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EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH SEPARATED BOUNDARY CONDITIONS

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ABSTRACT. This paper studies existence and uniqueness results in a Banach space for a two-point boundary value problem involving a nonlinear fractional differential equation given by

$$
\begin{gathered}
{}^cD^q x(t) = f(t, x(t)), \qquad 0 < t < 1, \quad 1 < q \le 2, \\
\alpha x(0) + \beta x'(0) = \gamma_1, \qquad \alpha x(1) + \beta x'(1) = \gamma_2.
\end{gathered}
$$

Our results are based on contraction mapping principle and Krasnoselskii's fixed point theorem.

Keywords: Nonlinear fractional differential equations, separated boundary conditions, existence, fixed point theorem

AMS Subject Classifications: 34A34, 34B15

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [1–11] and the references therein. In [1, 11], the authors have discussed the existence of positive solutions for boundary value problem of nonlinear fractional differential equations. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages. The recent surge in developing the theory of fractional differential equations has motivated the present work.

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In this paper, we consider the following boundary value problem for a nonlinear fractional differential equation with separated boundary conditions

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ 0 < t < 1, \ 1 < q \le 2, \\ \alpha x(0) + \beta x'(0) = \gamma_1, \qquad \alpha x(1) + \beta x'(1) = \gamma_2, \end{cases} \tag{1.1}$$

where ${}^cD$ is the Caputo's fractional derivative, $f : [0,1] \times X \to X$ and $\alpha > 0, \ \beta \ge 0, \ \gamma_{1,2}$ are real numbers. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathrm{C} = \mathrm{C}([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|_{\mathrm{C}}$.

2. **Preliminaries**

Let us recall some basic definitions.

Definition 2.1. For a function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, \ n-1 < q < n, \quad n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $f(t)$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Remark 2.1. The definition of Riemann-Liouville fractional derivative, which did certainly play an important role in the development of theory of fractional derivatives and integrals, could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. The same applies to the boundary value problems of fractional differential equations. It was Caputo's definition of fractional derivative which solved this problem. In fact, the Caputo derivative becomes the conventional $n-$th derivative of the function as $q \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero. For more details, see [10].

Lemma 2.1. ([11]) For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$. In view of Lemma 2.1, it follows that

$$I^q \, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$.

Now, we state a known result due to Krasnoselskii which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.1. Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$ (ii) A is compact and continuous (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.2. For a given $\zeta \in C[0,1]$, the unique solution of the boundary value problem

$$\begin{cases} {}^cD^q x(t) = \zeta(t), & 0 < t < 1, \ 1 < q \le 2, \\ \alpha x(0) + \beta x'(0) = \gamma_1, & \alpha x(1) + \beta x'(1) = \gamma_2, \end{cases} \tag{2.2}$$

is given by

$$x(t) = \int_0^1 G(t,s)\zeta(s)ds + \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2],$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{\alpha(t-s)^{q-1} + (\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, & s \le t, \\ \frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, & t \le s. \end{cases} \tag{2.3}$$

Proof. Using (2.1), for some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q \zeta(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\zeta(s)ds - c_0 - c_1 t.$$

In view of the relations ${}^cD^q \, I^q x(t) = x(t)$ and $I^q \, I^p x(t) = I^{q+p} x(t)$ for $q,\ p > 0$, $x \in L(0,1)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\zeta(s)ds - c_1.$$

Applying the boundary conditions for (2.2), we find that

$$c_0 = \frac{1}{\alpha^2}[\beta\gamma_2 - (\beta+\alpha)\gamma_1] - \frac{\beta}{\alpha\Gamma(q)}\int_0^1 (1-s)^{q-1}\zeta(s)ds - \frac{\beta^2}{\alpha^2\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\zeta(s)ds,$$

$$c_1 = \frac{1}{\alpha}(\gamma_1 - \gamma_2) + \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}\zeta(s)ds + \frac{\beta}{\alpha\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\zeta(s)ds.$$

Thus, the unique solution of (2.2)–(2.3) is

$$\begin{aligned}
x(t) &= \int_0^t [\frac{\alpha(t-s)^{q-1}+(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}]\zeta(s)ds \\
&\quad + \int_t^1 [\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}]\zeta(s)ds \\
&\quad + \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2] \\
&= \int_0^1 G(t,s)\zeta(s)ds + \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2],
\end{aligned}$$

where $G(t,s)$ is given by (2.3). This completes the proof.

3. **Main results**

Theorem 3.1. Let $f : [0,1]\times X \to X$ be a jointly continuous function mapping bounded subsets of $[0,1]\times X$ into relatively compact subsets of X, and

$$\|f(t,x)-f(t,y)\| \le L\|x-y\|, \forall t\in[0,1], \quad x,y\in X.$$

Then the boundary value problem (1.1) has a unique solution provided

$$L \le \frac{1}{2}[\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]^{-1}.$$

Proof. Define $\mathcal{F} : \mathrm{C} \to \mathrm{C}$ by

$$\begin{aligned}
(\mathcal{F}x)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))ds \\
&\quad + \int_0^1 [\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}]f(s,x(s))ds \\
&\quad + \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2], \quad t\in[0,1].
\end{aligned}$$

Setting $\sup_{t\in[0,1]}\|f(t,0)\| = M$ and Choosing

$$r \ge 2[M(\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}) + \frac{\alpha+\beta}{\alpha^2}(\gamma_1+\gamma_2)],$$

we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{x\in\mathrm{C} : \|x\|\le r\}$. For $x\in B_r$, we have

$$\begin{aligned}
\|(\mathcal{F}x)(t)\| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s))\|ds \\
&\quad + \int_0^1 |\beta-\alpha t|[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}]\|f(s,x(s))\|ds \\
&\quad + \frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|) \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\|f(s,x(s))-f(s,0)\| + \|f(s,0)\|]ds
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^1 |\beta-\alpha t|[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}] \\
\times\ & [\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|]ds+\frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|) \\
\leq\ & (Lr+M)[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
& +\int_0^1 |\beta-\alpha t|(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)})ds] \\
& +\frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|) \\
=\ & (Lr+M)[\frac{t^q}{\Gamma(q+1)}+|\beta-\alpha t|(\frac{1}{\alpha\Gamma(q+1)}+\frac{\beta}{\alpha^2\Gamma(q)})] \\
& +\frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|) \\
\leq\ & L[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]r \\
& +M[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]+\frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|)\leq r.
\end{aligned}
$$

Now, for $x, y \in \mathrm{C}$ and for each $t \in [0,T]$, we obtain

$$
\begin{aligned}
& \|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\| \\
& \quad \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s))-f(s,y(s))\|ds \\
& \qquad +\int_0^1 |\beta-\alpha t|[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}]\|f(s,x(s))-f(s,y(s))\|ds \\
& \leq L\|x-y\|_{\mathrm{C}}[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
& \qquad +\int_0^1 |\beta-\alpha t|(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)})ds \\
& \leq L\|x-y\|_{\mathrm{C}}[\frac{t^q}{\Gamma(q+1)}+|\beta-\alpha t|(\frac{1}{\alpha\Gamma(q+1)}+\frac{\beta}{\alpha^2\Gamma(q)})] \\
& \leq L[\frac{1}{\alpha\Gamma(q+1)}(2\alpha+\beta)+\frac{\beta^2+\alpha\beta)}{\alpha^2\Gamma(q)}]\|x-y\|_{\mathrm{C}} \\
& \leq \Lambda_{\alpha,\beta,q,L}\|x-y\|_{\mathrm{C}},
\end{aligned}
$$

where

$$\Lambda_{\alpha,\beta,q,L}=L[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}],$$

which depends only on the parameters involved in the problem. As $\Lambda_{\alpha,\beta,\gamma,L,T,q}<1$, therefore $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle.

Theorem 3.2. Assume that $f : [0,1] \times X \to X$ is a jointly continuous function and maps bounded subsets of $[0,1] \times X$ into relatively compact subsets of X. Further

($\mathbf{A_1}$) $\|f(t,x) - f(t,y)\| \le L\|x-y\|, \forall t \in [0,1],\ x,y \in X;$
($\mathbf{A_2}$) $\|f(t,x)\| \le \mu(t),\ \forall (t,x) \in [0,1] \times X,$ and $\mu \in L^1([0,1], R^+)$.

If $L(\frac{\alpha+\beta}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}) < 1$, then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Let us fix

$$r \ge \|\mu\|_{L^1}[\frac{2\alpha+\beta}{\alpha\Gamma(q)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q-1)}] + \frac{\alpha+\beta}{\alpha^2}(|\gamma_1| + |\gamma_2|),$$

and consider $B_r = \{x \in \mathrm{C} : \|x\| \le r\}$. We define the operators Φ and Ψ on B_r as

$$(\Phi x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds,$$

$$\begin{aligned}(\Psi x)(t) \quad = \quad & \int_0^1 \left[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\right] f(s,x(s))ds \\ & + \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2].\end{aligned}$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Psi y\| \le \|\mu\|_{L^1}[\frac{2\alpha+\beta}{\alpha\Gamma(q)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q-1)}] + \frac{\alpha+\beta}{\alpha^2}(|\gamma_1| + |\gamma_2|) \le r.$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (A_1) that Ψ is a contraction mapping for

$$L(\frac{\alpha+\beta}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}) < 1.$$

Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\| \le (\frac{\|\mu\|_{L^1}}{\Gamma(q)}.$$

Now we prove the compactness of the operator Φ. Setting $\Omega = [0,1] \times B_r$, we define $\sup_{(t,x)\in\Omega} \|f(t,x)\| = f_{\max}$, and consequently we have

$$\begin{aligned}\|(\Phi x)(t_1) - (\Phi x)(t_2)\| = \| & \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] f(s,x(s))ds \\ & + \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s,x(s))ds\| \le \frac{f_{\max}}{\Gamma(q+1)} |2(t_2-t_1)^q + t_1^q - t_2^q|,\end{aligned}$$

which is independent of x. So Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 2.1 are satisfied and the conclusion of Theorem 2.1 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$.

Example. Consider the following boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{\cos t}{(t+5)^2}\frac{|x|}{1+|x|}, \ t \in [0,1], \\ x(0) + x'(0) = 0, \qquad x(1) + x'(1) = 0. \end{cases} \tag{3.1}$$

Here, $f(t, x(t)) = \frac{\cos t}{(t+5)^2}\frac{|x|}{1+|x|}$, $\alpha = 1$, $\beta = 1$, $\gamma_1 = 0 = \gamma_2$. As $\|f(t,x) - f(t,y)\| \leq \frac{1}{25}\|x-y\|$, therefore, (A_1) is satisfied with $L = \frac{1}{25}$. Further,

$$2L\left(\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\right) = \frac{16}{25\sqrt{\pi}} < 1.$$

Thus, by Theorem 3.1, the boundary value problem (3.1) has a unique solution on $[0,1]$.

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