpha}{4+\alpha^2}, \frac{2+\alpha}{4+\alpha^2}\right),\ \left(\frac{2+\alpha}{4+\alpha^2}, \frac{2-\alpha}{4+\alpha^2}\right)\right\} \quad \textit{if} \quad \alpha < -2.$$

Proof. Using Conditions (2.1) and (2.3), a solution $\{y_n\}$ of Eq. (1.2) is periodic of minimal period 2 if and only if

$$2y_0 - \alpha|y_0| + 2y_{-1} - \alpha|y_{-1}| = 2 \quad \text{and} \quad 2y_{-1} - \alpha|y_{-1}| = 1 - 2(y_0 - y_{-1}).$$

Equivalently,

$$2y_{-1} - \alpha|y_0| = 1 \quad \text{and} \quad -\alpha|y_{-1}| + 2y_0 = 1.$$

If $\alpha \ge 0$, then both y_0 and y_{-1} must be positive, and so

$$2y_{-1} - \alpha y_0 = 1 \quad \text{and} \quad -\alpha y_{-1} + 2y_0 = 1.$$

Solving we get

$$(4-\alpha^2)y_0 = 2 + \alpha = (4-\alpha^2)y_{-1}$$

which leads to a contradiction.

Suppose that $\alpha < 0$. If both y_0 and y_{-1} are nonnegative, then

$$(4-\alpha^2)y_0 = 2+\alpha = (4-\alpha^2)y_{-1},$$

and if both y_0 and y_{-1} are negative, then

$$(4-\alpha^2)y_0 = 2-\alpha = (4-\alpha^2)y_{-1}.$$

If $\alpha \neq -2$, we reach a contradiction in either case. However, if $\alpha = -2$, we obtain periodic solutions of minimal period 2 whenever $y_0,\ y_{-1} \geq 0$ such that $y_0 \neq y_{-1}$ and $y_0 + y_{-1} = \frac{1}{2}$. Thus it remains to investigate the case $y_0 y_{-1} < 0$. In this case, the conditions of periodicity can be written in the matrix form

$$\begin{bmatrix} 2 & -\alpha u_0 \\ -\alpha u_{-1} & 2 \end{bmatrix}\begin{bmatrix} y_{-1} \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad u_j = \operatorname{sign}(y_j),\ j=0,\ -1.$$

Since $u_0 u_{-1} = -1$, we have

$$\begin{bmatrix} y_{-1} \\ y_0 \end{bmatrix} = \frac{1}{4+\alpha^2}\begin{bmatrix} 2 & \alpha u_0 \\ \alpha u_{-1} & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4+\alpha^2}\begin{bmatrix} 2+\alpha u_0 \\ 2+\alpha u_{-1} \end{bmatrix}.$$

Observe that the signs will be consistent if and only if $\alpha < -2$. This completes the proof. □

Lemma 2.4. *Eq.* (1.2) *has a periodic solution of minimal period* 3 *if and only if one of the following hold.*

1. $\alpha = -1$ *with* $y_0, y_{-1} \geq 0$ *and* $y_0 + y_{-1} \leq 1$,
2. $\alpha < -1$ *with*

$$\begin{aligned}(y_0, y_{-1}) \quad \in \quad & \left\{\left(\frac{1-\alpha}{\alpha^2-\alpha+2}, \frac{1-\alpha}{\alpha^2-\alpha+2}\right), \left(\frac{1+\alpha}{\alpha^2+\alpha+2}, \frac{1+\alpha}{\alpha^2+\alpha+2}\right)\right\} \\ & \bigcup\left\{\left(\frac{1+\alpha}{\alpha^2\pm\alpha+2}, \frac{1-\alpha}{\alpha^2\pm\alpha+2}\right), \left(\frac{1-\alpha}{\alpha^2\pm\alpha+2}, \frac{1+\alpha}{\alpha^2\pm\alpha+2}\right)\right\}\end{aligned}$$

Proof. First, using conditions (2.1) and (2.3), a solution $\{y_n\}_{n=-1}^{\infty}$ of Eq. (1.2) is periodic of period 3 if and only if

$$y_0 - y_{-1} + \alpha\left(|y_0| - |y_{-1}|\right) = 0 \tag{2.4}$$

and

$$y_0 + y_{-1} - \alpha\Big|\alpha|y_0| - y_{-1} + 1\Big| = 1. \tag{2.5}$$

Now, we consider four cases, namely

$$y_0 \geq 0\ \&\ y_{-1} \geq 0, \quad y_0 \geq 0\ \&\ y_{-1} < 0, \quad y_0 < 0\ \&\ y_{-1} < 0, \quad \text{and} \quad y_0 < 0\ \&\ y_{-1} \geq 0.$$

Due to similarity of arguments, we consider the first case and omit the other cases.

Suppose $y_0 \geq 0,\ y_{-1} \geq 0$. In this case Condition (2.4) holds if and only if

$$y_0 = y_{-1} \quad \text{or} \quad \alpha = -1.$$

If $y_0 = y_{-1}$, then Condition (2.5) reduces to

$$2y_0 - \alpha\Big|(\alpha-1)y_0+1\Big| = 1.$$

Furthermore, if $(\alpha-1)y_0+1 \geq 0$, then $y_0 = \dfrac{1}{2-\alpha}$. Since $y_0 \geq 0$, we need $\alpha < 2$. But in this case, we end up with an equilibrium solution! However, if $(\alpha-1)y_0+1 < 0$, then $y_0 = \dfrac{1-\alpha}{\alpha^2-\alpha+2}$. Since $y_0 \geq 0$, we need $\alpha \leq 1$. Moreover, we also need

$$0 > (\alpha-1)\frac{1-\alpha}{\alpha^2-\alpha+2} + 1 = \frac{1+\alpha}{\alpha^2-\alpha+2}$$

which is satisfied if $\alpha < -1$.

On the other hand, if $\alpha = -1$, then Condition (2.5) reduces to

$$(y_0 + y_{-1}) + |1 - (y_0 + y_{-1})| = 1$$

which has a solution if and only if $y_0 + y_{-1} \leq 1$. ☐

3. NECESSARY AND SUFFICIENT CONDITIONS FOR GLOBAL PERIODICITY

In this section, we first develop necessary conditions for all solutions of Eq. (1.2) to be periodic of the same minimal period p. However, in view of Lemmas 2.3 and 2.4, p is at least 4.

Lemma 3.1. *If every solution of Eq.* (1.2) *is periodic of minimal period* p*, then* $\alpha \in (-2, 2)$.

Proof. Observe that Condition (2.1) can be written as

$$2\sum_{j=-1}^{p-2} y_j = \alpha \sum_{j=-1}^{p-2} |y_j| + p. \tag{3.1}$$

Since $-|y_j| \leq y_j \leq |y_j|$,

$$-2\sum_{j=-1}^{p-2} |y_j| \leq \alpha \sum_{j=-1}^{p-2} |y_j| + p \leq 2\sum_{j=-1}^{p-2} |y_j|.$$

Furthermore, we can have at most two consecutive zero terms. Thus, we have $\sum_{j=-1}^{p-2} |y_j| > 0$, and so

$$-2 \leq \alpha + \frac{p}{\sum_{j=-1}^{p-2} |y_j|} \leq 2.$$

Since $\frac{p}{\sum_{j=-1}^{p-2} |y_j|} > 0$, we conclude that $\alpha < 2$. On the other hand, since y_0 and y_{-1} can be arbitrarily large, we conclude $\alpha \geq -2$. Furthermore, if $\alpha = -2$, then Eq. (3.1) reduces to

$$2\sum_{j=-1}^{p-2} (y_j + |y_j|) = p$$

which is impossible to hold for all solutions. Hence, $-2 < \alpha < 2$ as claimed. □

Extensive computer simulations made us believe that only for $\alpha = 0$, the global periodicity is preserved. Below, we present a proof that is of a computational and geometric nature.

Theorem 3.2. *Every solution of Eq.* (1.2) *is of minimal period* p *if and only if* $\alpha = 0$, *in which case* $p = 4$.

Proof. First, let $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ denote the standard basis of $\mathbb{R}^2$, $\mathbf{y}_n = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}$, and

$$A_n = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} & \text{if} \quad y_n \geq 0 \\ \\ \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix} & \text{if} \quad y_n \leq 0 \end{cases}.$$

Then Eq. (1.2) can be written in the form

$$\mathbf{y}_{n+1} = A_n \mathbf{y}_n + \mathbf{e}_2. \tag{3.2}$$

Using an inductive argument one can show that

$$\mathbf{y}_n = \left(\prod_{j=0}^{n-1} A_j \right) \mathbf{y}_0 + \sum_{i=0}^{n-1} \left(\prod_{j=n-i}^{n-1} A_j \right) \mathbf{e}_2 \tag{3.3}$$

where $\prod_{j=k}^{m} A_j = A_m \cdots A_k$ if $k \leq m$, and $\prod_{j=k}^{m} A_j = I$ if $k > m$.

Now, suppose that every solution is periodic of minimal period p, and assume to the contrary that $\alpha \neq 0$. Then by Eq. (3.3)

$$\left(I - \prod_{j=0}^{p-1} A_j \right) \mathbf{y}_0 = \sum_{i=0}^{p-1} \left(\prod_{j=p-i}^{p-1} A_j \right) \mathbf{e}_2 \quad \text{for all} \quad \mathbf{y_0} \in \mathbb{R}^2. \tag{3.4}$$

Since there can be at most 2^p products of the form $\prod_{j=0}^{p-1} A_j$, we can find at least two distinct positive constants c_1 and c_2 (say $0 < c_1 < c_2$) such that the matrices A_j for $\mathbf{y}_0 = c_1 \mathbf{e}_2$ and $\mathbf{y}_0 = c_2 \mathbf{e}_2$ are identical. Thus

$$\left(I - \prod_{j=0}^{p-1} A_j \right) c_1 \mathbf{e}_2 = \sum_{i=0}^{p-1} \left(\prod_{j=p-i}^{p-1} A_j \right) \mathbf{e}_2$$

and

$$\left(I - \prod_{j=0}^{p-1} A_j \right) c_2 \mathbf{e}_2 = \sum_{i=0}^{p-1} \left(\prod_{j=p-i}^{p-1} A_j \right) \mathbf{e}_2,$$

and so

$$\left(I-\prod_{j=0}^{p-1}A_j\right)\mathbf{e}_2=\mathbf{0}\quad\text{and}\quad\sum_{i=0}^{p-1}\left(\prod_{j=p-i}^{p-1}A_j\right)\mathbf{e}_2=\mathbf{0}.$$

But then

$$A_{p-1}\sum_{i=1}^{p-1}\left(\prod_{j=p-i}^{p-2}A_j\right)\mathbf{e}_2=-\mathbf{e}_2.$$

Since $A_{p-1}^{-1}\mathbf{e}_2=-\mathbf{e}_1$, by shifting the index i down by 1, we conclude

$$\sum_{i=0}^{p-2}\left(\prod_{j=p-1-i}^{p-2}A_j\right)\mathbf{e}_2=\mathbf{e}_1.$$

By Eq. (3.3),

$$\mathbf{y_{p-1}}=\left(\prod_{j=0}^{p-2}A_j\right)\mathbf{y}_0+\sum_{i=0}^{p-2}\left(\prod_{j=p-1-i}^{p-2}A_j\right)\mathbf{e}_2=\left(\prod_{j=0}^{p-2}A_j\right)\mathbf{y}_0+\mathbf{e}_1$$

for $\mathbf{y}_0=c_1\mathbf{e}_2$ or $c_2\mathbf{e}_2$. For ease of reference, we shall use the following labels:

$$\mathbf{q}_k=\left(\prod_{j=0}^{p-2}A_j\right)c_k\mathbf{e}_2,\quad k=1,\ 2.$$

To this end, since each A_j is nonsingular, $\mathbf{q}_1$ and $\mathbf{q}_2$ point in the same direction but they are of different lengths. This, in turn, implies that the vectors $\mathbf{q}_1+\mathbf{e}_1$ and $\mathbf{q}_2+\mathbf{e}_1$ do not point in the same direction unless both $\mathbf{q}_1$ and $\mathbf{q}_2$ are parallel to $\mathbf{e_1}$, that is in the same or opposite direction of $\mathbf{e}_1$. Also, by Lemma 3.1 we know that $-2<\alpha<2$. Thus, A_{p-1} is a rotation map, actually conjugate to a rotation map, with a rotation angle $\theta\in(0,\pi)\setminus\{\pi/2\}$.

If the vectors $\mathbf{q}_k+\mathbf{e}_1$, $k=1,2$ are parallel to $\mathbf{e}_1$, then the rotated vectors $A_{p-1}\left(\mathbf{q}_k+\mathbf{e}_1\right)$, $k=1,2$ are not parallel to $\mathbf{e}_2$. So for each $k\in\{1,2\}$, $\mathbf{y}_p=A_{p-1}\left(\mathbf{q}_k+\mathbf{e}_1\right)+\mathbf{e}_2$ is not pointing in the direction of $\mathbf{e}_2$ which is a contradiction. On the other hand, if the vectors $\mathbf{q}_k+\mathbf{e}_1$, $k=1,2$ do not point in the same direction, then neither do the rotated vectors $A_{p-1}\left(\mathbf{q}_k+\mathbf{e}_1\right)$, $k=1,2$. Thus, at most one of the vectors $\mathbf{y}_p=A_{p-1}\left(\mathbf{q}_k+\mathbf{e}_1\right)+\mathbf{e}_2$, $k=1,2$ will be parallel to $\mathbf{e}_2$, which is again a contradiction. Hence, α must be equal to 0.

Finally, if $\alpha=0$, then Eq. (1.2) reduces to $y_{n+1}=1-y_{n-1}$ for which, it is well-known, every solution is periodic of period 4. This completes the proof. □

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