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FIRST ORDER FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH PERIODIC BOUNDARY CONDITIONS

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ABSTRACT. In this paper, an existence theorem for a first order functional integro-differential inclusion in Banach algebras with the periodic boundary conditions is proved via a new fixed point principle of Leray-Schauder type under generalized Lipschitz and Carathéodory conditions. An existence theorem for the extremal solutions is also obtained under certain monotonicity conditions.

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1. INTRODUCTION

The origin of the nonlinear integral equations in Banach algebras or quadratic integral equations lies in the works of a famous physicist Chandrasekher [4] in his studies on radiative transfer in thermodynamics which gave birth to the well-known Chandrasekher's H-equation in thermodynamics. The method developed for studying the existence of the solutions for above quadratic H-equation is very cumbersome and involve many technicalities. Therefore, there was a need to establish a general tool for studying such type of quadratic integral equations involving the product of two nonlinearities. The present author in [6] proved a fixed point theorem in Banach algebras which is further applied to a certain nonlinear integral equation involving the product of two nonlinearities for proving the existence of solutions. The existence of the solutions to Chandrasekher's H-equation is also proved as a easy application of such fixed point theorems with a different method than previous ones (see Dhage [11] and the references therein). Similar results for quadratic integral equations may be found in the works of Banach and Lecho [3] and others. Furthermore, Dhage and O'Regan [15] established existence theorems for differential equations in Banach algebras via a new fixed point technique developed there. Since then, various differential and integral equations and inclusions in Banach algebras have been studied in the literature by several authors for different aspects of the solutions. See, for example, Dhage [10] and the references therein. Some of these quadratic equations are the generalizations of the integral equations that occur in queuing theory and biological

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processes etc. Quadratic differential equations and inclusions are gathered under the title differential equations and inclusions in Banach algebras even for the scalar case, and the terminology is patterned on the use of multiplicative structure of Banach algebras in the study of such equations and inclusions. Initial value problems of quadratic differential equations and inclusions have been studied much in the literature, but the study of periodic boundary value problems is relatively rare. In the present work, we deal with periodic boundary value problems of first order functional integro-differential inclusions and prove the existences results as well as as existence results for extremal solutions under suitable conditions.

2. FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS

Let $\mathbb{R}$ be the real line and let $\mathcal{P}_p(\mathbb{R})$ denote the class of all non-empty subsets of $\mathbb{R}$ with property p. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the first order functional integro-differential inclusion (in short FIGDI) with the periodic boundary condition

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right] \in G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big) \quad \text{a.e.} \quad t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{2.1}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous, $k : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$, and the functions $\theta, \mu, \sigma, \eta : J \to J$ are continuous with $\theta(0) = 0$ and $\theta(T) = T$.

By a *solution* of the FIGDI (2.1) we mean a function $x \in AC(J, \mathbb{R})$ satisfying

(i) the function $t \mapsto \left(\dfrac{x(t)}{f(t,x(t),x(\theta(t)))}\right)$ is absolutely continuous, and

(ii) there exists a function $v \in L^1(J, \mathbb{R})$ such that

$$v(t) \in G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big) \quad \text{a.e. } t \in J$$

satisfying

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x(\theta(t)))}\right] = v(t), \quad x(0) = x(T),$$

where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions on J.

The FIGDI (2.1) is new to the theory of differential inclusions and none of the special cases in the form of differential inclusion involving the product of two functions has been discussed in the literature. For example, the special case of FIGDI (2.1) in the simplest form

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] \in G(t, x(t)) \quad \text{a.e.} \quad t \in J, \\ x(0) = x(T), \end{aligned}\right\} \tag{2.2}$$

has also not been studied so far in the existing literature for the existence of solutions. If $f(t, x, y) = 1$, then the FIGDI (2.1) reduces to the FIGDI

$$\left.\begin{aligned} x'(t) \in G\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s)))\, ds\Big) \quad \text{a.e.} \quad t \in J, \\ x(0) = x(T). \end{aligned}\right\} \tag{2.3}$$

There is a considerable work available in the literature for some special cases of FIGDI (2.3). See Andres and Gorniewicz [2], Deimling [5], and Hu and Papageorgiou [19] etc. Similarly in the special case when $G(t, x, y) = \{g(t, x, y)\}$ we obtain the differential equation

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t, x(t), x(\theta(t)))}\right] = g\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s)))\, ds\Big) \quad \text{a.e.} \quad t \in J, \\ x(0) = x(T). \end{aligned}\right\} \tag{2.4}$$

The functional differential equation (2.4) is again new to the literature and a special case of differential equation (2.4) with $f(t, x, y) = f(t, x)$ and $g(t, x, y) = g(t, x)$ has been studied recently in Dhage *et al.* [14] for the existence of solutions. Thus FIGDI (2.1) is more general, and therefore, it of interest to discuss it for the various aspects of the solutions under suitable conditions. In this paper, we shall prove the existence of solutions as well as the existence of the extremal solutions for the FIGDI (2.1) under suitable conditions. We seek the solutions in the space $C(J, \mathbb{R})$ of continuous real-valued functions on J under mixed generalized Lipschitz and Carathéodory conditions.

3. AUXILIARY RESULTS

In this section, we develop a multi-valued fixed point theorem that is used as a basic tool for proving the main existence result for FIGDI (2.1). Before stating the main fixed point theorem, we give some useful definitions and preliminaries that will be used in the sequel. Let X be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Denote

$$\mathcal{P}_p(X) = \{A \subset X \mid A \ \text{ is non-empty and has a property } \ p\}.$$

Thus $\mathcal{P}_{bd}(X)$, $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{cp}(X)$, $\mathcal{P}_{cl,bd}(X)$, $\mathcal{P}_{cp,cv}(X)$ denote the classes of all bounded, closed, convex, compact, closed-bounded and compact-convex subsets of X, respectively. Similarly, $\mathcal{P}_{cl,cv,bd}(X)$ denotes the class of closed, convex and bounded subsets of X. A correspondence $Q : X \to \mathcal{P}_p(X)$ is called a multi-valued operator or multi-valued mapping on X. A point $u \in X$ is called a fixed point of Q if $u \in Qu$. The multi-valued operator Q is called lower semi-continuous (in short l.s.c.) if G is any open subset of X, then

$$Q^{-}(G) = \{x \in X \mid Qx \bigcap G \neq \emptyset\}$$

is an open subset of X. Similarly, the multi-valued operator Q is called upper semi-continuous (in short u.s.c.) if the set

$$Q^{+}(G) = \{x \in X \mid Qx \subset G\}$$

is open in X for every open set G in X. Finally, Q is called continuous if it is lower as well as upper semi-continuous on X. A multi-valued map $Q : X \to \mathcal{P}_{cp}(X)$ is called **compact** if $\overline{Q(X)}$ is a compact subset of X. Q is called **totally bounded** if for any bounded subset S of X, $Q(S) = \bigcup_{x\in S} Qx$ is a totally bounded subset of X. It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on a bounded subset of X. Finally, Q is called **completely continuous** if it is upper semi-continuous and totally bounded on X.

Let X be a Banach algebra. For any $A, B \in \mathcal{P}_p(X)$, let us denote

$$\begin{aligned} A \pm B &= \{a \pm b \mid a \in A, b \in B\}, \\ A \cdot B &= \{ab \mid a \in A, b \in B\}, \\ \lambda A &= \{\lambda a \mid a \in A\} \end{aligned}$$

for $\lambda \in \mathbb{R}$. Similarly, denote

$$|A| = \{|a| \mid a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{|a| \mid a \in A\}.$$

Let $A, B \in \mathcal{P}_{cl,bd}(X)$ and let $a \in A$. Then, denote

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $d_H : \mathcal{P}_{cl,bd}(X) \times \mathcal{P}_{cl,bd}(X) \to \mathbb{R}^{+}$ defined by

$$d_H(A, B) = \max\{\rho(A, B),\, \rho(B, A)\} \tag{3.1}$$

is metric and is called the Hausdorff metric on X. It is clear that

$$d_H(0, C) = \|C\|_{\mathcal{P}} = \sup\{\|c\| \mid c \in C\}$$

for any $C \in \mathcal{P}_{cl,bd}(X)$.

Lemma 3.1. *Let X be a Banach algebra. If $A, B \in P_{bd,cl}(X)$, then $d_H(AC, BC) \leq d_H(0, C)\, d_H(A, B)$.*

Proof. The proof appears in Dhage [7]. □

We need the following definition in sequel.

Definition 3.2. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathcal{D}$-**function** if it satisfies

(i) ψ is continuous,
(ii) ψ is nondecreasing, and
(iii) ψ is scalarly submultiplicative, that is, $\psi(\lambda r) \leq \lambda \psi(r)$ for all $\lambda > 0$ and $r \in \mathbb{R}^+$.

The class of all $\mathcal{D}$-functions on $\mathbb{R}^+$ is denoted by Ψ. There do exist $\mathcal{D}$-functions on $\mathbb{R}$. Indeed, the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\psi(r) = \ell r$, $\ell > 0$ satisfies the conditions (i)–(iii) mentioned above and hence is a $\mathcal{D}$-function on $\mathbb{R}^+$. Note that if $\psi \in \Psi$, then $\psi(0) = 0$.

Definition 3.3. Let $Q : X \to \mathcal{P}_{cl,bd}(X)$ be a multi-valued operator. Then Q is called a multi-valued $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi \in \Psi$ such that

$$d_H(Qx, Qy) \leq \psi(\|x - y\|)$$

for all $x, y \in X$. If $\psi(r) = qr$, then Q is called multi-valued Lipschitz operator on X and the constant q is called the Lipschitz constant of Q. Similarly, a single-valued mapping $Q : X \to X$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi \in \Psi$ such that

$$\|Qx - Qy\| \leq \psi(\|x - y\|)$$

for all $x, y \in X$. If $\psi(r) = qr$, then Q is called a single-valued Lipschitz operator on X and the constant q is called the Lipschitz constant of Q on X.

The Kuratowskii measure $\alpha(S)$ and the Hausdorff measure $\beta(S)$ of noncompactness of a bounded set S in a Banach space X are the nonnegative real numbers defined by

$$\alpha(S) = \inf \left\{ r > 0 : S \subset \bigcup_{i=1}^{n} S_i, \text{ and } \operatorname{diam}(S_i) \leq r, \forall\, i \right\} \tag{3.2}$$

and

$$\beta(S) = \inf \left\{ r > 0 : S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r), \text{ for some } x_i \in X \right\}, \tag{3.3}$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

The details on the Kuratowskii and Hausdorff measures of noncompactness and their properties appear in Akhmerov *et al.* [1] and the references therein. The following results appear in Akhmerov *et al.* [1].

Lemma 3.4. *Let α and β be respectively the Kuratowskii and Hausdorff measure of noncompactness in a Banach space X. Then for any bounded set S in X, we have $\alpha(S) \leq 2\,\beta(S)$.*

Lemma 3.5. *If $A : X \to X$ is a single-valued $\mathcal{D}$-Lipschitz map with the $\mathcal{D}$-function ψ, then we have $\alpha(A(S)) \leq \psi(\alpha(S))$ for any bounded subset S of X.*

Lemma 3.6 (Banas and Lecko [3])**.** *Let X be a Banach algebra. If $S_1, S_2 \in \mathcal{P}_{bd}(X)$, then*

$$\beta(S_1\, S_2) \leq \|S_1\|_{\mathcal{P}}\, \beta(S_2) + \|S_2\|_{\mathcal{P}}\, \beta(S_1).$$

Definition 3.7. A multi-valued mapping $Q : X \to \mathcal{P}_{bd}(X)$ is called β-**condensing** if for any $S \in \mathcal{P}_{bd}(X)$, we have that $\beta(Q(S)) < \beta(S)$ for $\beta(S) > 0$.

The following extension of the Leray-Schauder principle is well-known in the literature (see Granas and Dugundji [16]).

Theorem 3.8. *Let U and $\overline{U}$ be respectively open and closed subsets of the Banach space X such that $0 \in U$. Let $Q : \overline{U} \to P_{cp,cv}(X)$ be an upper semi-continuous and β-condensing multi-valued operator with $\bigcup Q(\overline{U})$ bounded. Then either*

(i) *the operator inclusion $x \in Qx$ has a solution in $\overline{U}$, or*
(ii) *there is an element $u \in \partial U$ such that $\mu u \in Qu$ for some $\mu > 1$, where ∂U is is the boundary of U.*

Very recently, the present author [13] proved the following improvement of Theorem 3.8 above; it has several nice applications in nonlinear analysis.

Theorem 3.9. *Let U and $\overline{U}$ be respectively open and closed subsets of the Banach space X such that $0 \in U$. Let $Q : \overline{U} \to P_{cp,cv}(X)$ be a closed graph and β-condensing multi-valued operator with $\bigcup Q(\overline{U})$ bounded. Then either*

(i) *the operator inclusion $x \in Qx$ has a solution in $\overline{U}$, or*
(ii) *there is an element $u \in \partial U$ such that $\mu u \in Qu$ for some $\mu > 1$, where ∂U is is the boundary of U.*

The following multi-valued hybrid fixed point theorem is an improvement of the multi-valued fixed point theorem of Dhage [9].

Theorem 3.10. *Let U and $\overline{U}$ be respectively open-bounded and closed-bounded subsets of the Banach space X such that $0 \in U$ and let $A : X \to X$, and $B : \overline{U} \to \mathcal{P}_{cp,cv}(X)$ be two multi-valued operators satisfying*

(a) *A is single-valued $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is completely continuous, and*
(c) *$2M\psi(r) < r$ for $r > 0$, where $M = \|\bigcup B(\overline{U})\|_{\mathcal{P}}$.*

Then either

(i) *the operator inclusion $x \in Ax\, Bx$ has a solution in $\overline{U}$, or*

(ii) *there is an element* $u \in \partial U$ *such that* $\mu u \in Au\,Bu$ *for some* $\mu > 1$, *where* ∂U *is is the boundary of* U.

Proof. Define a multi-valued mapping $Q : \overline{U} \to \mathcal{P}_p(X)$ by

$$Qx = Ax\,Bx,\ \ x \in \overline{U}.$$

We show that Q satisfies all the conditions of Theorem 3.10. First, we show that Q has convex and compact values on $\overline{U}$. Let $z_1, z_2 \in Ax\,Bx$ be any two elements. Then there are points $u_1, u_2 \in Bx$ such that $z_1 = (Ax)\,u_1$ and $z_2 = (Ax)\,u_2$. Now for any $\lambda \in [0,1]$, one has

$$\begin{aligned}\lambda z_1 + (1-\lambda) z_2 &= \lambda(Ax\,u_1) + (1-\lambda)(Ax\,u_2)\\ &= (Ax)\,(\lambda u_1) + Ax\big[(1-\lambda)\ u_2\big]\\ &= (Ax)\big[(\lambda u_1) + (1-\lambda)u_2\big]\\ &= (Ax)\,z.\end{aligned}$$

Since Bx is a convex set, one has $z = \lambda u_1 + (1-\lambda)u_2 \in Bx$, and hence Q has convex values on $\overline{U}$. Again, in view of Lemma 3.5, we obtain

$$\beta(Qx) = \beta(Ax\,Bx) \le \|Ax\|\beta(Bx) + \|Bx\|_{\mathcal{P}}\beta(Ax) = 0,$$

and therefore, Q has compact values on $\overline{U}$. As a result, Q defines a multi-valued mapping $Q : \overline{U} \to \mathcal{P}_{cp,cv}(X)$.

Now we shall show that the mapping Q has a closed graph on $\overline{U}$. Let $\{x_n\}$ be a sequence in $\overline{U}$ converging to the point $x^* \in \overline{U}$ and let $\{y_n\}$ be sequence defined by $y_n \in Qx_n$ converging to the point y^*. It is enough to prove that $y^* \in Qx^*$. Now for any $x, y \in \overline{U}$ we have

$$\begin{aligned}d_H(Qx, Qy) &= d_H(AxBx, AyBy)\\ &\le d_H(AxBx, AyBx) + d_H(AyBx, AyBy)\\ &\le d(Ax, Ay)\,d_H(0, Bx) + d(0, Ay)\,d_H(Bx, By)\\ &\le \psi(\|x-y\|)\,\|B(\overline{U})\| + \|Ay\|\ d_H(Bx, By)\\ &\le M\psi(\|x-y\|)\ + \|Ay\|\,d_H(Bx, By).\end{aligned}\tag{3.4}$$

Since B is u.s.c., it is d_H-upper semi-continuous and consequently

$$d_H(Bx_n, Bx^*) \to 0 \quad \text{whenever} \quad x_n \to x^*.$$

Therefore,

$$\begin{aligned}D(y^*, Qx^*) &\le \lim_{n\to\infty} D(y_n, Qx^*) \le d_H(Qx_n, Qx^*)\\ &\le M\psi(\|x_n - x^*\|) + \|Ay^*\|\,d_H(Bx_n, Bx^*)\\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This shows that the multi-valued Q is a closed graph operator on $\overline{U}$.

Next we show that Q is β-condensing on $\overline{U}$. Let S be any subset of U. Then $Q(S) \subset A(S)B(S) \subset A(U)\,B(U)$. First we show that $A(U)B(U)$ is bounded. To finish, it enough to prove that $A(U)$ is bounded. Since U is bounded, there is a real number $r > 0$ such that $\|x\| \le r$ for all $x \in U$. Now for any $x \in U$, one has

$$\|Ax\| \le \|Ax - A0\| + \|A0\| \le \psi(\|x\|) + \|A0\| \le \psi(r) + \|A0\|.$$

Hence, we have $\|A(U)\|_{\mathcal{P}} \le c$, where $c = \psi(r) + \|A0\|$, and so $A(U)$ is bounded subset of X. Consequently, $A(U)B(U)$ is bounded since $B(U)$ is totally bounded. As a result, $Q(U)$ is a bounded set in X. Now by Lemma 3.5 and 3.6,

$$\beta(Q(S)) \le \|A(S)\|_{\mathcal{P}}\,\beta(B(S)) + \|B(S)\|_{\mathcal{P}}\,\beta(A(S)) \le M\psi(\alpha(S)) \le 2M\psi(\beta(S))$$

and so, $\beta(Q(S)) < \beta(S)$ for all $\beta(S) > 0$ since $2M\psi(r) < r$ for $r > 0$. This shows that Q is β-condensing on $\overline{U}$. Now an application of Theorem 3.9 yields that either

(i) the operator inclusion $x \in Ax\,Bx$ has a solution in $\overline{U}$, or
(ii) there is an element $u \in \partial U$ such that $\mu u \in Au\,Bu$ for some $\mu > 1$, where ∂U is is the boundary of U.

This completes the proof. □

Corollary 3.11. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ denote respectively open and closed balls centered at origin of radius r in the Banach space X and let $A : X \to X$ and $B : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$ be two operators satisfying*

(a) *A is single-valued Lipschitz with the Lipschitz constant q,*
(b) *B is completely continuous, and*
(c) *$2Mq < 1$, where $M = \|\bigcup B(\overline{\mathcal{B}_r(0)})\|_{\mathcal{P}}$.*

Then either

(i) *the operator inclusion $x \in Ax\,Bx$ has a solution in $\overline{\mathcal{B}_r(0)}$, or*
(ii) *there is an element $u \in X$ such that $\|u\| = r$ satisfying $\mu u \in Au\,Bu$ for some $\mu > 1$.*

In the following section we prove our main existence theorem for the FIGDI (2.1) under suitable conditions.

4. EXISTENCE RESULTS

In this section we prove an existence theorem for the differential inclusion (2.1) in Banach algebras by an application of the abstract results of the previous section under generalized Lipschitz and Carathéodory conditions.

Let $B(J,\mathbb{R})$ denote the space of bounded real-valued functions on J and let $C(J,\mathbb{R})$, denote the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \quad \text{and} \quad (x\cdot y)(t) = x(t)y(t) \ \text{ for } \ t \in J.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with respect to above norm and multiplication. By $L^1(J,\mathbb{R})$ we denote the set of Lebesgue integrable functions on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\, ds.$$

The following useful lemma is obvious and the details may be found in Nieto [20, 21].

Lemma 4.1. *For any $h \in L^1(J,\mathbb{R}^+)$ and $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x' + h(t)x(t) &= \sigma(t) \ \text{ a. e. } t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{4.1}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T g_h(t,s)\sigma(s)\, ds \tag{4.2}$$

where

$$g_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)}}{1-e^{-H(T)}}, & 0 \le s \le t \le T, \\[2ex] \dfrac{e^{H(s)-H(t)-H(T)}}{1-e^{-H(T)}}, & 0 \le t < s \le T, \end{cases} \tag{4.3}$$

where $H(t) = \displaystyle\int_0^t h(s)\, ds$.

Notice that the Green's function g_h is nonnegative on $J \times J$ and the number

$$M_h := \max\{\, |g_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J,\mathbb{R}^+)$. Note also that $H(t) > 0$ for all $t > 0$ provided that h is not the identically zero function.

We need the following definition in the sequel.

Definition 4.2. A multi-valued mapping $\beta : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be **Carathéodory** if

(i) $t \mapsto \beta(t,x,y)$ is measurable for all $x, y \in \mathbb{R}$, and
(ii) $(x,y) \mapsto \beta(t,x,y)$ is upper semi-continuous almost everywhere for $t \in J$.

A Carathéodory mapping $\beta(t,x,y)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $m_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \le m_r(t), \quad a.e.\ t \in J$$

for all $x, y \in \mathbb{R}$ with $|x| \le r$ and $|y| \le r$.

Let $\beta : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multi-valued mapping with nonempty compact values. Assign to β, the multi-valued operator $S^1_\beta : C(J, \mathbb{R}) \to \mathcal{P}(L^1(J, \mathbb{R}))$ defined by

$$S^1_\beta(x) = \Big\{ v \in L^1(J, \mathbb{R}) : v(t) \in \beta\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s)))\, ds\Big) \text{ a.e. } \ t \in J \Big\}.$$

The operator S^1_β is called the Niemytsky operator associated with the multi-valued mapping β and $S^1_\beta(x)$ is called the *selection set* of functions of the multi-valued mapping β at $x \in C(J, \mathbb{R})$.

Then we have the following lemmas due to Lasota and Opial [18].

Lemma 4.3. *Let E be a Banach space. If $\dim(E) < \infty$ and $\beta : J \times E \times E \to \mathcal{P}_{cp}(E)$ L^1-Carathéodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

Lemma 4.4. *Let E be a Banach space. Let $\beta : J \times E \times E \to \mathcal{P}_{cp}(E)$ be an L^1-Carathéodory multi-valued map with $S^1_\beta \neq \emptyset$, and let $\mathcal{L}$ be a linear continuous mapping from $L^1(J, E)$ into $C(J, E)$. Then the multi-valued composition mapping $\mathcal{L} \circ S^1_\beta$: $C(J, E) \to \mathcal{P}_{cp,cv}(C(J, E))$ defined by $u \mapsto (\mathcal{L} \circ S_\beta)(u) := \mathcal{L}(S_\beta(u))$ is a closed graph operator in $C(J, E) \times C(J, E)$.*

We will use the following hypotheses in the sequel.

(A_0) The mapping $t \mapsto f(t, x, y)$ is periodic of period T for all $x, y \in \mathbb{R}$.

(A_1) The mapping $x \mapsto \dfrac{x}{f(0, x, x)}$ is injective in $\mathbb{R}$.

(A_2) The mapping $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and there exists a bounded function $\ell : J \to \mathbb{R}$ with bound $\|\ell\|$ satisfying

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \ell(t) \max\{|x_1 - y_1|,\, |x_2 - y_2|\} \quad \text{a.e.} \quad t \in J$$

for all $x, y \in \mathbb{R}$.

(B_1) G is Carathéodory.

Note that hypotheses (A_0) through (A_2) are common in the literature on the theory of nonlinear differential equations. Actually, the mapping $f : J \times \mathbb{R} \to \mathbb{R}$ defined by $f(t, x, y) = \alpha + \beta(x + y)$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta(x + y) \neq 0$ satisfies the hypotheses (A_0)-(A_2).

Now consider the FIGDI with periodic boundary condition

$$\left.\begin{aligned}&\Big(\frac{x(t)}{f(t,x(t),x(\theta(t)))}\Big)'+h(t)\Big(\frac{x(t)}{f(t,x(t),x(\theta(t)))}\Big)\\&\qquad\in G_h\Big(t,x(\mu(t)),\int_0^{\sigma(t)}k(t,s,x(\eta(s)))\,ds\Big)\quad\text{a.e.}\quad t\in J,\\&x(0)=x(T),\end{aligned}\right\}\tag{4.4}$$

where $h\in L^1(J,\mathbb{R}^+)$ is bounded and the mapping $G_h: J\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ is defined by

$$G_h(t,x,y)=G(t,x,y)+h(t)\Big(\frac{x}{f(t,x,y)}\Big).\tag{4.5}$$

Remark 4.5. Note that the FIGDI (2.1) is equivalent to the FIGDI (4.4) and a solution of the FIGDI (2.1) is the solution for the FIGDI (4.4) on J and vice versa.

Remark 4.6. If the mapping f is continuous on $J\times\mathbb{R}\times\mathbb{R}$ and the hypothesis (B_1) holds, then the mapping G_h defined by (4.5) is Carathéodory on $J\times\mathbb{R}\times\mathbb{R}$.

Lemma 4.7. *Assume that hypotheses (A_0) and (A_1) hold. Then for any bounded $h\in L^1(J,\mathbb{R}^+)$, x is a solution to the differential inclusion* (4.4) *if and only if it is a solution of the integral inclusion*

$$x(t)\in\big[f(t,x(t),x(\theta(t)))\big]\left(\int_0^T g_h(t,s)G_h\Big(s,x(\mu(s)),\int_0^{\sigma(s)}k(s,\tau,x(\eta(\tau))\,d\tau\Big)\,ds\right),\tag{4.6}$$

where the Green's function $g_h(t,s)$ is defined by (4.3).

Proof. Let $y(t)=\dfrac{x(t)}{f(t,x(t),x(\theta(t)))}$. Since $f(t,x,x)$ is periodic in t of period T for all $x\in\mathbb{R}$, we have

$$y(0)=\frac{x(0)}{f(0,x(0),x(0))}=\frac{x(T)}{f(T,x(T),x(T))}=y(T).$$

Now an application of Lemma 4.1 yields that the solution to differential inclusion (4.4) is the solution to integral inclusion (4.6). Conversely, suppose that x is any solution to the integral inclusion (4.6), then

$$y(0)=\frac{x(0)}{f(0,x(0),x(0))}=\frac{x(T)}{f(0,x(T),x(T))}=y(T).$$

Since the function $x\mapsto\dfrac{x}{f(0,x,x)}$ is injective in $\mathbb{R}$, one has $x(0)=x(T)$ and so, x is a solution to FIGDI (2.1). The proof of the lemma is complete. $\square$

We make use of the following hypotheses in the sequel.

(B$_2$) The function $k : J \times J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\alpha \in L^1(J, \mathbb{R}^+)$, such that

$$|k(t,s,y)| \leq \alpha(t)|y| \text{ a.e. } t,s \in J \text{ and } y \in \mathbb{R}.$$

(B$_3$) There exists a function $\gamma \in L^1(J, \mathbb{R}^+)$ and a $\mathcal{D}$-function $\psi \in \Psi$ such that

$$\|G_h(t,x,y)\|_{\mathcal{P}} \leq \gamma(t)\psi(|x|+|y|) \ \text{ a.e. } t \in J$$

for each $x, y \in \mathbb{R}$.

We frequently make use of the following estimate concerning the multi-valued function $G(t,x,y)$ in the sequel. If the hypotheses (B$_2$)–(B$_3$) hold, then for any $x \in C(J,\mathbb{R})$ with $\|x\| \leq r$, one has

$$\begin{aligned}
\Big\| G_h\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\Big\|_{\mathcal{P}} \\
&\leq \gamma(t)\psi\Big(|x(\mu(t))| + \int_0^{\sigma(t)} |k(t,s,x(\eta(s)))|\,ds\Big) \\
&\leq \gamma(t)\psi\Big(\|x\| + \int_0^{\sigma(t)} \alpha(s)|x(\eta(s))|\,ds\Big) \\
&\leq \gamma(t)\psi\Big(\|x\| + \int_0^{T} \alpha(s)\|x\|\,ds\Big) \\
&\leq \gamma(t)\psi\Big([1+\|\alpha\|_{L^1}]\,\|x\|\Big) \\
&\leq \gamma(t)(1+\|\alpha\|_{L^1})\psi(r)
\end{aligned} \tag{4.7}$$

for all $t \in J$.

Theorem 4.8. *Assume that the hypotheses (A$_0$)–(A$_2$)and (B$_1$)–(B$_3$) hold. If there exists a real number $r > 0$ such that*

$$r > \frac{FM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)}{1 - \|\ell\|\,[M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)]} \tag{4.8}$$

where, $\|\ell\|\,[M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)] < 1/2$ and $F = \sup_{t\in J}|f(t,0,0)|$, then the FIGDI (2.1) *has a solution on J.*

Proof. Let $X = C(J,\mathbb{R})$ and define an open ball $\mathcal{B}_r(0)$ in X, where the real number r satisfies the inequality (4.8). Now consider two operators $A : X \to X$ and $B : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_p(\mathbb{R})$ defined by

$$Ax(t) = f(t, x(t), x(\theta(t))) \tag{4.9}$$

and

$$Bx = \left\{ u \in X \,\middle|\, u(t) = \int_0^T g_h(t,s)v(s)\,ds, \quad v \in S^1_{G_h}(x) \right\} \tag{4.10}$$

for all $t \in J$.

Then the FIGDI (2.1) is equivalent to the operator inclusion

$$x(t) \in Ax(t)\,Bx(t), \ t \in J. \tag{4.11}$$

We shall show that the multi-valued operators A and B satisfy all the conditions of Corollary 3.11. Clearly the operator B is well defined since $S^1_{G_h}(x) \neq \emptyset$ for each $x \in X$.

Step I : We first show that the operators A and B define respectively the single-valued and the multi-valued operators $A : X \to X$ and $B : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$. The case of A is obvious since the function f is continuous on $J \times \mathbb{R} \times \mathbb{R}$. We only prove the claim for the operator B. It is shown as in the Step III below that the multi-valued operator B has compact values on $\overline{\mathcal{B}_r(0)}$.

Again, let $u_1, u_2 \in Ax$. Then there are $v_1, v_2 \in S^1_{G_h}(x)$ such that

$$u_1(t) = \int_0^T g_h(t,s) v_1(s)\, ds, \ t \in J,$$

and

$$u_2(t) = \int_0^T g_h(t,s) v_2(s)\, ds, \ t \in J.$$

Now for any $\lambda \in [0,1]$,

$$\begin{aligned} \lambda u_1(t) + (1-\lambda) u_2(t) &= \lambda \left(\int_0^T g_h(t,s) v_1(s)\, ds \right) + (1-\lambda) \left(\int_0^T g_h(t,s) v_2(s)\, ds \right) \\ &= \int_0^T [\lambda g_h(t,s) v_1(s) + (1-\lambda) g_h(t,s) v_2(s)]\, ds. \end{aligned}$$

Since $S^1_{G_h}$ has convex values on X (because G has convex values), we have that $v(t) = \lambda v_1(t) + (1-\lambda) v_2(t) \in S^1_{G_h}(x)(t)$ for all $t \in J$. Hence, $\lambda u_1 + (1-\lambda) u_2 \in Bx$ and consequently Bx is convex for each $x \in X$. As a result, B defines a multi-valued operator $B : X \to \mathcal{P}_{cp,cv}(X)$.

Step II : To show A a Lipschitz on X, let $x, y \in X$. Then,

$$\begin{aligned} \|Ax - Ay\| &= \sup_{t \in J} |Ax(t) - Ay(t)| \\ &= \sup_{t \in J} |f(t, x(t), x(\theta(t))) - f(t, y(t), y(\theta(t)))| \\ &\leq \sup_{t \in J} \ell(t) \max\{|x(t) - y(t)|, |x(\theta(t)) - x(\theta(t))|\} \\ &\leq \|\ell\| \|x - y\|, \end{aligned}$$

showing that A is a Lipschitz on X with the Lipschitz constant $\|\ell\|$.

Step III : Next we show that B is completely continuous on $\overline{\mathcal{B}_r(0)}$. First, we prove that $B(\overline{\mathcal{B}_r(0)})$ is totally bounded subset of X. To do this, it is enough to prove that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equi-continuous set in X. To see this, let $u \in B(\overline{\mathcal{B}_r(0)})$ be arbitrary. Then there is a $v \in S^1_{G_h}(x)$ such that

$$u(t) = \int_0^T g_h(t,s)v(s)\,ds.$$

for some $x \in \overline{\mathcal{B}_r(0)}$. Hence,

$$\begin{aligned}
|u(t)| &\le \int_0^T g_h(t,s)|v(s)|\,ds \\
&\le \int_0^T g_h(t,s)\Big\|G_h\Big(s, x(\mu(s)), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau))\,d\tau\Big)\Big\|_{\mathcal{P}}\,ds \\
&\le \int_0^T g_h(t,s)\gamma(s)(1+\|\alpha\|_{L^1})\psi(r)\,ds \\
&= M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)
\end{aligned}$$

for all $t \in J$, and so $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded set in X. Next we show that $B(\overline{\mathcal{B}_r(0)})$ is an equicontinuous set. To finish, it is enough to show that u' is bounded on $[0,T]$. Now for any $t \in [0,T]$, one has

$$\begin{aligned}
|u'(t)| &= \Big|\int_0^T \frac{\partial}{\partial t} g_h(t,s)v(s)\,ds\Big| \\
&= \Big|\int_0^T (-h(t))g_h(t,s)v(s)\,ds\Big| \\
&\le H\,M_h\|\gamma\|_{L^1}\psi(r) \\
&= c,
\end{aligned}$$

where $H = \max_{t\in J} h(t)$. Hence, for any $t,\tau \in [0,T]$, one has

$$|Bx(t) - Bx(\tau)| \le c\,|t-\tau| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $B(\overline{\mathcal{B}_r(0)})$ is a equi-continuous set in X. Hence, $B(\overline{\mathcal{B}}_r(0))$ is compact by the Arzela-Ascoli theorem. Thus, we have $B : \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$ is totally bounded.

Next we show that B is an upper semi-continuous multi-valued operator on X. Let $\{x_n\}$ be a sequence in X such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Bx_n$ and $y_n \to y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_{G_h}(x_n)$, $n = 1,2,\ldots$, such that

$$y_n(t) = \int_0^T g_h(t,s)v_n(s)\,ds, \;\; t \in J.$$

We must prove that there is a $v_* \in S^1_{G_h}(x_*)$ such that

$$y_*(t) = \int_0^T g_h(t,s)v_*(s)\,ds, \;\; t \in J.$$

Consider the continuous linear operator $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^T g_h(t,s)v(s)\,ds, \ \ t \in J.$$

Now we have $\|y_n - y_*\| \to 0$ as $n \to 0$. From Lemma 4.4, it follows that $\mathcal{K} \circ S^1_{G_h}$ is a closed graph operator. Also, from the definition of $\mathcal{K}$, we have $y_n \in (\mathcal{K} \circ S^1_{G_h})(x_n)$. Since $y_n \to y_*$, there is a point $v_* \in S^1_{G_h}(x_*)$ such that

$$y_*(t) = \int_0^T g_h(t,s)v_*(s)ds, \ \ t \in J.$$

This shows that B is a completely continuous operator on $\overline{\mathcal{B}_r(0)}$. Thus, B is an upper semi-continuous and compact operator on X.

Step IV : Finally, from the condition given in the statement of the theorem, it follows that

$$2Mq = 2\|\ell\| M_h \|\gamma\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r) < 1.$$

Thus all the conditions of Corollary 3.11 are satisfied so, either conclusion (i) or conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be such that $\|u\| = r$. Then, for any $\mu = \frac{1}{\lambda} > 1$, for some $\lambda \in (0,1)$, one has

$$\begin{aligned}
&\mu u(t) \in Au(t)Bu(t) \\
&\quad = [f(t,u(t),u(\theta(t))]\left(\int_0^T g_h(t,s)G_h\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big))\,ds\right),
\end{aligned}$$

for all $t \in J$. Therefore,

$$\mu u(t) \in \Big[f(t,u(t),u(\theta(t)))\Big]\Big(\int_0^T g_h(t,s)G_h\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau)))\,d\tau\Big)\,ds\Big)$$

or

$$u(t) = \lambda\Big[f(t,u(t),u(\theta(t)))\Big]\Big(\int_0^T g_h(t,s)v(s)\,ds\Big)$$

for some $v \in S^1_{G_h}(u)$. Therefore, by (4.7),

$$\begin{aligned}
|u(t)| &= \Big|\lambda\Big[f(t,u(t),u(\theta(t)))\Big]\Big|\ \Big|\int_0^T g_h(t,s)v(s)\,ds\Big)\Big| \\
&\le \Big|\Big[f(t,u(t),u(\theta(t)))\Big]\Big|\ \Big(\int_0^T g_h(t,s)|v(s)|\,ds\Big) \\
&\le \Big[|f(t,u(t),u(\theta(t))) - f(t,0,0)| + |f(t,0,0)|\Big] \\
&\qquad \times \Big(\int_0^T g_h(t,s)\Big\|G_h\Big(s,u(\mu(s)),\int_0^{\sigma(s)} k(s,\tau,u(\eta(\tau))\,d\tau\Big)\Big\|_{\mathcal{P}}\,ds\Big) \\
&\le |\ell(t)|\max\{|u(t)|,u(\theta(t))\}\Big(\int_0^T g_h(t,s)\gamma(s)(1+\|\alpha\|_{L^1})\psi(r)\,ds\Big) \\
&\qquad + F\Big(\int_0^T g_h(t,s)\gamma(s)(1+\|\alpha\|_{L^1})\psi(r)\,ds\Big)
\end{aligned}$$

$$\leq \|\ell\| \max\{|u(t)|, u(\theta(t))\} \left(M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)\right)$$
$$+ F\left(M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)\right). \tag{4.12}$$

Taking the supremum over t, we obtain

$$\|u\| \leq \frac{FM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)}{1-\|\ell\|\left[M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)\right]}$$

or,

$$r \leq \frac{FM_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)}{1-\|\ell\|\left[M_h\|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\psi(r)\right]}$$

which is a contradiction to (4.8). Hence, the conclusion (ii) of Corollary 3.11 does not hold. Therefore, the operator inclusion $x \in AxBx$, and consequently the FIGDI (2.1), has a solution in $\overline{\mathcal{B}_r(0)}$ defined on J. This completes the proof. □

5. EXISTENCE OF EXTREMAL SOLUTIONS

A non-empty and closed set K in a Banach algebra X is called a **cone** if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called **positive** if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in K as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called **normal** if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. Details on cones and their properties appear in Guo and Lakshmikantham [17].

We equip the space $C(J,\mathbb{R})$ with the order relation $\leq$ defined by the cone

$$K = \{x \in C(J,\mathbb{R}) : x(t) \geq 0 \text{ for all } t \in J\}. \tag{5.1}$$

It is well known that the cone K is positive and normal in $C(J,\mathbb{R})$. As a result of positivity of the cone K in $C(J,\mathbb{R})$, we have:

Lemma 5.1 (Dhage [8])**.** *Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1u_2 \leq v_1v_2$.*

For any $a, b \in X = C(J,\mathbb{R})$ with $a \leq b$, the order interval $[a, b]$ is a set in X defined by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

Definition 5.2. A multi-valued operator $Q : X \to \mathcal{P}_p(X)$ is called strictly monotone increasing if $x, y \in X$ with $x < y$ implies $Qx \leq Qy$.

We use the following fixed point theorem of Dhage [8] for proving existence of the extremal solutions for the FIGDI(2.1) under certain monotonicity conditions.

Theorem 5.3 (Dhage [8]). *Let $[a, b]$ be an order interval in a Banach algebra X. Suppose that $A : [a, b] \to K$ is nondecreasing and $B : [a, b] \to \mathcal{P}_{cl}(K)$ is strictly monotone increasing such that*

(a) *A is Lipschitz with the Lipschitz constant q,*
(b) *B is completely continuous, and*
(c) *$Ax\,Bx \subset [a, b]$ for each $x \in [a, b]$.*

If the cone K is positive and normal, then the operator equation $x \in Ax\,Bx$ has a least and a greatest positive solution in $[a, b]$, whenever $2Mq < 1$, where $M = \| \cup B([a, b])\|_{\mathcal{P}} := \sup\{\|Bx\|_{\mathcal{P}} : x \in [a, b]\}$.

We need the following definitions in the sequel.

Definition 5.4. A function $a \in C(J, \mathbb{R})$ is called a strict lower solution of the FIGDI (2.1) on J if the function $t \mapsto \left(\dfrac{x(t)}{f(t, x(t), x(\theta(t)))}\right)$ is absolutely continuous, and for all $v \in S^1_{G_h}(a)$, we have that

$$\frac{d}{dt}\left[\frac{a(t)}{f(t, a(t), a(\theta(t)))}\right] \le v(t), \quad \text{a.e } t \in J, \quad \text{and} \quad a(0) \le a(T).$$

Similarly, a strict upper solution $b \in C(J, \mathbb{R})$ for the FIGDI (2.1) on J is defined.

Definition 5.5. A solution x_M of the FIGDI (2.1) is said to be maximal if for any other solution x to FIGDI (2.1) one has $x(t) \le x_M(t)$, for all $t \in J$. Again a solution x_m of the FIGDI (2.1) is said to be minimal if $x_m(t) \le x(t)$, for all $t \in J$, where x is any solution of the FIGDI (2.1) on J.

Remark 5.6. If a is a strict lower solution for the FIGDE (2.1), then it is also a strict lower solution for the FIGDI (4.4) and vice-versa. The same is true for a strict upper solution for the FIGDE (2.1) on J. Similarly, a minimal solution for the FIGDE (2.1) is a minimal solution for the for the FIGDE (4.4) and vice-versa. Again, the same is true for maximal solution for the FIGDE (2.1) on J.

Definition 5.7. A multi-valued mapping $\beta(t, x, y)$ is said to be strictly increasing in x if for all $t \in J$ and $y \in \mathbb{R}$, we have $\beta(t, x_1, y) \le \beta(t, x_2, y)$ for all $x_1, x_2 \in \mathbb{R}$ for which $x_1 < x_2$. Similarly, the strict monotonicity of $\beta(t, x, y)$ in the argument y is defined.

We consider the following set of assumptions:

(B_4) $f : J \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ - \{0\}$ and $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R}^+)$.
(B_5) G_h is L^1-Carathéodory.
(B_6) $f(t, x, y)$ is nondecreasing in x and y and $k(t, s, x)$ is monotone increasing in x almost everywhere for $t \in J$.
(B_7) $G_h(t, x, y)$ is strictly monotone increasing in x and y almost everywhere for $t \in J$.

(B_8) The FIGDI (2.1) has a lower solution a and an upper solution b on J with $a \leq b$.

Remark 5.8. *Assume that (B_5)–(B_8) hold. Define a function $m : J \to \mathbb{R}^+$ by*

$$m(t) = \Big\| G_h\Big(t, b(\mu(t)), \int_0^{\sigma(t)} k(t,s,b(\eta(s)))\,ds\Big)\Big\|_{\mathcal{P}}$$

for all $t \in J$. Then m is Lebesgue integrable and

$$\Big\| G_h\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\Big\|_{\mathcal{P}} \leq m(t), \quad a.e.\ t \in J,$$

for all $x \in [a,b]$.

Theorem 5.9. *Suppose that the assumptions (A_0)–(A_2) and (B_4)–(B_8) hold. If $\|\ell\| M_h \|m\|_{L^1} < 1/2$, and h is given in Remark* 5.8, *then FIGDI* (2.1) *has a minimal and a maximal positive solution in* $[a,b]$ *defined on J.*

Proof. Now FIGDI (2.1) is equivalent to functional integral inclusion (4.6) on J. Let $X = C(J,\mathbb{R})$ and consider the order interval $[a,b]$ in X. Define two operators A and B on $[a,b]$ by (4.9) and (4.10) respectively. Then FIGDI (2.1) is transformed into an operator inclusion $x(t) \in Ax(t)Bx(t)$ in a Banach algebra X. Notice that (B_4) implies $A : [a,b] \to K$ and $B : [a,b] \to \mathcal{P}_{cl}(K)$. Since the cone K in X is normal, $[a,b]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 4.8, that A is Lipschitz with the Lipschitz constant $\|\ell\|$ and B is completely continuous operator on $[a,b]$. Again the hypotheses (B_6) and (B_7) imply that A is nondecreasing and B is strictly monotone increasing on $[a,b]$. To see this, let $x,y \in [a,b]$ be such that $x \leq y$. Then by (B_6),

$$Ax(t) = f(t,x(t),x(\theta(t))) \leq f(t,y(t),y(\theta(t))) = Ay(t)$$

for all $t \in J$. Similarly, let $x,y \in [a,b]$ be such that $x < y$; then we have

$$\begin{aligned} Bx(t) &= \int_0^T g_h(t,s) G_h\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\,ds \\ &\leq \int_0^T g_h(t,s) G_h\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\,ds \\ &= By(t) \end{aligned}$$

for all $t \in J$. Again, Lemma 4.1 and hypothesis (B_7) together imply that

$$\begin{aligned} a(t) &\leq [f(t,a(t),a(\theta(t)))]\Big(\int_0^T g_h(t,s) G_h\Big(t, a(\mu(t)), \int_0^{\sigma(t)} k(t,s,a(\eta(s)))\,ds\Big)\,ds\Big) \\ &\leq [f(t,x(t),x(\theta(t)))]\Big(\int_0^T g_h(t,s) G_h\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big)\,ds\Big) \\ &\leq [f(t,b(t),b(\theta(t)))]\Big(\int_0^T g_h(t,s) G_h\Big(t, b(\mu(t)), \int_0^{\sigma(t)} k(t,s,b(\eta(s)))\,ds\Big)\,ds\Big) \\ &\leq b(t), \end{aligned}$$

for all $t \in J$ and $x \in [a, b]$. As a result, $a(t) \le Ax(t)\, Bx(t) \le b(t)$, for all $t \in J$ and $x \in [a, b]$. Hence, $Ax\, Bx \subset [a, b]$ for all $x \in [a, b]$. Again, we have

$$\begin{aligned} M &= \|\bigcup B([a,b])\|_{\mathcal{P}} \\ &= \sup\{\|Bx\|_{\mathcal{P}} : x \in [a,b]\} \\ &\le \sup_{x\in[a,b]} \left\{ \sup_{t\in J} \int_0^T g_h(t,s) \Big\| G_h\Big(t, x(\mu(t)), \int_0^{\sigma(t)} k(t,s,x(\eta(s)))\,ds\Big) \Big\|_{\mathcal{P}} ds \right\} \\ &\le \int_0^T g_h(t,s) m(s)\,ds \\ &= M_h \|m\|_{L^1}. \end{aligned}$$

Since $2Mq \le 2\|\ell\| M_h \,\|m\|_{L^1} < 1$, we apply Theorem 5.3 to the operator inclusion $x \in AxBx$ to yield that the FIGDI (2.1) has a minimal and a maximal positive solution in $[a, b]$ defined on J. □

6. AN EXAMPLE

Given the closed and bounded interval $J = [0, \pi]$ in $\mathbb{R}$, consider the first order periodic boundary value problem of FIGDI,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{1+\frac{|\sin t|}{12}(|x(t)|+|x(t^2/\pi)|)}\right] &\in -\left(\frac{x(t)}{1+\frac{|\sin t|}{12}(|x(t)|+|x(t^2/\pi)|)}\right) \\ &\qquad + G\Big(t, x(t/2), \int_0^{\pi-t} k(t,s,x(s/3))\,ds\Big) \text{ a.e. } t\in J, \\ x(0) &= x(\pi), \end{aligned}\right\} \tag{6.1}$$

where, $p \in L^1(J, \mathbb{R}^+)$, and the functions $k : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$, $\theta, \mu, \sigma, \eta : J \to J$ are given by

$$G(t,x,y) = \begin{cases} \left[\dfrac{p(t)x}{1+|x|}, p(t)\right], & \text{if } \; x \neq 0, y = 0, \\ \left[\dfrac{p(t)x}{1+|x|}, \dfrac{p(t)x}{1+|x|} + |y|\right], & \text{if } \; x \neq 0, y \neq 0 \\ [0, p(t) + |y|]\,, & \text{if } \; x = 0, y \neq 0, \end{cases},$$

and

$$k(t,s,x) = \frac{x}{4\pi(1+|x|)}.$$

Here,

$$\theta(t) = t^2/\pi, \;\; \mu(t) = t/2, \;\; \sigma(t) = \pi - t, \quad \text{and} \quad \eta(t) = t/3$$

for $t \in J$. Clearly the functions $k : J \times J \times \mathbb{R} \to \mathbb{R}$ and $\theta, \mu, \sigma, \eta : J \to J$ are continuous with $\theta(0) = 0$ and $\theta(\pi) = \pi$.

Here, the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is defined by

$$f(t, x, y) = 1 + \frac{|\sin t|}{12}\,(|x| + |y|).$$

Obviously $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$. It is easy to verify that f is continuous and satisfies the hypotheses (A$_0$)–(A$_2$) on $J \times \mathbb{R} \times \mathbb{R}$ with $\ell(t) = \frac{1}{6}$ for all $t \in J$. To see this, let $x, y \in \mathbb{R}$; then we have

$$\begin{aligned}
|f(t, x_1, x_2) - f(t, y_1, y_2)| &= \left| \left[1 + \frac{|\sin t|}{12}\,(|x_1| + |x_2|)\right] - \left[1 + \frac{|\sin t|}{12}\,(|y_1| + |y_2|)\right] \right| \\
&\le \frac{1}{12}\,\big|(\,|x_1| - |y_1| + |x_2| - |y_2|\,)\big| \\
&\le \frac{1}{12}\,\big(|x_1 - y_1| + |x_2 - y_2|\big) \\
&\le \frac{1}{6}\max\{|x_1 - y_1|\,,\,|x_2 - y_2|\}.
\end{aligned}$$

Again the function $G(t, x, y)$ is measurable in t for all $x, y \in \mathbb{R}$ and upper semi-continuous in x and y almost everywhere for $t \in J$, and so G defines a Carathéodory multi-valued mapping $G : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$. Further, G_1 is also Carathéodory on $J \times \mathbb{R} \times \mathbb{R}$, and

$$\begin{aligned}
\|G_1(t, x, y)\|_{\mathcal{P}} &= \left| \frac{p(t)\,x(t)}{1 + |x(t)|} + \int_0^{\pi - t} \frac{x(s/3)}{4\pi(1 + |x(s/2)|)}\,ds \right| \\
&\le \left| \frac{p(t)\,x(t)}{1 + |x(t)|} \right| + \left| \int_0^{\pi - t} \frac{x(s/3)}{4\pi(1 + |x(s/2)|)}\,ds \right| \\
&\le |p(t)| + \frac{1}{4}
\end{aligned}$$

Hence, the function G_1 is $L^1_{\mathbb{R}}$-Carathéodory and satisfies all the hypotheses (B$_1$) through (B$_3$) on $J \times \mathbb{R} \times \mathbb{R}$ with $\gamma(t) = |p(t)| + \frac{1}{4}$ on J and $\psi(r) = 1$ for all $r \in \mathbb{R}^+$. Therefore, if $\|p\|_{L^1} < 5$ and $r = 2$, then by Theorem 4.8, then the FIGDI (6.1) has a solution in $\overline{\mathcal{B}_2(0)}$ defined on J.

Remark 6.1. While concluding this paper, we mention that our existence results of this paper can be extended to infinite dimensional Banach algebras with appropriate modifications. Also, the existence results of this paper include the existence results for the differential inclusions (2.2), (2.3) and (2.4) as special cases that are again new to the literature on quadratic differential inclusions. Our results also extend the existence results proved in Dhage *et al.* [14] for the periodic boundary value problem

$$\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] = g(t, x(t)) \text{ a.e. } t \in J, \quad x(0) = x(T), \tag{6.2}$$

to the corresponding quadratic ordinary differential inclusions with periodic boundary conditions.

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