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ON THE OSCILLATIONS OF FOURTH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

SAID R. GRACE[1], RAVI P. AGARWAL[2], AND SANDRA PINELAS[3]

[1]Department of Engineering Mathematics, Faculty of Engineering
Cairo University, Orman, Giza 12221, Egypt
E-mail: srgrace@eng.cu.eg

[2]Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901, U. S. A.
E-mail: agarwal@fit.edu

[3]Department of Mathematics, Azores University, R. Mãe de Deus
9500-321 Ponta Delgada, Portugal
E-mail: sandra.pinelas@clix.pt

ABSTRACT. We establish some sufficient conditions for the oscillations of all solutions of fourth order functional differential equations

$$\frac{d}{dt}\left(a(t)\left(\frac{d^3}{dt^3}x(t)\right)^{\alpha}\right)+q(t)f(x[g(t)])=0$$

and

$$\frac{d}{dt}\left(a(t)\left(\frac{d^3}{dt^3}x(t)\right)^{\alpha}\right)=q(t)f(x[g(t)])+p(t)h(x[\sigma(t)])$$

when $\int^{\infty} a^{-1/\alpha}(s)\,ds<\infty$. The case when $\int^{\infty} a^{-1/\alpha}(s)\,ds=\infty$ is also included.

1. Introduction

This paper deals with the oscillatory behavior of solutions of fourth order functional differential equations

$$\frac{d}{dt}\left(a(t)\left(\frac{d^3}{dt^3}x(t)\right)^{\alpha}\right)+q(t)f(x[g(t)])=0 \tag{1}$$

and

$$\frac{d}{dt}\left(a(t)\left(\frac{d^3}{dt^3}x(t)\right)^{\alpha}\right)=q(t)f(x[g(t)])+p(t)h(x[\sigma(t)]) \tag{2}$$

where the following conditions are assumed to hold:

(i) α is the ratio of two positive odd integers;
(ii) $a(t)$, $p(t)$ and $q(t)\in C([t_0,\infty),(0,\infty))$;
(iii) $g(t)$ and $\sigma(t)\in C^1([t_0,\infty),\mathbb{R})$, $g(t)<t$, $\sigma(t)>t$, $g'(t)\geqslant 0$ and $\sigma'(t)\geqslant 0$ for $t\geqslant t_0$ and $\lim_{t\to\infty} g(t)=\infty$;

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(iv) f, $h \in C(\mathbb{R},\mathbb{R})$, $xf(x) > 0$, $xh(x) > 0$, $f'(x) \geqslant 0$ and $h'(x) \geqslant 0$ for $x \neq 0$ and f and g satisfy

$$-f(-xy) \geqslant f(xy) \geqslant f(x)\, f(y), \quad \text{for} \quad xy > 0 \tag{3}$$

and

$$-h(-xy) \geqslant h(xy) \geqslant h(x)\, h(y), \quad \text{for} \quad xy > 0. \tag{4}$$

By a solution of Equation (1) (respectively (2)) is meant a function $x : [T_x, \infty) \to \mathbb{R}$, $T_x \geqslant t_0$ such that $x(t)$, $x'(t)$, $x''(t)$ and $a(t)\left(\frac{d^3}{dt^3}x(t)\right)^\alpha$ are continuously differentiable and satisfy equation (1) (respectively (2)) on $[T_x, \infty)$. Our attention will be restricted to those solutions of equations (1) and (2) which satisfy $\sup\{x(t) : t \geqslant T\} > 0$ for any $T \geqslant T_x$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity and nonoscillatory otherwise.

In the last three decades there has been an increasing interest in studying the oscillatory and nonoscillatory behavior of solutions of functional differential equations. Most of the work on the subject, however, has been restricted to first and second order differential equations of the form

$$\left(a(t)\,(x'(t))^\alpha\right)' + q(t)\, f(x[g(t)]) = 0$$

when

$$\int^\infty a^{-1/\alpha}(s)\, ds = \infty, \tag{5}$$

as well as higher order equations with $n > 2$. For recent contributions we refer to [1]–[9] and the references cited therein.

It seems that little is known concerning the oscillations of equations (1) and (2) particularly when

$$\int^\infty a^{-1/\alpha}(s)\, ds < \infty. \tag{6}$$

Therefore, our aim is to present some sufficient conditions for the oscillation of equations (1) and (2) via comparison with first order delay and/or advanced equations whose oscillatory characters are known.

In Section 2 we establish results for the oscillation of equation (1) when (6) holds, and from these deduce the results when (5) holds. Similar results for equation (2) are presented in Section 3.

2. **Oscillations for equation (1.1)**

In this section we shall establish sufficient conditions in which all solutions of equation (1) are oscillatory.

For $t \geqslant t_0$, we let

$$A[t, t_0] = \int_{t_0}^t \int_{t_0}^s u a^{-1/\alpha}(u)\, du ds$$

and

$$m(t) = \int_t^\infty a^{-1/\alpha}(s)\,ds.$$

Theorem 1. *Let conditions (i)–(iv) and* (6) *hold and assume that there exists a function* $\xi(t) \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\xi'(t) \geqslant 0$ *and* $g(t) < \xi(t) < t$ *for* $t \geqslant t_0$. *If both the first order delay equations*

$$y'(t) + cq(t) f(A[g(t), t_1]) f\left(y^{1/\alpha}[g(t)]\right) = 0 \tag{7}$$

for every $t \geqslant t_1 \geqslant t_0$ *and any constant c,* $0 < c < 1$ *and*

$$z'(t) + \overline{c}q(t) f(g(t)) f\left(\int_{g(t)}^{\xi(t)} (s - g(t))\, a^{-1/\alpha}(s)\,ds\right) f\left(z^{1/\alpha}[\xi(t)]\right) = 0 \tag{8}$$

for any constant $\overline{c} > 0$, *are oscillatory,*

$$\int^\infty \left(\frac{1}{a(s)} \int_{t_0}^s q(u) f\left(g^2(u)\right) f(m[g(u)])\,du\right)^{1/\alpha} ds = \infty \tag{9}$$

and

$$\int^\infty \left(\frac{1}{a(s)} \int_{t_0}^s q(u) f\left(\frac{[\xi(u) - g(u)]^2}{2!}\right) f(m[\xi(u)])\,du\right)^{1/\alpha} ds = \infty \tag{10}$$

then the equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0 \geqslant 0$. Now, $\left(a(t)\left(x^{(3)}(t)\right)^\alpha\right)' \leqslant 0$ for $t \geqslant t_0$. There exists a $t_1 \geqslant t_0$ such that $x^{(i)}(t)$, $i = 1,2,3$ are of one sign for all $t \geqslant t_1$. There are eight possibilities to consider. The following four cases hold:

(I) $x^{(i)}(t) > 0$, $i = 1,2,3$ for $t \geqslant t_1$;
(II) $(-1)^{i+1} x^{(i)}(t) > 0$, $i = 1,2,3$ for $t \geqslant t_1$;
(III) $x^{(3)}(t) < 0$, $x^{(i)}(t) > 0$, $i = 1,2$ for $t \geqslant t_1$
and
(IV) $(-1)^i x^{(i)}(t) > 0, i = 1,2,3$ for $t \geqslant t_1$;

while the other four cases, namely

$x^{(i)}(t) > 0$, $i = 2,3$ and $x'(t) < 0$ for $t \geqslant t_1$;

$x^{(3)}(t) > 0$, $x^{(i)}(t) < 0$, $i = 1,2$ for $t \geqslant t_1$:

$x^{(i)}(t) < 0$, $i = 2,3$ and $x'(t) > 0$ for $t \geqslant t_1$;

and

$x^{(i)}(t) < 0$, $i = 1,2,3$ for $t \geqslant t_1$;

are obviously disregarded. Next, we consider:

Case (I). There exist a constant k, $0 < k < 1$ and a $t_2 \geqslant t_1$ such that

$$\begin{aligned} x''(t) &\geqslant ktx^{(3)}(t) \quad \text{for} \quad t \geqslant t_2 \\ &=: k\frac{t}{a^{1/\alpha}(t)} y^{1/\alpha}(t) \quad \text{for} \quad t \geqslant t_2 \end{aligned} \tag{11}$$

where $y(t) = a(t)\left(x^{(3)}(t)\right)^{\alpha}$, $t \geqslant t_2$.

Integrating (11) twice from t_2 to t we have

$$x(t) \geqslant kA[t, t_2]\, y^{1/\alpha}(t) \quad \text{for} \quad t \geqslant t_2. \tag{12}$$

There exists a $t_3 \geqslant t_2$ such that

$$x[g(t)] \geqslant kA[g(t), t_2]\, y^{1/\alpha}[g(t)] \quad \text{for} \quad t \geqslant t_3. \tag{13}$$

Using (13) and (3) in equation (1) we have

$$y'(t) + \overline{c}q(t) f(A[g(t), t_2]) f\left(y^{1/\alpha}[g(t)]\right) \leqslant 0 \quad \text{for} \quad t \geqslant t_3 \tag{14}$$

where $\overline{c} = f(k)$, $0 < \overline{c} < 1$.

Integrating (14) from $t \geqslant t_3$ to $u > t$ and letting $u \to \infty$ we find

$$y(t) \geqslant \overline{c} \int_t^{\infty} q(s) f(A[g(s), t_2]) f\left(y^{1/\alpha}[g(s)]\right) ds.$$

The function $y(t)$ is strictly decreasing on $[t_3, \infty)$. Hence by Theorem 1 in [9], we conclude that there exists a positive solution $y(t)$ of equation (7) with $\lim_{t\to\infty} y(t) = 0$, which is a contradiction.

Case (II). There exist a $t_2 \geqslant t_1$ and a constant k, $0 < k < 1$, such that

$$x(t) \geqslant ktx'(t) \quad \text{for} \quad t \geqslant t_2.$$

There exist a $t_3 \geqslant t_2$ such that

$$x[g(t)] \geqslant kg(t)\, x'[g(t)] \quad \text{for} \quad t \geqslant t_3. \tag{15}$$

Using (15) and (3) in equation (1), we have

$$\left(a(t)\left(y''(t)\right)^{\alpha}\right)' + \overline{\overline{c}}q(t) f(g(t)) f(y[g(t)]) \leqslant 0 \quad \text{for} \quad t \geqslant t_3 \tag{16}$$

where $y(t) = x'(t)$ for $t \geqslant t_3$ and $\overline{\overline{c}} = f(k)$. Clearly

$$y''(t) > 0,\ y'(t) < 0 \text{ and } y(t) > 0 \quad \text{for} \quad t \geqslant t_3.$$

By Taylor's series, for $t \geqslant s \geqslant t_3$ we find

$$y(s) \geqslant \int_s^t (u - s)\, y''(u)\, du.$$

Replacing s and t by $g(t)$ and $\xi(t)$ respectively we get

$$\begin{aligned} y[g(t)] &\geqslant \int_{g(t)}^{\xi(t)} (u-g(t))\, a^{-1/\alpha}(u) \left(a(u)(y''(u))^{\alpha}\right)^{1/\alpha} du \quad \text{for} \quad t \geqslant t_4 \geqslant t_3 \\ &=: \left(\int_{g(t)}^{\xi(t)} (u-g(t))\, a^{-1/\alpha}(u)\, du\right) z^{1/\alpha}[\xi(t)] \quad \text{for} \quad t \geqslant t_4 \qquad (17) \end{aligned}$$

where $z(t) = a(t)(y''(t))^{\alpha}$ for $t \geqslant t_4$.

Using (17) and (3) in (16), we have

$$z'(t) + \overline{\overline{c}} q(t) f(g(t)) f\left(\int_{g(t)}^{\xi(t)} (u-g(t))\, a^{-1/\alpha}(u)\, du\right) f\left(z^{1/\alpha}[\xi(t)]\right) \leqslant 0 \quad \text{for} \quad t \geqslant t_4. \qquad (18)$$

The rest of the proof is similar to the Case (I) and hence omitted.

Case (III). There exist a constant k, $0 < k < 1$, and a $t_2 \geqslant t_1$ such that

$$x'(t) \geqslant ktx''(t) \quad \text{for} \quad t \geqslant t_2.$$

Integrating this inequality from t_2 to t, there exist a constant $\overline{k}$, $0 < \overline{k} < 1$ and a $t_3 \geqslant t_2$ such that

$$x[g(t)] \geqslant \overline{k} g^2(t)\, x''[g(t)] \quad \text{for} \quad t \geqslant t_3. \qquad (19)$$

Using (19) and (3) in equation (1), one can easily find

$$\left(a(t)(w'(t))^{\alpha}\right)' + bq(t) f\left(g^2(t)\right) f(w[g(t)]) \leqslant 0 \quad \text{for} \quad t \geqslant t_3, \qquad (20)$$

where $b = f\left(\overline{k}\right)$ and $w(t) = x''(t)$, $t \geqslant t_3$. Clearly, $w(t) > 0$ and $w'(t) < 0$ for $t \geqslant t_3$.

For $s \geqslant t \geqslant t_3$, we have

$$a(s)(-w'(s))^{\alpha} \geqslant a(t)(-w'(t))^{\alpha},$$

or

$$-w'(s) \geqslant \frac{1}{a^{1/\alpha}(s)} \left(-a^{1/\alpha}(t)\, w'(t)\right). \qquad (21)$$

Integrating (21) from $t \geqslant t_3$ to $u > t$ and letting $u \to \infty$ we obtain

$$w(t) \geqslant \left(\int_t^{\infty} a^{-1/\alpha}(s)\, ds\right) \left(-a^{-1/\alpha}(t)\, w'(t)\right) \quad \text{for} \quad t \geqslant t_3. \qquad (22)$$

Combining (22) with the inequality

$$-a^{1/\alpha}(t)\, w'(t) \geqslant -a^{1/\alpha}(t_3)\, w'(t_3) \quad \text{for} \quad t \geqslant t_3,$$

there exist a $t_4 \geqslant t_3$ and a constant $\ell > 0$ such that

$$w[g(t)] \geqslant \ell m[g(t)] \quad \text{for} \quad t \geqslant t_4. \qquad (23)$$

Using (23) and (3) in (1), we get

$$-\left(a(t)(w'(t))^{\alpha}\right)' \geqslant bf(\ell)\, q(t) f\left(g^2(t)\right) f(m[g(t)]) \quad \text{for} \quad t \geqslant t_4. \qquad (24)$$

Integrating (24) from t_3 to t, we obtain

$$a(t_3)(w'(t_3))^\alpha - a(t)(w'(t))^\alpha \geqslant bf(\ell)\int_{t_3}^{t} q(s) f\left(g^2(s)\right) f(m[g(s)])\,ds$$

or

$$-a(t)(w'(t))^\alpha \geqslant bf(\ell)\int_{t_3}^{t} q(s) f\left(g^2(s)\right) f(m[g(s)])\,ds$$

or

$$-w'(t) \geqslant (bf(\ell))^{1/\alpha}\left(\frac{1}{a(t)}\int_{t_3}^{t} q(s) f\left(g^2(s)\right) f(m[g(s)])\,ds\right)^{1/\alpha} \quad \text{for} \quad t \geqslant t_3. \tag{25}$$

Integrating (25) from t_3 to t, we have

$$\begin{aligned} \infty &> w(t_3) \geqslant w(t_3) - w(t) \\ &\geqslant (bf(\ell))^{1/\alpha}\int_{t_3}^{t}\left(\frac{1}{a(s)}\int_{t_3}^{s} q(u) f\left(g^2(u)\right) f(m[g(u)])\,du\right)^{1/\alpha} ds \\ &\to \infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

which is a contradiction.

Case (IV). By Taylor's expansion, for $t \geqslant s \geqslant t_1$ we find

$$x(s) \geqslant \frac{(t-s)^2}{2!}x''(t).$$

Replacing s and t by $g(t)$ and $\xi(t)$ respectively, we obtain

$$\begin{aligned} x[g(t)] &\geqslant \frac{[\xi(t)-g(t)]^2}{2!}x''[\xi(t)] \quad \text{for} \quad t \geqslant t_2 \geqslant t_1 \\ &=: \frac{[\xi(t)-g(t)]^2}{2!}v[\xi(t)] \quad \text{for} \quad t \geqslant t_2 \end{aligned} \tag{26}$$

where $v(t) = x''(t)$ for $t \geqslant t_2$.

Using (26) and (3) in equation (1), we have

$$\left(a(t)(v'(t))^\alpha\right)' + q(t) f\left(\frac{[\xi(t)-g(t)]^2}{2!}\right) f(v[\xi(t)]) \leqslant 0 \quad \text{for} \quad t \geqslant t_2.$$

The rest of the proof is similar to the Case (III) above and hence omitted. □

We note that when condition (5) holds, Cases (III) and (IV) in the proof of Theorem 1 are disregarded. In fact, we have the following result.

Theorem 2. *Let conditions (i)–(iv) and* (5) *hold and assume that there exists a function* $\xi(t) \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\xi'(t) \geqslant 0$ *and* $g(t) < \xi(t) < t$ *for* $t \geqslant t_0$. *If both the first order delay equations* (7) *and* (8) *are oscillatory, then the equation* (1) *is oscillatory.*

By using known results for the oscillation of first order delay equations (see [8]), the following corollary is immediate.

Corollary 3. *Let conditions (i)–(iv),* (6), (9) *and* (10) *hold and assume that there exists a nondecreasing function* $\xi(t) \in C^1([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \geqslant t_0$. *Then equation* (1) *is oscillatory if one of the following conditions holds*

(I₁)

$$\frac{f\left(u^{1/\alpha}\right)}{u} \geqslant k > 0 \quad \textit{for} \quad u \neq 0 \quad \textit{where} \quad k \textit{ is a constant}$$

and

$$\liminf_{t\to\infty} \int_{\xi(t)}^{t} Q(s)\,ds > \frac{1}{ek},$$

where

$$Q(t) = \min\left\{cq(t)\,f(A[g(t),t_1]),\,\bar{c}q(t)\,f(g(t))\,f\left(\int_{g(t)}^{\xi(t)} (s-g(t))\,a^{-1/\alpha}(s)\,ds\right)\right\},$$

$\bar{c} > 0$ *is any constant and* $0 < c < 1$.

(I₂)

$$\int_{\pm 0} f^{-1}\left(u^{1/\alpha}\right)du < \infty \quad \textit{and} \quad \int^{\infty} Q(s)\,ds = \infty.$$

Remark 4. The technique of the proof of Theorem 1 may allow to obtain criteria for equations similar to (1) on time scale, for example the dynamic equation

$$\left(a\left(x^{\Delta\Delta\Delta}\right)^{\alpha}\right)^{\Delta} + q(t)\,x^{\beta}(g(t)) = 0,$$

where β is the ratio of two positive odd integers. Also, it will be of interest to consider the forced equation

$$\left(a\left(x^{(3)}\right)^{\alpha}\right)' + q(t)\,f(x[g(t)]) = e(t),$$

where $e(t) \in C([t_0,\infty),\mathbb{R})$.

For bounded solutions of equation (1) one can easily prove the following result.

Theorem 5. *Let conditions (i)–(iv),* (6) *and* (10) *hold and assume that there exists a nondecreasing function* $\xi(t) \in C([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \geqslant t_0$. *If the delay first order equation* (8) *is oscillatory, then all bounded solutions of equation* (1) *are oscillatory.*

3. Oscillations for equation (1.2)

In this section we are interested in obtaining criteria for the oscillation of all solutions of equation (2).

Theorem 6. *Let conditions (i)–(iv) and* (6) *hold and assume that there exist nondecreasing functions* $\xi(t)$ *and* $\zeta(t) \in C^1([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *and* $\sigma(t) > \zeta(t) > t$ *for* $t \geqslant t_0$. *If the advanced first order equation*

$$y'(t) - p(t)\,h\left(\int_{\zeta(t)}^{\sigma(t)} \frac{[\sigma(t)-s]^2}{2!} a^{-1/\alpha}(s)\,ds\right) h\left(y^{1/\alpha}[\zeta(t)]\right) = 0 \qquad (27)$$

both delay first order equations

$$z'(t) + cq(t) f\left(g^2(t)\right) f\left(\frac{\xi(t) - g(t)}{a^{1/\alpha}[\xi(t)]}\right) f\left(z^{1/\alpha}[\xi(t)]\right) = 0 \qquad (28)$$

for every constant c*,* $0 < c < 1$ *and*

$$w'(t) + q(t) f\left(\int_{g(t)}^{\xi(t)} \frac{[s - g(t)]^2}{2!} a^{-1/\alpha}(s)\, ds\right) f\left(w^{1/\alpha}[\xi(t)]\right) = 0 \qquad (29)$$

are oscillatory, and

$$\int_{t_0}^{\infty} \left(\frac{1}{a(s)} \int_{t_0}^{s} q(u) f(g(u)) f(\xi(u) - g(u)) f(m[\xi(u)])\, du\right)^{1/\alpha} ds = \infty \qquad (30)$$

then equation (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (2), say $x(t) > 0$, $x[g(t)] > 0$ and $x[\sigma(t)] > 0$ for $t \geqslant t_0 \geqslant 0$. Now, $\left(a(t)\left(x^{(3)}(t)\right)^{\alpha}\right)' \geqslant 0$ for $t \geqslant t_0$. There exists a $t_1 \geqslant t_0$ such that $x^{(i)}(t)$, $i = 1, 2, 3$ are of one sign for all $t \geqslant t_1$. There are eight possibilities to consider. The following four cases hold:

(I) $x^{(i)}(t) > 0$, $i = 1, 2, 3$ for $t \geqslant t_1$;
(II) $x^{(3)}(t) < 0$, $x^{(i)}(t) > 0$, $i = 1, 2$ for $t \geqslant t_1$;
(III) $(-1)^i x^{(i)}(t) > 0$, $i = 1, 2, 3$ for $t \geqslant t_1$
and
(IV) $(-1)^{i+1} x^{(i)}(t) > 0$, $i = 1, 2, 3$ for $t \geqslant t_1$.

It is easy to see that the following four cases are obviously disregarded:

$x^{(i)}(t) > 0$, $i = 2, 3$ and $x'(t) < 0$ for $t \geqslant t_1$;

$x^{(3)}(t) > 0$, and $x^{(i)}(t) < 0$, $i = 1, 2$ for $t \geqslant t_1$;

$x^{(i)}(t) < 0$, $i = 2, 3$ and $x'(t) > 0$ for $t \geqslant t_1$;

and

$x^{(i)}(t) < 0$, $i = 1, 2, 3$ for $t \geqslant t_1$.

Now, we consider:

Case (I). By the Taylor's expansion, for $t \geqslant s \geqslant t_1$ we get

$$\begin{aligned} x(t) &\geqslant \int_s^t \frac{(t-u)^2}{2!} x^{(3)}(u)\, du \\ &=: \int_s^t \frac{(t-u)^2}{2!} a^{-1/\alpha}(u) \left(a(u)\left(x^{(3)}(u)\right)^{\alpha}\right)^{1/\alpha} du \\ &\geqslant \left(\int_s^t \frac{(t-u)^2}{2!} a^{-1/\alpha}(u)\, du\right) y^{1/\alpha}(s) \end{aligned} \qquad (31)$$

where $y(t) = a(t)\left(x^{(3)}(t)\right)^{\alpha}$ for $t \geqslant t_1$. Replacing t and s in (31) by $\sigma(t)$ and $\zeta(t)$ respectively, we have

$$x[\sigma(t)] \geqslant \left(\int_{\zeta(t)}^{\sigma(t)} \frac{[\sigma(t)-u]^2}{2!} a^{-1/\alpha}(u)\,du\right) y^{1/\alpha}(\zeta(t)) \quad \text{for} \quad t \geqslant t_2 \geqslant t_1. \tag{32}$$

Using (32) and (4) in equation (2) we obtain

$$y'(t) \geqslant p(t)\,h\left(\int_{\zeta(t)}^{\sigma(t)} \frac{[\sigma(t)-u]^2}{2!} a^{-1/\alpha}(u)\,du\right) h\left(y^{1/\alpha}[\zeta(t)]\right) \quad \text{for} \quad t \geqslant t_2.$$

By known results, see [2], [3] and [9], we arrive at the desired contradiction.

Case (II). There exist a constant k, $0 < k < 1$ and a $t_2 \geqslant t_1$ such that

$$x'(t) \geqslant k t x''(t) \quad \text{for} \quad t \geqslant t_2.$$

Integrating the above inequality from t_2 to t we have

$$x(t) \geqslant \frac{k}{2}\left(t^2 - t_2^2\right) x''(t) \quad \text{for} \quad t \geqslant t_2.$$

Now, there is a constant $\overline{c}$, $0 < \overline{c} < 1$ and a $t_3 \geqslant t_2$ such that

$$x[g(t)] \geqslant \overline{c} g^2(t)\, y[g(t)] \quad \text{for} \quad t \geqslant t_3 \tag{33}$$

where $y(t) = x''(t)$ for $t \geqslant t_3$. Using (33) and (3) in equation (2) we have

$$\left(a(t)(y'(t))^{\alpha}\right)' \geqslant f(\overline{c})\, q(t)\, f\left(g^2(t)\right) f(y[g(t)]) \quad \text{for} \quad t \geqslant t_3. \tag{34}$$

Clearly $y(t) > 0$ and $y'(t) < 0$ for $t \geqslant t_3$. Thus there exists a $t_4 \geqslant t_3$ such that

$$y[g(t)] \geqslant (\xi(t) - g(t))\left(-y'[\xi(t)]\right) \quad \text{for} \quad t \geqslant t_4,$$

or

$$y[g(t)] \geqslant (\xi(t) - g(t))\, a^{-1/\alpha}[\xi(t)]\left(z^{1/\alpha}[\xi(t)]\right) \quad \text{for} \quad t \geqslant t_4, \tag{35}$$

where $z(t) = -a(t)(y'(t))^{\alpha} > 0$ for $t \geqslant t_4$.

Using (35) and (3) in (34) we get

$$z'(t) + f(\overline{c})\, q(t)\, f\left(g^2(t)\right) f\left(\frac{\xi(t) - g(t)}{a^{1/\alpha}[\xi(t)]}\right) f\left(z^{1/\alpha}[\xi(t)]\right) \leqslant 0 \quad \text{for} \quad t \geqslant t_4.$$

The rest of the proof is similar to the Theorem 1-Case (I) and hence omitted.

Case (III). By Taylor's expansion, one can easily see that there exists a $t_2 \geqslant t_1$ such that

$$x[g(t)] \geqslant \left(\int_{g(t)}^{\xi(t)} \frac{[s-g(t)]^2}{2!} a^{-1/\alpha}(s)\,ds\right)\left(w^{1/\alpha}[\xi(t)]\right) \quad \text{for} \quad t \geqslant t_2, \tag{36}$$

where

$$w(t) = -a(t)\left(x^{(3)}(t)\right)^{\alpha} > 0 \quad \text{for} \quad t \geqslant t_2.$$

Now, using (36) and (3) in equation (2) we find

$$w'(t) + q(t) f\left(\int_{g(t)}^{\xi(t)} \frac{[s-g(t)]^2}{2!} a^{-1/\alpha}(s)\, ds\right) f\left(w^{1/\alpha}[\xi(t)]\right) \leqslant 0 \quad \text{for} \quad t \geqslant t_2.$$

The rest of the proof is similar to the Theorem 1-Case (I) and hence omitted.

Case (IV). There exist a constant k, $0 < k < 1$ and a $t_2 \geqslant t_1$ such that

$$x[g(t)] \geqslant k g(t) y[g(t)] \quad \text{for} \quad t \geqslant t_2 \tag{37}$$

where $y(t) = x'(t) > 0$ for $t \geqslant t_2$. Using (37) and (3) in equation (2) we get

$$\left(a(t)(y''(t))^{\alpha}\right)' \geqslant f(k) q(t) f(g(t)) f(y[g(t)]) \quad \text{for} \quad t \geqslant t_2. \tag{38}$$

Clearly $y(t) > 0$, $y'(t) < 0$ and $y''(t) > 0$ for $t \geqslant t_2$. Thus, there exists a $t_3 \geqslant t_2$ such that

$$y[g(t)] \geqslant (\xi(t) - g(t)) z[\xi(t)] \quad \text{for} \quad t \geqslant t_3, \tag{39}$$

where $z(t) = -y'(t) > 0$ for $t \geqslant t_3$. Using (39) and (3) in equation (38) we have

$$\left(a(t)(z'(t))^{\alpha}\right)' + f(k) q(t) f(g(t)) f(\xi(t) - g(t)) f(z[\xi(t)]) \leqslant 0.$$

The rest of the proof is exactly the same as that of Theorem 1-Case (III) and hence omitted. This completes the proof. □

From the proof of the above theorem, one can easily obtain the following result when condition (5) holds.

Theorem 7. *Let conditions (i)–(iv) and* (5) *hold and assume that there exist nondecreasing functions $\xi(t)$ and $\zeta(t) \in C^1([t_0,\infty),\mathbb{R})$ such that $g(t) < \xi(t) < t$ and $\sigma(t) > \zeta(t) > t$ for $t \geqslant t_0$. If the advanced first order equation* (27) *and both the delay first order equations* (28) *and* (29) *are oscillatory, then equation* (2) *is oscillatory.*

By applying well known criteria for the oscillation of first order equations, the following corollary is immediate.

Corollary 8. *Let conditions (i)–(iv),* (6) *and* (30) *hold and assume that there exist nondecreasing functions $\xi(t)$ and $\zeta(t) \in C^1([t_0,\infty),\mathbb{R})$ such that $g(t) < \xi(t) < t$ and $\sigma(t) > \zeta(t) > t$ for $t \geqslant t_0$. Then equation* (2) *is oscillatory if one of the following conditions holds*

(II_1)

$$\frac{h\left(u^{1/\alpha}\right)}{u} \geqslant k > 0 \quad \textit{for} \quad u \neq 0 \quad \textit{and} \quad k \textit{ is a constant},$$

$$\liminf_{t\to\infty} \int_t^{\zeta(t)} p(s) h\left(\int_{g(t)}^{\xi(t)} \frac{[\sigma(s)-v]^2}{2!} a^{-1/\alpha}(v)\, dv\right) ds > \frac{1}{ek},$$

$$\frac{f\left(u^{1/\alpha}\right)}{u} \geqslant k_1 > 0 \quad \textit{for} \quad u \neq 0 \quad \textit{and} \quad k_1 \textit{ is a constant},$$

and

$$\liminf_{t\to\infty} \int_{\xi(t)}^{t} \tilde{Q}(s)\,ds > \frac{1}{ek_1}$$

where

$$\tilde{Q}(t) = \min\left\{cq(t)\,f\left(g^2(t)\right) f\left(\frac{\xi(t)-g(t)}{a^{1/\alpha}[\xi(t)]}\right), q(t)\,f\left(\int_{g(t)}^{\xi(t)} \frac{[s-g(t)]^2}{2!a^{1/\alpha}(s)}ds\right)\right\}$$
$$\text{for} \quad t \geqslant t_0. \tag{40}$$

(I₂)

$$\int^{\pm\infty} h^{-1}\left(u^{1/\alpha}\right) du < \infty.$$

and

$$\int^{\infty} p(s)\,h\left(\int_{g(s)}^{\xi(t)} \frac{[\sigma(s)-v]^2}{2!} a^{-1/\alpha}(v)\,dv\right) ds = \infty,$$

$$\int_{\pm 0} f^{-1}\left(u^{1/\alpha}\right) du < \infty \quad \text{and} \quad \int^{\infty} \tilde{Q}(s)\,ds = \infty.$$

We note that many other criteria similar to above can be obtained. The details are left to the reader.

When we are concerned with bounded solutions of equation (2), the term $p(t)h(x[\sigma(t)])$ may be disregarded. In this case we have

Theorem 9. *Let conditions (i)–(iv),* (6) *and* (30) *hold and assume that there exist a nondecreasing functions* $\xi(t) \in C([t_0,\infty),\mathbb{R})$ *such that* $g(t) < \xi(t) < t$ *for* $t \geqslant t_0$. *If the first order delay equation* (29) *is oscillatory, all bounded solutions of equation* (2) *are oscillatory.*

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