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THE SIZE OF THE LARGEST FLUCTUATIONS IN A MARKET MODEL WITH MARKOVIAN SWITCHING

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ABSTRACT. This paper considers the size of the large fluctuations of a stochastic differential equation with Markovian switching. We concentrate on processes which obey the Law of the Iterated Logarithm, or obey upper and lower iterated logarithm growth bounds on their almost sure partial maxima. The results are applied to financial market models which are subject to random regime shifts. We prove that the security exhibits the same long–run growth properties and deviations from the trend rate of growth as conventional geometric Brownian motion, and also that the returns, which are non–Gaussian, still exhibit the same growth rate in their almost sure large deviations as stationary continuous–time Gaussian processes.

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1. Introduction

We study the stochastic differential equation with Markovian switching

$$dX(t) = f(X(t), Y(t), t)\,dt + g(X(t), Y(t), t)\,dB(t) \tag{1.1}$$

where $g(x, y, t)$ and $xf(x, y, t)$ are uniformly bounded above and below in (x, y, t), and Y is an irreducible continuous–time Markov chain with finite state space $\mathbb{S}$ independent of the Brownian motion B. If the lower bound on $xf(x, y, t)$ is sufficiently

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large, we show that X obeys upper and lower laws of the iterated logarithm, in the sense that

$$\sqrt{K_2} \le \limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \le \sqrt{K_1}, \quad \text{a.s.}$$

where $g^2(x,y,t) \in [K_2, K_1]$. In the case when g additionally obeys $g(x,y,t) = \gamma(y)$, it can be shown that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = \sigma_*, \quad \text{a.s.}$$

where $\sigma_*^2 = \sum_{j\in\mathbb{S}} \gamma^2(j)\pi_j$ and $\pi = (\pi_j)_{j\in\mathbb{S}}$ is the stationary distribution of Y. The proofs rely on time change and comparison arguments, constructing upper and lower bounds on $|X|$ which, under appropriate changes of time and scale, are recurrent and stationary processes whose dynamics are not determined by Y. The large deviations of these processes are determined by means of a classical theorem of Motoo [20].

These large deviation results are then applied to a security price model, where the security price S obeys

$$dS(t) = \mu S(t)\,dt + S(t)\,dX(t), \quad t \ge 0, \tag{1.2}$$

and X obeys (1.1). The problem is motivated by the observations from financial market econometrics that security prices often move from bearish to bullish (or other) regimes. These regimes are modelled by the presence of the Markov process Y. One of the seminal contributions on the econometric analysis of financial times series subject to these regime shifts is [10], and a recent monograph covering this topic, amongst others, is [7].

The classical Geometric Brownian motion model of stock evolution assumes that the market is informationally efficient, following forms of the Efficient Market Hypothesis (EMH). A classical statement and discussion about the EMH and its ramifications may be found in e.g., Fama [4] or the seminal volume edited by Cootner [3]. However, in recent times, econometric evidence suggesting that financial markets might be inefficient has accumulated (see e.g., [15]). The model (1.1) is an inefficient market model, since the increments of the cumulative returns process $\mu t + X(t)$ are not independent. However, the fact that $xf(x,y,t)$ is uniformly bounded means that the process X does not depart too much (in some sense) from Brownian motion, thereby placing limits on the inefficiency of the market, particularly when the price departs too far from its trend rate of growth. Therefore, the assumption that $xf(x,y,t)$ be bounded can be seen as hypothesising that the market is not "too inefficient". Finally, the assumption that the movement between regimes is not influenced by the stock price or returns, is accommodated by presuming that Y and the driving Brownian motion B are independent.

Despite the presence of regime shifts and inefficiency, we can still deduce that the new market model enjoys some of the properties of standard Geometric Brownian motion models. Having established the existence of a trend rate of growth in the price, we use results about the solution of (1.1) to show that the large deviations of the price from this trend rate of growth obey a law of the iterated logarithm, just as in standard models. Finally, although the returns are non–Gaussian, we can nevertheless show that the partial maxima of the returns have the same almost sure rate of growth as those of a stationary Gaussian process.

Recently, there has been increasing attention devoted to hybrid systems, in which continuous dynamics are intertwined with discrete events. One of the distinct features of such systems is that the underlying dynamics are subject to changes with respect to certain configurations. A convenient way of modelling these dynamics is to use continuous–time Markov chains to delineate many practical systems where they may experience abrupt changes in their structure and parameters. Such hybrid systems have been considered for the modelling of electric power systems by Willsky and Levy [28] as well as for the control of a solar thermal central receiver by Sworder and Rogers [24]. Athans [2] suggested to use hybrid systems control-related issues in Battle Management Command, Control and Communications (BM/C^3) systems. Sethi and Zhang used Markovian structure to describe hierarchical control of manufacturing systems [23]. Yin and Zhang examined probabilistic structure and developed a two-time-scale approach for control of hybrid dynamic systems [25]. Optimal control of switching diffusions and applications to manufacturing systems were studied in Ghosh, Arapostathis, and Marcus [8] and [9]. In addition, Markovian hybrid systems have also been used in emerging applications in financial engineering [26, 27, 29]. For a detailed treatment of the hybrid stochastic differential equations we refer the reader to the new book [18].

The paper is organised as follows. Notation and detailed formulation of the equations being studied are presented in Section 2. The main results on iterated logarithm growth rates for the solution of (1.1) are given in Section 3. In Section 4, these results are applied to a stock price model. The proofs of all results are postponed to the final two sections: proofs of results from Section 3 are given in Section 5, while those from Section 4 are given in Section 6.

2. **Mathematical Preliminaries**

Throughout the paper we use $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$ to denote the complete filtered probability space. The set of non-negative real numbers is denoted by $\mathbb{R}^+$. Let $\mathcal{L}^1[a,b]$ be the family of Borel measurable functions $h : [a,b] \to \mathbb{R}$ such that $\int_a^b |h(x)|\,dx < \infty$. The abbreviation a.s. stands for *almost surely.* If a stochastic process with state space $\mathbb{R}$ is the solution of an autonomous stochastic differential equation with drift coefficient $f : \mathbb{R} \to \mathbb{R}$ and non-zero diffusion coefficient $g : \mathbb{R} \to \mathbb{R}$, then the scale function p and speed measure m of this process are defined by

$$p(x) = \int_a^x e^{-2\int_a^y \frac{f(u)}{g^2(u)}\,du}\,dy, \qquad a \in \mathbb{R}, \tag{2.1}$$

$$m(dx) = \frac{2}{g^2(x)}.\frac{1}{p'(x)}\,dx, \qquad x > 0. \tag{2.2}$$

Motoo's theorem [20] is an important tool for determining the largest deviations for stationary solutions of scalar autonomous stochastic differential equations. We state it here for future use.

Theorem 2.1. *Let $f : (l,\infty) \to \mathbb{R}$ and $g : (l,\infty) \to \mathbb{R}$ and X be the unique continuous adapted process satisfying*

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t), \qquad t \geq 0.$$

Suppose that X is a recurrent process on (l,∞) with the scale function p and speed measure m of X, defined by (2.1) and (2.2) respectively, satisfying

$$p(l) = -\infty, \quad p(\infty) = \infty \quad \text{and} \quad m(l,\infty) < \infty. \tag{2.3}$$

If $h : (0,\infty) \to (0,\infty)$ is an increasing function with $h(t) \to \infty$ as $t \to \infty$, then

$$\mathbb{P}\left[\limsup_{t\to\infty} \frac{X(t)}{h(t)} \geq 1\right] = 1 \quad \text{or} \quad 0$$

depending on whether

$$\int_c^\infty \frac{1}{p(h(t))}\,dt = \infty \quad \text{or} \quad \int_c^\infty \frac{1}{p(h(t))}\,dt < \infty$$

for some $c \in \mathbb{R}$.

Before going further we clarify on our terminology; in particular when we refer to *stationarity*. Econometric studies of financial markets and asset prices often concentrate on detecting underlying stationary processes which may drive the asset prices, such as stock volatility or returns. The *stationarity* of such a process (say U) should not be confused with the a.s. *point stability* of U. If we suppose that the process $U = \{U(t) : t \geq 0\}$ is the solution of a stochastic differential equation defined on $t \geq 0$, then U would moreover be a stationary solution of the equation if

$$\begin{aligned}\mathbb{P}[U(t+t_1) \leq x_1, U(t+t_2) \leq x_2, \ldots, U(t+t_n) \leq x_n] \\ = \mathbb{P}[U(t_1) \leq x_1, U(t_2) \leq x_2, \ldots, U(t_n) \leq x_n] \\ \forall\, t \geq 0, \forall\, n \in \mathbb{N}, \forall\, t_j \geq 0, x_j \in \mathbb{R}, \quad j = 1\ldots, n,\end{aligned} \tag{2.4}$$

and U would be an *asymptotically stationary* solution if (2.4) holds with the left–hand side replaced by the limit as $t \to \infty$.

A very special case of a stationary solution is a point equilibrium x^* where $U(0) = x^*$ implies $U(t) = x^*$ for all $t \geq 0$ a.s., in which case the stationary distribution of the process, starting from x^*, is a Dirac δ–function concentrated at x^*. Such an equilibrium is said to be *a.s. (globally) asymptotically stable* if

$$\lim_{t\to\infty} U(t) = x^*, \quad \text{a.s.} \tag{2.5}$$

for all initial conditions $U(0)$. In finance, and in this paper in particular, it is usual to be concerned with stationary (or asymptotically stationary) processes rather than with stable (or asymptotically stable) equilibria, and consequently no results about stability in the sense of (2.5) appear in the paper. In fact, for the class of equations studied we do *not* establish stationarity (or asymptotic stationarity) in the increments of the returns. Nonetheless, we show that they enjoy recurrence and large deviation properties like those of the stationary increments of the returns resulting from the standard geometric Brownian motion market model. We use this as a reference against which to compare our model.

We now state some known results on the distribution of standard Gaussian random variables that will be useful in the sequel. Let Φ be the distribution of a standard normal (i.e., $\mathcal{N}(0,1)$) random variable N, so that $\Phi(x) := \mathbb{P}[N \le x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du$, $x \in \mathbb{R}$. Mill's estimate gives us that

$$\frac{1}{\sqrt{2\pi}}\frac{x}{x^2+1}e^{-\frac{x^2}{2}} \le 1-\Phi(x) \le \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}, \quad x>0. \tag{2.6}$$

In this paper, we consider the asymptotic behaviour of a scalar non–autonomous stochastic differential equation with Markovian switching. Let Y be a continuous–time Markov chain with state space $\mathbb{S}$, and let B be a standard one–dimensional Brownian motion independent of Y. To make our theory more understandable, we assume the state space of the Markov chain is finite, say $\mathbb{S} = \{1,2,\ldots,N\}$ and the Markov chain has its generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{Y(t+\Delta)=j|Y(t)=i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i\ne j, \\ 1+\gamma_{ii}\Delta + o(\Delta) & \text{if } i=j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i}\gamma_{ij}$. It is known (see e.g. [1]) that almost every sample path of $Y(t)$ is a right-continuous step function with a finite number of jumps in any finite subinterval of $[0,\infty)$. As a standing hypothesis we assume in this paper that the Markov chain is *irreducible*. This is equivalent to the condition that for any $i, j \in \mathbb{S}$, one can find finite numbers $i_1, i_2, \ldots, i_k \in \mathbb{S}$ such that $\gamma_{i,i_1}\gamma_{i_1,i_2}\cdots\gamma_{i_k,j} > 0$. Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\text{rank}(\Gamma) = N-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving the following linear equation

$$\pi\Gamma = 0 \quad \text{subject to } \sum_{j=1}^{N} \pi_j = 1 \text{ and } \pi_j > 0 \quad \forall j \in \mathbb{S}. \tag{2.7}$$

Moreover, the Markov chain has the very nice ergodic property which states that for any mapping $\phi : \mathbb{S} \to \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \phi(Y(s))ds = \sum_{j=1}^{N} \phi(j)\pi_j \quad a.s. \tag{2.8}$$

Let $f, g : \mathbb{R} \times \mathbb{S} \times [0,\infty) \to \mathbb{R}$ be continuous functions obeying local Lipschitz continuity and linear growth conditions. Let $X(0) = x_0$ and consider the stochastic differential equation with Markovian switching given by

$$dX(t) = f(X(t), Y(t), t)\, dt + g(X(t), Y(t), t)\, dB(t). \tag{2.9}$$

Under the above conditions, there is a unique continuous and adapted process which satisfies (2.9) (see e.g. [18]). We make the standing assumption throughout the paper that f and g obey these continuity and growth restrictions, and that Y is an irreducible continuous–time Markov chain with finite state space $\mathbb{S}$. For economy of exposition these assumptions are not explicitly repeated in the statement of theorems in this paper.

3. Statement and Discussion of Main Results

In this section we give sufficient conditions ensuring law of the iterated logarithm–type behaviour for the solution of (2.9). All proofs are found in Section 5. The first two theorems deal with upper and lower estimates on the asymptotic growth rate of the partial maxima respectively.

Theorem 3.1. *Let X be the unique adapted continuous solution satisfying* (2.9)*. If there exist positive real numbers ρ, K_1 and K_2 such that*

$$xf(x, y, t) \le \rho, \quad \textit{for all } (x, y, t) \in \mathbb{R} \times \mathbb{S} \times [0,\infty); \tag{3.1a}$$

$$K_2 \le g^2(x, y, t) \le K_1, \quad \textit{for all } (x, y, t) \in \mathbb{R} \times \mathbb{S} \times [0,\infty) \tag{3.1b}$$

then X satisfies

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t \log\log t}} \le \sqrt{K_1}, \quad a.s. \tag{3.2}$$

The result and hypotheses of this theorem are similar to those in a theorem in Mao [16], in which no switching process is present. Here, in Theorem 3.1, a sharper upper bound on the solution is obtained, at the expense of a two–sided bound on the diffusion coefficient g. The proof in [16] employs martingale and integral inequalities, while Theorem 3.1 is proven by means of a comparison result. An advantage of this comparison approach is that a similar argument also yields a lower estimate on the large fluctuations of the solution, which we have been unable to obtain using the methods in [16].

Theorem 3.2. *Let X be the unique adapted continuous solution satisfying* (2.9). *If there exist real numbers K_1 and K_2 such that* (3.1b) *holds, and there is an $L \in \mathbb{R}$ such that*

$$\inf_{(x,y,t)\in\mathbb{R}\times\mathbb{S}\times[0,\infty)} \frac{xf(x,y,t)}{g^2(x,y,t)} =: L > -\frac{1}{2}, \tag{3.3}$$

then X satisfies

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq \sqrt{K_2}, \quad a.s. \tag{3.4}$$

We can combine the arguments used to prove these results to obtain a general result on the exact size of the large fluctuations, under the assumption that the diffusion coefficient depends only on the process Y. The result plays a role later in the paper when we consider applications of these pathwise large deviation results to finance.

Corollary 3.3. *Let X be the unique continuous adapted process satisfying the equation*

$$dX(t) = f(X(t), Y(t), t)\,dt + \gamma(Y(t))\,dB(t) \tag{3.5}$$

with $X(0) = x_0$, where $\gamma : \mathbb{S} \to \mathbb{R}\setminus\{0\}$. If there exists a real number $\rho > 0$ such that

$$\sup_{(x,y,t)\in\mathbb{R}\times\mathbb{S}\times[0,\infty)} \frac{xf(x,y,t)}{\gamma^2(y)} \leq \rho \quad \textit{and} \quad \inf_{(x,y,t)\in\mathbb{R}\times\mathbb{S}\times[0,\infty)} \frac{xf(x,y,t)}{\gamma^2(y)} > -\frac{1}{2}, \tag{3.6}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = \sigma_*, \quad a.s. \tag{3.7}$$

where

$$\sigma_*^2 = \sum_{j\in\mathbb{S}} \gamma^2(j)\pi_j, \tag{3.8}$$

and π is the stationary probability distribution of Y defined by (2.7).

The first condition in (3.6) is equivalent to (3.1a). The second condition is more subtle. Although it is sufficient to establish an iterated logarithm–type result, it is not a necessary condition to do so: Theorem 3.4 which follows justifies the second part of this remark. However, examples of equations (2.9) exist in which the second condition in (3.6) is false, and the solutions do not obey iterated logarithm type growth bounds.

We supply such an example now. Suppose in (2.9) that $f(x,y,t) = f(x)$ and $g(x,y,t) = \sigma \neq 0$, and let f obey $\lim_{x\to\infty} xf(x) = \lim_{x\to-\infty} xf(x) = L < -\sigma^2/2$. Then, provided f is continuous, the first condition in (3.6) is true, but $\inf_{x\in\mathbb{R}} xf(x) < -\sigma^2/2$, and so the second condition in (3.6) is false. Routine calculations show that the conditions of Motoo's theorem hold. Moreover, by determining the asymptotic behaviour of the scale function, we can use Motoo's theorem to show that

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \quad \text{exists a.s.}$$

is deterministic and is strictly less than 1/2. Therefore a solution of (2.9) under these conditions cannot obey the law of the iterated logarithm. It can be seen that the second part of condition (3.6) is quite a sharp hypothesis, since in the case that $L > -\sigma^2/2$ we can find functions f such that the second part of (3.6) holds, and hence the law of the iterated logarithm holds also.

We observe that (3.7) provides an *exact* rate of growth of the partial maxima of $|X|$. This is in contrast with the results of Theorems 3.1 and 3.2, in which only *bounds* on the growth rate are determined. We also notice that the presence of the switching process Y influences the rate of growth, because the value of σ_* in (3.8) depends on the stationary distribution of Y. On the other hand, it is not immediately clear from Theorems 3.1 and 3.2 that the switching process can influence the asymptotic behaviour so directly, because the bounds on the diffusion coefficients K_1 and K_2 are independent of the switching state Y.

Finally, not only is the a.s. rate of growth of the partial maxima deterministic, but it also can be computed explicitly once the generator of Y and the diffusion coefficient γ are known. The stronger conclusion of Corollary 3.3 relies upon the stronger assumption that the diffusion coefficient depends only on the Markov process Y.

In Theorem 3.1, 3.2 and in Corollary 3.3, we assume that f obeys a pointwise bound that depends on x. We can allow f to violate such a bound, provided any "spikes" that may be present in f are sufficiently narrow. This is achieved by the choice of hypothesis (3.10) in the statement of Theorem 3.4 below.

Theorem 3.4. *Let X be the unique continuous adapted process satisfying*

$$dX(t) = f(X(t), Y(t), t)\, dt + g(X(t), Y(t), t)\, dB(t), \tag{3.9}$$

with $X(0) = x_0$. If there exist positive real numbers K_1, K_2 such that (3.1b) *holds, and there is a locally Lipschitz continuous function $\tilde{f}$ such that*

$$\frac{|f(x,y,t)|}{g^2(x,y,t)} \le \tilde{f}(x), \quad \tilde{f} \in \mathcal{L}^1(\mathbb{R};\mathbb{R}^+), \tag{3.10}$$

then X almost surely obeys

$$\frac{\sqrt{K_2}e^{-2\sup_{x\in\mathbb{R}}\int_0^x(-\tilde{f}(y))dy}}{e^{-2\int_0^\infty(-\tilde{f}(y))dy}} \le \limsup_{t\to\infty}\frac{X(t)}{\sqrt{2t\log\log t}} \le \frac{\sqrt{K_1}e^{-2\inf_{x\in\mathbb{R}}\int_0^x\tilde{f}(y)dy}}{e^{-2\int_0^\infty\tilde{f}(y)dy}} \tag{3.11a}$$

$$\frac{-\sqrt{K_1}e^{-2\inf_{x\in\mathbb{R}}\int_0^x(-\tilde{f}(y))dy}}{e^{2\int_{-\infty}^0(-\tilde{f}(y))dy}} \le \liminf_{t\to\infty}\frac{X(t)}{\sqrt{2t\log\log t}} \le \frac{-\sqrt{K_2}e^{-2\sup_{x\in\mathbb{R}}\int_0^x\tilde{f}(y)dy}}{e^{2\int_{-\infty}^0\tilde{f}(y)dy}}. \tag{3.11b}$$

We notice in this result that both positive and negative large fluctuations obey an iterated logarithm growth bound: this contrasts with the results of Theorem 3.1, 3.2 and Corollary 3.3, in which the growth bounds are for the absolute value of the process. While the estimates on the normalising constants $\sqrt{K_1}$ and $\sqrt{K_2}$ in Theorems 3.1 and 3.2 are sharper than those obtained in Theorem 3.4, we are able to dispense with the pointwise bounds required in (3.6).

4. **Application to Financial Market Models**

In this section, we consider the application of the results from the previous section to a variant of Geometric Brownian motion (GBM) which involves Markovian switching. In the first subsection, we state and discuss some properties of standard models, and then do likewise for analogous results for the switching model. These results concentrate on the long run growth rate, the size of the largest departures from the trend, and the large fluctuations of the incremental returns. In the second subsection, we specialise our results to a market in which there are only two regimes of “high” and “low” volatility. Some conjectures are also stated and their possible proofs outlined.

4.1. **Discussion of main results.** We begin by reviewing briefly some mathematical and economic properties of GBM. GBM is one of the canonical models used to describe the stochastic evolution of asset prices (see e.g., Karatzas and Shreve [13]), and is behind the classical Black–Scholes–Merton option pricing formula (see e.g., Merton [19]). This work has given rise to a great variety of alternative market models and has lead to an explosion in the variety of financial instruments that can be priced; a flavour of this activity can be gleaned from the popular textbook [11].

As is well–known, GBM can be characterised as the unique solution of the linear stochastic differential equation

$$dS^*(t) = \mu S^*(t)\,dt + \sigma S^*(t)\,dB(t), \quad t \geq 0 \tag{4.1}$$

where $S^*(0) > 0$. In the context of financial economics, μ is the instantaneous mean rate of growth of the price, and σ its instantaneous volatility. The importance of the GBM model is embodied by the following fact: if security returns are stationary and independent (so that the market is informationally efficient) and the stock price process S^* varies continuously in continuous time, then S^* *must* obey (4.1). It is well–known that the logarithm of S^* is a Brownian motion with drift, having mean and variance at time t of $(\mu - \frac{\sigma^2}{2})t$ and $\sigma^2 t$ respectively, and that S^* grows exponentially according to

$$\lim_{t\to\infty} \frac{\log S^*(t)}{t} = \mu - \frac{1}{2}\sigma^2, \quad \text{a.s.} \tag{4.2}$$

Furthermore the maximum size of the large deviations from this growth trend obey the law of the iterated logarithm:

$$\limsup_{t\to\infty} \frac{|\log S^*(t) - (\mu - \frac{1}{2}\sigma^2)t|}{\sqrt{2t\log\log t}} = \sigma, \quad \text{a.s.} \tag{4.3}$$

Before discussing other properties of S^*, we explore the significance and implications of the result (4.3) in terms of finance. Since S^* represents the price of a *risky* asset, we cannot expect that S^* grows at *exactly* the rate $\exp[(\mu - \sigma^2/2)t]$ as $t \to \infty$. Indeed, as real stock prices experience departures from such steady growth rates (for example in market crashes or bubbles), it is advantageous for any model of these prices to also have this property and to be able to determine how *large* these bubbles or crashes are likely to be from the perspective of both long–term investment and portfolio management.

This leads us to consider the size of the *largest* fluctuations from the trend rate of growth. We can study these large fluctuations by first removing the exponential trend from the stock price, leaving us with the process $\log S^*(t) - (\mu - \sigma^2/2)t$, which gives the logarithm of the departure from the trend. The largest deviations of this departure obey a law of the iterated logarithm, according to (4.3). In terms of the stock price itself, roughly speaking, this means that the stock can be *bigger* than the smooth exponential trend by a factor of $\exp[\sigma\sqrt{2t\log\log t}]$, or can be *smaller* by a factor of $\exp[-\sigma\sqrt{2t\log\log t}]$ as $t \to \infty$, a.s.

Moreover the Δ–increments of $\log S^*$ are stationary and Gaussian, with the mean and variance of the increments depending linearly on Δ. These Δ–increments, defined by $R^*_\Delta(t) = \log(S^*(t)/S^*(t-\Delta))$, therefore obey

$$\limsup_{t\to\infty} \frac{R^*_\Delta(t)}{\sqrt{2\log t}} = \sigma\sqrt{\Delta}, \quad \text{a.s.} \tag{4.4}$$

In the following section, we propose a variant of (4.1) in which the stock price S is the solution of a stochastic differential equation where the driving Brownian motion in (4.1) is replaced by a semi–martingale which partly depends on a continuous–time Markov chain. The model departs from (4.1) in that the returns are no longer stationary nor independent. To make this precise, note that if the cumulative returns on the security with price $S = \{S(t) : t \geq 0\}$ up to time t are defined by $R(t)$, then

$$R(t) = \log\big(S(t)/S(0)\big), \quad t \geq 0 \tag{4.5}$$

and the (log) returns of the security over the time interval $[t-\Delta, t]$ are defined by

$$R_\Delta(t) = R(t) - R(t-\Delta) = \log(S(t)/S(t-\Delta)), \quad t \geq \Delta. \tag{4.6}$$

With these definitions we show that the processes S and R_Δ obey analogous properties to (4.2), (4.3) and (4.4). Therefore, the stock price process grows exponentially, experiences large deviations from the trend growth rate of iterated logarithm type, and incremental returns have the same rate of growth as those of stationary Gaussian processes, despite R_Δ being non–Gaussian. The above claims are made precise in the following Theorems in this section, whose proofs are supplied in Section 6.

Theorem 4.1. *Let Y be a continuous–time Markov process with state space $\mathbb{S}$. Let X be the unique continuous adapted process governed by*

$$dX(t) = f(X(t), Y(t), t)\,dt + \sigma\,dB(t) \quad t \geq 0 \tag{4.7}$$

with $X(0) = 0$. Let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R} \setminus \{0\}$, and S be the unique continuous adapted process defined by

$$dS(t) = \mu S(t)\,dt + S(t)\,dX(t) \quad t \geq 0 \tag{4.8}$$

with $S(0) = s_0 > 0$. Suppose that f obeys (3.6). Then:

(i)

$$\lim_{t\to\infty} \frac{\log S(t)}{t} = \mu - \frac{\sigma^2}{2}, \quad a.s.$$

(ii)

$$\limsup_{t\to\infty} \frac{|\log S(t) - (\mu - \frac{\sigma^2}{2})t|}{\sqrt{2t\log\log t}} = |\sigma|, \quad a.s. \tag{4.9}$$

(iii) *If R_Δ is given by (4.6), then for each $0 < \Delta < \infty$*

$$\limsup_{t\to\infty} \frac{|R_\Delta(t)|}{\sqrt{2\log t}} = |\sigma|\sqrt{\Delta}, \quad a.s.$$

Despite the presence of the Markov process Y (which introduces regime shifts) and the X–dependent drift term f in (4.7) (which introduces inefficiency), we see that S obeys the same asymptotic properties as S^*, namely (4.2), (4.3) and (4.4). These properties of S^* are shared by S because condition (3.6) guarantees that f becomes small for large values of X, thereby forcing S and S^* to remain close, in some sense. Indeed, if f is identically zero, we see that S and S^* actually coincide.

On the other hand, the analysis is now more complicated because the increments are neither independent nor Gaussian, and it is not possible to write down an explicit formula for S in terms of B and Y. This complication is worthwhile, however, because it stems from the addition of inefficiency and regime shifts into the market model.

We now give a result in the case when the diffusion coefficient depends on the switching process Y. This is an important special case for two related economic reasons. The first is the principal economic rationale for switching models in finance: namely that market sentiment occasionally changes, leading to differing volatility or growth rates. The incorporation of sentiment in this manner is one of the important motivations behind the discipline of behavioural finance (see e.g., the survey paper [5]). Secondly, it makes the volatility a stochastic process which cannot be explained purely in terms of the current market returns. This places the model within the framework of stochastic volatility (SV) models, particularly as the volatility process is stationary and ergodic. One of the first such SV models was presented in [12], and a recent textbook devoted to stochastic volatility models is [6]. A common feature of SV models is that the volatility is described by the stationary solution of a stochastic differential equation which is driven by a Brownian motion which is correlated with, but not equal to, the Brownian motion which drives the stock price. This renders the market incomplete, as there are more sources of randomness than tradable securities. In the model proposed here the volatility is also a stationary stochastic process, but unlike processes in SV models, it can assume only finitely many values, does not change from instant to instant, and is also uncorrelated with the Brownian motion which drives the stock process. However, if employed to price options, the model analysed here should lead to both incomplete markets and the presence of volatility smiles. Volatility smiles have been shown to exist for other stochastic volatility models in which the volatility assumes a finite number of values (see e.g., Renault and Touzi [22]).

The first result shows that when the volatility depends on the switching process alone, there is a well–defined growth rate, and the fluctuations around this growth rate still obey a law of the iterated logarithm.

Theorem 4.2. *Let S be the unique continuous adapted process governed by* (4.8) *with $S(0) = s_0 > 0$, where X satisfies*

$$dX(t) = f(X(t), Y(t), t)\, dt + \gamma(Y(t))\, dB(t) \quad t \geq 0 \tag{4.10}$$

with $X(0) = 0$ and $\gamma : \mathbb{S} \to \mathbb{R} \setminus \{0\}$. Suppose that f obeys (3.6)*.*

(i) *If $\sigma_* > 0$ is defined by* (3.8)*, then*

$$\lim_{t\to\infty} \frac{\log S(t)}{t} = \mu - \frac{\sigma_*^2}{2}, \quad a.s.$$

(ii) *If $\sigma_* > 0$ is defined by* (3.8)*, then*

$$\limsup_{t\to\infty} \frac{|\log S(t) - (\mu t - \frac{1}{2}\int_0^t \gamma^2(Y(s))\, ds)|}{\sqrt{2t \log\log t}} = \sigma_*, \quad a.s. \tag{4.11}$$

Before proceeding further, we pause to examine the relevance of (4.11) and its connection with (4.9) in Theorem 4.1. The limit in (4.11) gives, at least superficially, a weaker result than the limit in (4.9). As explained earlier, (4.9) can be interpreted in terms of the size of the fluctuations of the price around its *deterministic* exponential rate of growth $G(t) := \exp[(\mu - \sigma^2/2)t]$. Hence the log trend is $\log G(t) = (\mu - \sigma^2/2)t$, so (4.9) can be written

$$\limsup_{t\to\infty} \frac{|\log S(t) - \log G(t)|}{\sqrt{2t \log\log t}} = \sigma, \quad \text{a.s.}$$

Similarly, (4.11) can be written in this form with σ_* being the limit on the right–hand side, and the log trend, $\log G_*(t)$, in this case is *stochastic* and given by

$$\log G_*(t) = \mu t - \frac{1}{2}\int_0^t \gamma^2(Y(s))ds. \tag{4.12}$$

Moreover, despite $G_*(t)$ being stochastic, we have

$$\lim_{t\to\infty} \frac{\log G(t)}{t} = \mu - \frac{1}{2}\sigma^2, \quad \lim_{t\to\infty} \frac{\log G_*(t)}{t} = \mu - \frac{1}{2}\sigma_*^2 \quad \text{a.s.}$$

The fact that G_* is stochastic does not by itself create a difficulty in (4.11) but rather the fact that it depends on the switching process Y which cannot be observed directly from market data. Therefore it is certainly more cumbersome, and perhaps infeasable, to remove this stochastic growth trend as easily as in (4.9). However, it may be possible to recover the full strength of (4.9) by introducing a *deterministic* log trend $\log G_1(t) := (\mu - \sigma_*^2/2)t$. Since the ergodic theorem for jump processes implies

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma^2(Y(s))ds = \sigma_*^2 \quad \text{a.s.},$$

if we can show that the convergence rate to the limit is so fast that

$$\lim_{t\to\infty} \sqrt{t\log\log t}\left\{\frac{1}{t}\int_0^t \gamma^2(Y(s))ds - \sigma_*^2\right\} = 0 \quad \text{a.s.}, \tag{4.13}$$

then (4.11) implies

$$\limsup_{t\to\infty} \frac{|\log S(t) - (\mu - \sigma_*^2/2)t|}{\sqrt{2t\log\log t}} = \sigma_*, \quad \text{a.s.},$$

and we can interpret a large value of σ_* as giving rise to larger fluctuations from the *deterministic* exponential growth trend $\exp[(\mu - \sigma_*^2/2)t]$.

We conjecture that the rate of convergence needed in (4.13) is in fact attained under the hypotheses of Theorem 4.2 because the jump process is irreducible and has a finite state space. We sketch a rough and tentative proof of the steps involved. To determine the convergence rate a.s., by means of a Borel–Cantelli argument, it is enough to know that the convergence of $\int_0^t \gamma^2(Y(s))ds/t$ to σ_*^2 is sufficiently fast in t as $t \to \infty$ in, for example, the fourth moment. The idea is to consider a discrete Markov chain embedded in the jump process, and to use analysis of the independent excursions of the chain from each state. By irreducibility and the finiteness of the state space, the excursion time should have finite moments and the standard Strong Law of Large Numbers with fourth moment condition can be applied. This analysis should establish convergence of the numbers of visits of the chain to each state as a function of discrete time both in fourth mean and a.s. The connection between discrete–time for the embedded chain, and continuous–time for Y, is asymptotically linear by virtue of the independent and exponentially distributed holding time distributions for each state for Y. Therefore, this connection should enable us to recover the continuous–time result (4.13). Even if correct, this outline falls far short of a proof but nonetheless we feel adds weight to our claim.

Finally, a result can be proven about the large fluctuations of the Δ–returns, even when the diffusion coefficient depends on X, Y and t. Once again, the large fluctuations of the Δ–returns grow at a rate $\sqrt{\Delta}\sqrt{2\log t}$ times a constant which depends on the volatility. This rate of growth is consistent with the Δ–increments of a stationary Gaussian process.

Theorem 4.3. *Let S be the unique continuous adapted process governed by* (4.8) *with $S(0) = s_0 > 0$, where X satisfies*

$$dX(t) = f(X(t), Y(t), t)\,dt + g(X(t), Y(t), t)\,dB(t) \quad t \geq 0 \tag{4.14}$$

with $X(0) = 0$. Suppose moreover that f obeys (3.6)*, and g obeys* (3.1b)*. Let $\Delta > 0$. If R_Δ is the process defined by* (4.6)*, then*

$$\sqrt{K_2}\sqrt{\Delta} \leq \limsup_{t\to\infty} \frac{\sup_{0\leq\delta\leq\Delta} |R_\delta(t)|}{\sqrt{2\log t}} \leq \sqrt{K_1}\sqrt{\Delta}, \quad a.s. \tag{4.15}$$

We note that the argument used to prove these results can also be applied to an inefficient market model in which the diffusion coefficient in X depends not only on the Markovian switching term but also on a delay term, once that diffusion coefficient remains bounded.

4.2. **Results for a two–state volatility model.** In this subsection, we explore further the case when X is given by (4.10), in which the diffusion coefficient depends only on the switching process Y. In this example, Y is a two-state Markov chain. To capture this in the notation of the previous subsection we let the state space $\mathbb{S} = \{H, L\}$ so the diffusion coefficient can take the values $\gamma(H) = \sigma_H$ or $\gamma(L) = \sigma_L$. This represents a market model where the volatility can be either "high" or "low", with values $\sigma_H > \sigma_L > 0$ respectively. The generator of Y, denoted Γ, is given by

$$\Gamma = \begin{pmatrix} -\gamma_1 & \gamma_1 \\ \gamma_2 & -\gamma_2 \end{pmatrix}$$

where γ_1 is the rate of transition from the high state to the low state, and γ_2 is the transition rate from the low state to the high state. In a typical situation one would have $\gamma_2 < \gamma_1$ so that the process spends more time in the low volatility state in the long run. We give calculations and interpretations in this case and we note that this can easily be generalised to a finite number of volatility levels. However, econometric evidence indicates that a two–state model is very often sufficient.

Define a process R^* such that $R^*(0) = 0$ and $dR^*(t) = \mu dt + dX(t)$, where X is again given by (4.10). Then we can reformulate (4.8) as $dS(t) = S(t)dR^*(t)$. We call R^* the gains process; it is intimately related to the cumulative returns R, given by (4.5). Indeed, the increments of the gains $R^*_\Delta(t) := R^*(t) - R^*(t-\Delta)$ are once again similar to Brownian motion since $R^*_\Delta(t) = X(t) - X(t-\Delta) + \mu\Delta$ and it has already been shown that the increments of X are similar to those of Brownian motion. Recalling Corollary 3.3 we have that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = \sigma_* \quad \text{a.s., where } \sigma_*^2 = \sum_{j\in\mathbb{S}} \gamma^2(j)\pi_j \tag{4.16}$$

and $\pi = (\pi_H, \pi_L)$ is the stationary probability distribution of Y. Moreover, in the situation that $\int_0^t \gamma^2(Y(s))\,ds/t \to \sigma_*^2$ at the rate conjectured in (4.13), we have

$$\limsup_{t\to\infty} \frac{|\log S(t) - (\mu - \sigma_*^2/2)t|}{\sqrt{2t\log\log t}} = \sigma_*, \quad \text{a.s.} \tag{4.17}$$

$\pi = (\pi_H, \pi_L)$ can be found by solving $\pi\Gamma = 0$ (or equivalently $-\pi_H\gamma_1 + \pi_L\gamma_2 = 0$) subject to the constraint $\pi_H + \pi_L = 1$. Solving these equations we arrive at

$$\pi_H = \frac{\gamma_2}{\gamma_1+\gamma_2}, \quad \pi_L = \frac{\gamma_1}{\gamma_1+\gamma_2}.$$

Thus, σ_*^2 is now simply the weighted average of the different volatility levels

$$\sigma_*^2 = \sigma_H^2\frac{\gamma_2}{\gamma_1+\gamma_2} + \sigma_L^2\frac{\gamma_1}{\gamma_1+\gamma_2}.$$

As mentioned earlier, if $\gamma_2 < \gamma_1$ then more weight will be placed on the lower volatility regime as more time will be spent in the low volatility state. This means that σ_* will be small and thus the fluctuations of $|X|$ will be relatively small. On the other hand, if π_H is relatively close to unity then σ_* can be quite large also and so periods in the high volatility regime can have a big impact on the fluctuations. Moreover, if σ_* is large then the growth rate, given by $\mu - \sigma_*^2/2$, is adversely affected. These important features are somewhat concealed in the statement of (4.16).

Since we are considering the simpler case where the diffusion coefficient is $t-$ and $X-$independent, the conclusion (4.15) of Theorem 4.3 applies and we get

$$\sigma_L\sqrt{\Delta} \le \limsup_{t\to\infty} \frac{\sup_{0\le\delta\le\Delta}|R_\delta(t)|}{\sqrt{2\log t}} \le \sigma_H\sqrt{\Delta}, \quad \text{a.s.}$$

In fact, we conjecture that in this case (where $\mathbb{S} = \{H, L\}$ and $\gamma(H) = \sigma_H$) we have

$$\limsup_{t\to\infty} \frac{|R_\Delta(t)|}{\sqrt{2\log t}} = \sigma_H\sqrt{\Delta}, \quad \text{a.s.} \tag{4.18}$$

This suggests that the "high" volatility periods are entirely responsible for the largest fluctuations in the absolute Δ–returns. This phenomena cannot be observed from (4.16) and (4.17) which deal with the cumulative returns, which include accumulated contributions from both high and low volatility periods. Note that the upper bound obtained in (4.15) gives the inequality

$$\limsup_{t\to\infty} \frac{|R_\Delta(t)|}{\sqrt{2\log t}} \le \sigma_H \sqrt{\Delta}, \quad \text{a.s.} \tag{4.19}$$

We are lead to the conjecture (4.18) by the following argument. First, we have

$$R_\Delta(t) = \log(S(t)/S(t-\Delta)) = X(t) - X(t-\Delta) - \int_{t-\Delta}^t \{\mu - \frac{1}{2}\gamma^2(Y(s))\}\, ds,$$

so because the limits exist we have

$$\limsup_{t\to\infty} \frac{|R_\Delta(t)|}{\sqrt{2\log t}} = \limsup_{t\to\infty} \frac{|X(t-\Delta) - X(t)|}{\sqrt{2\log t}}, \quad \text{a.s.}$$

and since

$$X(t) - X(t-\Delta) = \int_{t-\Delta}^t f(X(s), Y(s), s)\, ds + \int_{t-\Delta}^t \gamma(Y(s))\, dB(s), \quad t \ge \Delta$$

we have

$$\limsup_{t\to\infty} \frac{|R_\Delta(t)|}{\sqrt{2\log t}} = \limsup_{t\to\infty} \frac{|\int_{t-\Delta}^t \gamma(Y(s))\, dB(s)|}{\sqrt{2\log t}}, \quad \text{a.s.}$$

In particular, with $U_n = \int_{n\Delta}^{(n+1)\Delta} \gamma(Y(s))\, dB(s)$ we have

$$\limsup_{t\to\infty} \frac{|R_\Delta(t)|}{\sqrt{2\log t}} \ge \limsup_{n\to\infty} \frac{|R_\Delta((n+1)\Delta)|}{\sqrt{2\log((n+1)\Delta)}} = \limsup_{n\to\infty} \frac{|U_n|}{\sqrt{2\log n}}. \tag{4.20}$$

Since Y is stationary, the probability that $Y(n\Delta) = H$ is $\pi_H = \gamma_2/(\gamma_1+\gamma_2)$. Define the event $A_n := \{Y(s) = H, \text{ for all } s \in [n\Delta, (n+1)\Delta]\}$. Then

$$\begin{aligned}\mathbb{P}[A_n] &= \mathbb{P}[Y(n\Delta) = H]\mathbb{P}[\text{no jump from state } H \text{ for at least } \Delta \text{ time units}]\\ &= \gamma_2/(\gamma_1+\gamma_2)\cdot e^{-\gamma_1\Delta} =: \pi(\Delta).\end{aligned}$$

Note also that the process $\{I_{A_n} : n \ge 1\}$ is stationary and that $\mathrm{Cov}(I_{A_n}, I_{A_{n+m}}) \to 0$ as $m \to \infty$ (here I_C is the indicator random variable of an event C, and $\mathrm{Cov}(U, V)$ is the covariance of the random variables U and V). Define $T_n = \sum_{j=1}^n I_{A_j}$. Then by a corollary of the ergodic theorem we have $T_n/n \to \mathbb{E}[I_{A_1}] = \pi(\Delta)$ as $n \to \infty$ a.s. Let $L_n = \min\{l \ge n : \sum_{j=1}^l I_{A_j} = n\}$. By definition $I_{A_{L_n}} = 1$. Then if we consider the collection of $\{U_j : j = 1, \dots, n\}$ for which $I_{A_j} = 1$ we have

$$\max_{1\le j\le n} |U_j| \ge \max_{1\le k\le T_n} |U_{L_k}|.$$

Next, if $I_{A_n} = 1$ then $Y(s) = H$ for all $s \in [n\Delta, (n+1)\Delta]$, we have
$U_n = \int_{n\Delta}^{(n+1)\Delta} \gamma(H)\, dB(s) = \sigma_H(B((n+1)\Delta) - B(n\Delta))$. Hence we get

$$\max_{1\le j\le n} |U_j| \ge \max_{1\le k\le T_n} |U_{L_k}| = \max_{1\le k\le T_n} |\sigma_H\{B((L_k+1)\Delta) - B(L_k\Delta)\}|.$$

Therefore with $\xi(k) := B((L_k+1)\Delta) - B(L_k\Delta)$ we have

$$\limsup_{n\to\infty} \frac{\max_{1\le j\le n}|U_j|}{\sqrt{2\log n}} \ge \limsup_{n\to\infty} \frac{\max_{1\le k\le T_n}|\sigma_H \xi(k)|}{\sqrt{2\log T_n}} \cdot \sqrt{\frac{\log T_n}{\log n}}$$
$$= \limsup_{n\to\infty} \frac{\max_{1\le k\le T_n}|\sigma_H \xi(k)|}{\sqrt{2\log T_n}} = \sigma_H \limsup_{n\to\infty} \frac{\max_{1\le k\le n}|\xi(k)|}{\sqrt{2\log n}},$$

where we used the fact that $T_n \to \infty$ as $n \to \infty$ a.s. at the last step. Since B and Y are independent, it follows that $L = \{L_n : n \ge 1\}$ and B are independent. Also $L_{k+1} - L_k \ge 1$. Thus $\{\xi(k) : k \ge 1\}$ is a sequence of independently and identically distributed normal random variables with mean zero and variance Δ. Therefore

$$\limsup_{n\to\infty} \frac{\max_{1\le k\le n}|\xi(k)|}{\sqrt{2\log n}} = \sqrt{\Delta}, \quad \text{a.s.}$$

Hence

$$\Lambda := \limsup_{n\to\infty} \frac{\max_{1\le j\le n}|U_j|}{\sqrt{2\log n}} \ge \sigma_H\sqrt{\Delta}, \quad \text{a.s.}$$

This implies that

$$\limsup_{n\to\infty} \frac{|U_n|}{\sqrt{2\log n}} = \Lambda \ge \sigma_H\sqrt{\Delta}, \quad \text{a.s.} \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) gives (4.18).

5. Proofs of Theorems from Section 3

5.1. **Proof of Theorem 3.1.** Applying Itô's formula to (2.9) we get

$$dX^2(t) = \left[2X(t)f(X(t),Y(t),t) + g^2(X(t),Y(t),t)\right]dt$$
$$+ 2X(t)g(X(t),Y(t),t)\,dB(t), \quad t \ge 0. \tag{5.1}$$

Let N be the local martingale defined by $N(t) = \int_0^t 2X(s)g(X(s),Y(s),s)\,dB(s)$. It has quadratic variation given by $\langle N\rangle(t) = \int_0^t 4X^2(s)g^2(X(s),Y(s),s)\,ds$. Then by Doob's martingale representation theorem (cf. e.g., Theorem 3.4.2 in [14]), there exists another Brownian motion β in an extended probability space with measure $\widetilde{\mathbb{P}}$ such that

$$N(t) = \int_0^t 2|g(X(s),Y(s),s)|\sqrt{X^2(s)}d\beta(s) \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

Now let $Z(t) = X^2(t)$ and let $\phi(t) = 2X(t)f(X(t),Y(t),t) + g^2(X(t),Y(t),t)$ so that we can write equation (5.1) as

$$dZ(t) = \phi(t)\,dt + 2|g(X(t),Y(t),t)|\sqrt{Z(t)}\,d\beta(t). \tag{5.2}$$

Let $M(t) = \int_0^t |g(X(s),Y(s),s)|\,d\beta(s)$, so $\langle M\rangle(t) = \int_0^t g^2(X(s),Y(s),s)\,ds$. Then by the martingale time–change theorem (cf., e.g., Theorem 3.4.6 in [14]), we may define a new Brownian motion $\tilde{\beta}$ by $\tilde{\beta}(\langle M\rangle(t)) = M(t)$ and the stopping time τ by $\tau(t) = \inf\{s > 0 : \langle M\rangle(s) > t\}$. Since $g^2(x,y,t) \ge K_2$, we have $\tau(t) = \langle M\rangle^{-1}(t)$. Moreover,

$M(\tau(t)) = \tilde{\beta}(t)$ and we introduce the processes $\tilde{X}(t) = X(\tau(t))$, $\tilde{Y}(t) = Y(\tau(t))$ and $\tilde{Z}(t) = Z(\tau(t))$. So now, applying this time-change to (5.2) we get:

$$\tilde{Z}(t) = Z(\tau(t)) = Z(\tau(0)) + \int_0^{\tau(t)} \phi(s)\, ds + \int_0^{\tau(t)} 2|g(X(s), Y(s), s)| \sqrt{Z(s)}\, d\beta(s). \tag{5.3}$$

To deal with the stochastic integral above, we use Proposition 3.4.8 from [14], which states that if $\tilde{\eta}(t) = \eta(\tau(t))$ and η is $\mathcal{F}^\beta$–adapted, then $\int_0^{\tau(s)} \eta(u)\, dM(u) = \int_0^s \tilde{\eta}(u)\, d\tilde{\beta}(u)$. In this case, we set $\eta(t) = 2\sqrt{Z(t)}$ and set M equal to the martingale defined above. Therefore

$$\int_0^{\tau(t)} 2\sqrt{Z(s)}\, |g(X(s), Y(s), s)|\, d\beta(s) = \int_0^{\tau(t)} 2\sqrt{Z(s)}\, dM(s) = \int_0^t 2\sqrt{\tilde{Z}(s)}\, d\tilde{\beta}(s).$$

To deal with the Riemann integral term in (5.3), we use Problem 3.4.5 from [14], which states that if G is a bounded measurable function, and $[a, b] \subset [0, \infty)$ then $\int_a^b G(s)\, d\langle M\rangle(s) = \int_{\langle M\rangle(a)}^{\langle M\rangle(b)} G(\tau(s))\, ds$. In this case, we set

$$G(t) = \phi(t)/g^2(X(t), Y(t), t)$$

and as $d\langle M\rangle(t) = g^2(X(t), Y(t), t)\, dt$, we obtain

$$\begin{aligned} \int_0^{\tau(t)} \phi(s)\, ds = \int_0^{\tau(t)} G(s)\, d\langle M\rangle(s) &= \int_{\langle M\rangle(0)}^{\langle M\rangle(\tau(t))} G(\tau(s))\, ds \\ &= \int_0^t \frac{\tilde{\phi}(s)}{g^2(\tilde{X}(s), \tilde{Y}(s), \tau(s))}\, ds, \end{aligned}$$

where $\tilde{\phi}(t) = \phi(\tau(t))$. So we can now write (5.3) as:

$$\tilde{Z}(t) = \tilde{Z}(0) + \int_0^t \frac{\tilde{\phi}(s)}{g^2(\tilde{X}(s), \tilde{Y}(s), \tau(s))}\, ds + \int_0^t 2\sqrt{\tilde{Z}(s)}\, d\tilde{\beta}(s). \tag{5.4}$$

Now, using conditions (3.1a) and (3.1b), it is easy to see that the drift coefficient of (5.4) is bounded above by $(K_2 + 2\rho)/K_2$. Define the process which is uniquely determined by the stochastic differential equation

$$dU(t) = C_u\, dt + 2\sqrt{|U(t)|}\, d\tilde{\beta}(t) \tag{5.5}$$

with $U(0) \ge \tilde{Z}(0) \ge 0$, where $C_u = (K_2 + 2\rho)/K_2$. We will now show, using a stochastic comparison technique, that for all $t \ge 0$, $\tilde{Z}(t) \le U(t)$ a.s.

First, we apply a stochastic comparison theorem (cf., e.g., Proposition 5.2.18 in [14]) to (5.5) and to the equation $dU_1(t) = 2\sqrt{|U_1(t)|}\, d\tilde{\beta}(t)$ with $U_1(0) = 0$; this shows that $U(t) \ge U_1(t)$ a.s., and since the process U_1 has the unique solution $U_1(t) = 0$, it follows that $U(t) \ge 0$ a.s. Finally, we can apply the comparison theorem to (5.4) and (5.5) to conclude that for all $t \ge 0$, $\tilde{Z}(t) \le U(t)$ a.s. Now we can approximate an upper bound for $\tilde{Z}$ by getting an upper bound for U. However, before we do that we will apply a time-change and a change of scale to U to get a process with finite speed measure. Consider $V(t) = e^{-t}U(e^t - 1)$. By using the product rule and introducing a new Brownian motion $\bar{\beta}$, we can show that

$$dV(t) = [-V(t) + C_u]\ dt + 2\sqrt{V(t)}\, d\bar{\beta}(t). \tag{5.6}$$

A scale function of V is given by $p_V(x) = \mu \int_a^x e^{y/2}y^{-C_u/2}\,dy$, $a > 0$, for some positive real number μ. It is easy to check that V satisfies (2.3). Hence Theorem 2.1 can be applied to V. Now, there exists $y_0 > a$ such that for all $y \geq y_0$, $y \mapsto e^{y/2}y^{-C_u/2}$ is increasing. Thus for all $x \geq y_0 + 1$, $e^{(x-1)/2}(x-1)^{-C_u/2} \leq p_V(x)$, then $1/p_V(x) \leq e^{-(x-1)/2}(x-1)^{C_u/2}$. Let $\beta > 1$ and define $h(t) = 2\beta \log t$ for $t \geq e^{(y_0+1)/(2\beta)}$. Hence

$$\frac{1}{p_V(h(t))} \leq e^{-\beta \log t + \frac{1}{2}}(2\beta \log t - 1)^{\frac{C_u}{2}},$$

and so $\limsup_{t\to\infty} \log(1/p_V(h(t)))/\log t \leq -\beta$. So for any $\beta - 1 > \epsilon > 0$, there exists t_ϵ such that for all $t > t_\epsilon$, $\log(1/p_V(h(t))) \leq (-\beta+\epsilon)\log t$, which implies $1/p_V(h(t)) \leq t^{-\beta+\epsilon}$. Since $\beta - \epsilon > 1$, it follows that $\int_{t_\epsilon}^\infty 1/p_V(h(s))\,ds \leq \int_{t_\epsilon}^\infty 1/s^{\beta-\epsilon}\,ds < \infty$. Therefore $\limsup_{t\to\infty} V(t)/2\log t \leq \beta$ a.s. Letting $\beta \downarrow 1$ through the rational numbers, we have

$$\limsup_{t\to\infty} \frac{V(t)}{2\log t} \leq 1, \quad \text{a.s.}$$

Using the fact that $V(t) = e^{-t}U(e^t - 1)$, we find that

$$\limsup_{t\to\infty} \frac{U(t)}{2t\log\log t} \leq 1, \quad \text{a.s.}$$

So

$$\limsup_{t\to\infty} \frac{Z(\tau(t))}{2t\log\log t} = \limsup_{t\to\infty} \frac{\tilde{Z}(t)}{2t\log\log t} \leq 1, \quad \text{a.s.}$$

By definition, $\tau(t) = \langle M\rangle^{-1}(t)$ and $\tau(\cdot)$ is monotone, so it follows that

$$\limsup_{t\to\infty} \frac{Z(t)}{2\langle M\rangle(t)\log\log\langle M\rangle(t)} \leq 1, \quad \text{a.s.} \tag{5.7}$$

Since $K_2 t \leq \langle M\rangle(t) \leq K_1 t$, $t \geq 0$, we can show that

$$\lim_{t\to\infty} \frac{\log\log\langle M\rangle(t)}{\log\log t} = 1 \quad \text{and} \quad \frac{t}{\langle M\rangle(t)} \geq \frac{t}{K_1 t} = \frac{1}{K_1}, \text{ a.s. for all } t > 0.$$

Therefore (5.7) implies $\limsup_{t\to\infty} Z(t)/(2t\log\log t) \leq K_1$ a.s. By taking square roots on both sides we get the assertion.

5.2. **Proof of Theorem 3.2.** Following the same argument as the previous proof, we arrive at (5.4). Therefore

$$d\tilde{Z}(t) = \frac{\phi(\tau(t))}{g^2(\tilde{X}(t), \tilde{Y}(t), \tau(t))}\,dt + 2\sqrt{\tilde{Z}(t)}\,d\tilde{\beta}(t).$$

By (3.3), it is easy to see that the drift coefficient of the above equation is bounded below by some positive number, say C_l. Consider the process governed by the following equation

$$dU(t) = C_l\, dt + 2\sqrt{|U(t)|}\, d\tilde{\beta}(t)$$

with $U(0) \leq \tilde{Z}(0)$. Then it can be shown that for all $t \geq 0$, $\tilde{Z}(t) \geq U(t) \geq 0$. Applying changes in both time and scale again, let $V(t) = e^{-t}U(e^t - 1)$, to get

$$dV(t) = (-V(t) + C_l)\, dt + 2\sqrt{V(t)}\, d\bar{\beta}(t) \quad t \geq 0.$$

We proceed as before; the process V obeys (2.3), and so we may apply Theorem 2.1 to it. Since a scale function of V is given by $p_V(x) = \mu \int_a^x e^{\frac{1}{2}y} y^{-C_l/2}\, dy$ for some positive real number μ, then by L'Hôpital's Rule $\lim_{x\to\infty} p_V(x)/e^{x/2} = 0$. This implies that there exists $x_* > 0$ such that for all $x > x_*$, $p_V(x) < e^{x/2}$. Hence if we let $h(t) = 2\log t$, there exists $t_* > 0$, such that for all $t > t_*$, $h(t) > x_*$, so $p_V(h(t)) < t$, thus $\int_{t_*}^\infty 1/p_V(h(s))\, ds > \int_{t_*}^\infty 1/s\, ds = \infty$. Therefore $\limsup_{t\to\infty} V(t)/2\log t \geq 1$ a.s. Since $V(t) = e^{-t}U(e^t - 1)$, we get $\limsup_{t\to\infty} U(t)/(2t\log\log t) \geq 1$ a.s. Since $\tilde{Z}(t) \geq U(t)$, we get $\limsup_{t\to\infty} \tilde{Z}(t)/(2t\log\log t) \geq 1$ a.s. Hence, as in the previous proof, we have

$$\limsup_{t\to\infty} \frac{Z(t)}{2\langle M\rangle(t)\log\log\langle M\rangle(t)} \geq 1, \quad \text{a.s.} \tag{5.8}$$

Proceeding as in the end of the last proof, we get the desired result (3.4).

5.3. **Proof of Corollary 3.3.** By (5.7) and (5.8), as $Z(t) = X^2(t)$, we have

$$\limsup_{t\to\infty} \frac{X^2(t)}{2\langle M\rangle(t)\log\log\langle M\rangle(t)} = 1, \quad \text{a.s.} \tag{5.9}$$

By analogy to the proof of Theorem 3.1, we have $\langle M\rangle(t) = \int_0^t \gamma^2(Y(s))\, ds$. From (2.8) and (3.8), it follows that $\langle M\rangle(t)/t \to \sigma_*^2$ a.s., which together with (5.9), proves the result.

5.4. **Proof of Theorem 3.4.** For all $t \geq 0$

$$X(t) = x_0 + \int_0^t f(X(s), Y(s), s)ds + \int_0^t g(X(s), Y(s), s)\, dB(s).$$

Let $M_1(t) = \int_0^t g(X(s), Y(s), s)\, dB(s)$, so $\langle M_1\rangle(t) = \int_0^t g^2(X(s), Y(s), s)\, ds$. Hence for all $t \geq 0$, $K_2 t \leq \langle M_1\rangle(t) \leq K_1 t$ and $\lim_{t\to\infty}\langle M_1\rangle(t) = \infty$ almost surely. Moreover $\langle M_1\rangle$ is increasing on $(0, \infty)$. Again we use the time-change theorem for martingales: for each $0 \leq t < \infty$, define the stopping time $\lambda(t) := \inf\{s > 0 : \langle M_1\rangle(s) > t\}$. Thus $\langle M_1\rangle(\lambda(t)) = t$ and $\lambda(t) = \langle M_1\rangle^{-1}(t)$. A process defined by $W(t) := M(\lambda(t)), \forall\, t \geq$

0 is a standard Brownian motion with respect to the filtration $\mathcal{G}(t) := \mathcal{F}(\lambda(t))$. Therefore, as in the proof of Theorem 3.1, we get

$$\begin{aligned}\widetilde{X}(t) := X(\lambda(t)) &= x_0 + \int_0^{\lambda(t)} f(X(s), Y(s), s)ds + \int_0^{\lambda(t)} g(X(s), Y(s), s)dB(s) \\ &= x_0 + \int_0^t \frac{f(\widetilde{X}(s), \widetilde{Y}(s), \lambda(s))}{g^2(\widetilde{X}(s), \widetilde{Y}(s), \lambda(s))} ds + W(t)\end{aligned}$$

where $\widetilde{Y}(t) := Y(\lambda(t))$. Due to (3.10), we have

$$\forall\, (x, y, t) \in \mathbb{R} \times \mathbb{S} \times [0, \infty), \quad -\tilde{f}(x) \le \frac{f(x, y, t)}{g^2(x, y, t)} \le \tilde{f}(x).$$

Consider two processes Z_1 and Z_2 governed by the following two equations, for $t \ge 0$

$$dZ_1(t) = \tilde{f}(Z_1(t))\, dt + dW(t), \quad dZ_2(t) = -\tilde{f}(Z_2(t))\, dt + dW(t)$$

with $Z_2(0) \le x_0 \le Z_1(0)$. Then again by the comparison theorem, we can show that for all $t \ge 0$, $Z_2(t) \le \widetilde{X}(t) \le Z_1(t)$ a.s. Consider the scale function of Z_1 defined as the following

$$p_Z(x) = \int_0^x e^{-2\int_0^y \tilde{f}(z)dz} dy, \quad x \in \mathbb{R}.$$

Then $p_Z \in C^2(\mathbb{R}; \mathbb{R})$ and for all $x \in \mathbb{R}$, we have

$$p_Z'(x)\tilde{f}(x) + \frac{1}{2} p_Z''(x) = 0. \tag{5.10}$$

Since $\tilde{f} \in \mathcal{L}^1$, there exist real numbers k_1, k_2 such that $\int_0^\infty \tilde{f}(z)dz = k_1$ and $\int_{-\infty}^0 \tilde{f}(z)dz = k_2$, which implies $\lim_{x\to\infty} p_Z'(x) = e^{-2k_1}$ and $\lim_{x\to-\infty} p_Z'(x) = e^{2k_2}$. So $p_Z(\infty) = \infty$ and $p(-\infty) = -\infty$. Thus $\limsup_{t\to\infty} Z_1(t) = \infty$ and $\liminf_{t\to\infty} Z_1(t) = -\infty$ a.s. Also by L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{p_Z(x)}{x} = e^{-2k_1}, \quad \lim_{x\to-\infty} \frac{p_Z(x)}{x} = e^{2k_2}. \tag{5.11}$$

Let $H(t) = p_Z(Z_1(t))$. Then by Itô's Rule and (5.10)

$$dH(t) = p_Z'(Z_1(t))dW(t), \quad t \ge 0,$$

with $H(0) = p_Z(Z_1(0))$. Now since p_Z is strictly increasing, the above equation can be written as

$$dH(t) = l(H(t))dW(t), \quad t \ge 0,$$

where $l(x) = p_Z'(p_Z^{-1}(x))$, for all $x \in \mathbb{R}$. H is also a recurrent process on $\mathbb{R}$. Moreover, (5.11) gives

$$\lim_{t\to\infty} \frac{\sup_{0\le s\le t} H(s)}{\sup_{0\le s\le t} Z_1(s)} = e^{-2k_1} \quad \text{and} \quad \lim_{t\to\infty} \frac{\inf_{0\le s\le t} H(s)}{\inf_{0\le s\le t} Z_1(s)} = e^{2k_2}, \quad \text{a.s.} \tag{5.12}$$

For each $t \ge 0$, define the continuous local martingale Q given by

$$Q(t) := \int_0^t l(H(s))\, dW(s),$$

which has quadratic variation $\langle Q\rangle(t) := \int_0^t l^2(H(s))\,ds$. Thus $\langle Q\rangle'(t) > 0$ for $t > 0$ and $\langle Q\rangle$ is an increasing function. Now

$$\inf_{x\in\mathbb{R}} l^2(x) = \inf_{x\in\mathbb{R}} p_Z'(p_Z^{-1}(x))^2 = \inf_{x\in\mathbb{R}} e^{-4\int_0^{p_Z^{-1}(x)} \tilde{f}(z)dz} = e^{-4\sup_{x\in\mathbb{R}}\int_0^x \tilde{f}(z)dz} > 0.$$

Similarly, $\sup_{x\in\mathbb{R}} l^2(x) = e^{-4\inf_{x\in\mathbb{R}}\int_0^x \tilde{f}(z)dz} < \infty$. Let $l_1^2 = \inf_{x\in\mathbb{R}} l^2(x)$ and $l_2^2 = \sup_{x\in\mathbb{R}} l^2(x)$, so for all $t \geq 0$,

$$l_1^2 t \leq \langle Q\rangle(t) \leq l_2^2 t, \tag{5.13}$$

which implies $\lim_{t\to\infty}\langle Q\rangle(t) = \infty$ almost surely. Now define, for each $0 \leq s < \infty$, the stopping time $\kappa(s) = \inf\{t \geq 0; \langle Q\rangle(t) > s\}$. It is obvious that κ is continuous and tends to infinity almost surely. Furthermore $\langle Q\rangle(\kappa(t)) = t$, and $\kappa^{-1}(t) = \langle Q\rangle(t)$ for $t \geq 0$. Then the time-changed process $\widetilde{W}(t) := Q(\kappa(t))$ is a standard one-dimensional Brownian motion with respect to the filtration $\mathcal{J}(t) := \mathcal{G}(\kappa(t))$. Hence we have

$$\widetilde{H}(t) := H(\kappa(t)) = H(\kappa(0)) + \int_0^{\kappa(t)} l(H(s))dW(s) = \widetilde{H}(0) + \widetilde{W}(t)$$

where $\widetilde{H}$ is $\mathcal{J}(t)$-adapted. So the law of the iterated logarithm holds for $\widetilde{H}$, that is

$$1 = \limsup_{t\to\infty} \frac{H(\kappa(t))}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{H(t)}{\sqrt{2\langle Q\rangle(t)\log\log\langle Q\rangle(t)}}, \quad \text{a.s.}$$

Note by (5.13) for all $t \geq 0$, that $\log l_1^2 + \log t \leq \log\langle Q\rangle(t) \leq \log l_2^2 + \log t$, so we have

$$\lim_{t\to\infty} \frac{\log\log\langle Q\rangle(t)}{\log\log t} = 1, \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{H(t)}{\sqrt{2\langle Q\rangle(t)\log\log t}} = 1, \quad \text{a.s.}$$

Similarly

$$\liminf_{t\to\infty} \frac{H(t)}{\sqrt{2\langle Q\rangle(t)\log\log t}} = -1, \quad \text{a.s.}$$

Now as $\langle Q\rangle(t) \leq l_2^2 t$, we have

$$\limsup_{t\to\infty} \frac{H(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \sqrt{\frac{\langle Q\rangle(t)}{t}} \cdot \frac{H(t)}{\sqrt{2\langle Q\rangle(t)\log\log t}} \leq l_2, \quad \text{a.s.}$$

Similarly

$$\limsup_{t\to\infty} \frac{H(t)}{\sqrt{2t\log\log t}} \geq l_1, \quad \text{a.s.}$$

and

$$-l_2 \leq \liminf_{t\to\infty} \frac{H(t)}{\sqrt{2t\log\log t}} \leq -l_1, \quad \text{a.s.}$$

Combining the above results with (5.12), we get

$$e^{2k_1}l_1 \le \limsup_{t\to\infty} \frac{Z_1(t)}{\sqrt{2t\log\log t}} \le e^{2k_1}l_2, \quad \text{a.s.}$$
$$-e^{-2k_2}l_2 \le \liminf_{t\to\infty} \frac{Z_1(t)}{\sqrt{2t\log\log t}} \le -e^{-2k_2}l_1, \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{X(\lambda(t))}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{\widetilde{X}(t)}{\sqrt{2t\log\log t}} \le \limsup_{t\to\infty} \frac{Z_1(t)}{\sqrt{2t\log\log t}}$$
$$\le \frac{e^{-2\inf_{x\in\mathbb{R}}\int_0^x \tilde{f}(y)\,dy}}{e^{-2\int_0^\infty \tilde{f}(y)\,dy}}, \quad \text{a.s.}$$

Similarly,

$$\liminf_{t\to\infty} \frac{X(\lambda(t))}{\sqrt{2t\log\log t}} \le \frac{-e^{-2\sup_{x\in\mathbb{R}}\int_0^x \tilde{f}(y)dy}}{e^{2\int_{-\infty}^0 \tilde{f}(y)dy}}, \quad \text{a.s.}$$

By an analogous argument to that given in the proof of Theorem 3.1, we get

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le \frac{\sqrt{K_1}e^{-2\inf_{x\in\mathbb{R}}\int_0^x \tilde{f}(y)\,dy}}{e^{-2\int_0^\infty \tilde{f}(y)dy}}, \quad \text{a.s.},$$
$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le \frac{-\sqrt{K_2}e^{-2\sup_{x\in\mathbb{R}}\int_0^x \tilde{f}(y)dy}}{e^{2\int_{-\infty}^0 \tilde{f}(y)dy}}, \quad \text{a.s.}$$

By considering Z_2 in a similar manner, we deduce the lower estimates on $\limsup$ and $\liminf$ of X in (3.11).

6. **Proofs of Theorems from Section 4**

6.1. **Proof of Theorem 4.1.** Combining (4.8) and (4.7), we have

$$dS(t) = [\mu S(t) + f(X(t), Y(t), t)S(t)]\,dt + \sigma S(t)\,dB(t) \quad t \ge 0.$$

Thus $S(t) = s_0 e^{(\mu-\frac{\sigma^2}{2})t+X(t)}$, $t \ge 0$, which implies $\log S(t)/t = \log s_0/t + \mu - \sigma^2/2 + X(t)/t$. Now by Corollary 3.3, we have $\lim_{t\to\infty} X(t)/t = 0$, a.s. Therefore by letting $t\to\infty$, the second part of the conclusion is obtained. Since $X(t) = \log S(t) - \log s_0 - (\mu - \frac{\sigma^2}{2})t$, $t \ge 0$, also by Corollary 3.3, we get the third part of the conclusion. For the first part, we observe that the assertion is equivalent to

$$\limsup_{t\to\infty} \frac{|X_\Delta(t)|}{\sqrt{2\log t}} = |\sigma|\sqrt{\Delta}, \quad \text{a.s.}$$

where $X_\Delta(t) = \int_{t-\Delta}^t f(X(s), Y(s), s)\,ds + \sigma(B(t) - B(t-\Delta))$. Now since for all $(x, y, t) \in \mathbb{R}\times\mathbb{S}\times\mathbb{R}^+$, $-\rho/|x| < f(x, y, t) < \rho/|x|$, then for any $y \in \mathbb{S}$ we have $\lim_{|x|\to\infty} f(x, y, t) = 0$. Also because f is continuous on $\mathbb{R}$, there exists a global upper bound, say K, such that for all $(x, y, t) \in \mathbb{R}\times\mathbb{S}\times\mathbb{R}^+$, $|f(x, y, t)| < K$. Thus

$$\lim_{t\to\infty} \frac{\int_{t-\Delta}^t f(X(s), Y(s), s)ds}{\sqrt{2\log t}} = 0, \quad \text{a.s.}$$

Hence it remains to show that

$$\limsup_{t\to\infty} \frac{|B(t)-B(t-\Delta)|}{\sqrt{2\log t}} = \sqrt{\Delta}, \quad \text{a.s.}$$

Consider $Z_\Delta(n) := (B(n)-B(n-\Delta))/\sqrt{\Delta}$, $n \in \mathbb{N}$. Then $\{Z_\Delta(n)\}_{n\in\mathbb{N}}$ is a sequence of standard normal random variables. For every $\varepsilon > 0$, Mill's estimate gives

$$\mathbb{P}[|Z_\Delta(n)| > \sqrt{2(1+\varepsilon)\log n}] \leq \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{2(1+\varepsilon)\log n}} \frac{1}{n^{1+\varepsilon}}.$$

Therefore by the Borel-Cantelli lemma and by letting $\varepsilon \downarrow 0$ through the rational numbers, we have $\limsup_{n\to\infty} |Z_\Delta(n)|/\sqrt{2\log n} \leq 1$, a.s. Now choose $\{n_i\}_{i\in\mathbb{N}}$ in $\{n\}$ such that for any fixed Δ and $i \in \mathbb{N}$, $n_{i+1} > n_i + \Delta$. So $\{Z_\Delta(n_i)\}_{n,i\in\mathbb{N}}$ is a sequence of independent $\mathcal{N}(0,1)$ random variables. By (2.6),

$$\lim_{i\to\infty} \frac{\mathbb{P}[|Z_\Delta(n_i)| > \sqrt{2\log n_i}]}{\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\log n_i}}\frac{1}{n_i}} = 1. \tag{6.1}$$

Since the denominator of the left–hand side of (6.1) is not summable, using the Borel-Cantelli lemma again, we get $\limsup_{i\to\infty} |Z_\Delta(n_i)|/\sqrt{2\log n_i} \geq 1$ a.s., and so

$$\limsup_{n\to\infty} \frac{|Z_\Delta(n)|}{\sqrt{2\log n}} = 1, \quad \text{a.s.} \tag{6.2}$$

It immediately follows that

$$\limsup_{t\to\infty} \frac{|B(t)-B(t-\Delta)|}{\sqrt{2\log t}} \geq \limsup_{n\to\infty} \frac{|B(n)-B(n-\Delta)|}{\sqrt{2\log n}} = \sqrt{\Delta}, \quad \text{a.s.} \tag{6.3}$$

For the upper estimate, by the triangle inequality

$$|B(t)-B(t-\Delta)| \leq |B(t)-B(n^\varepsilon)| + |B(t-\Delta)-B(n^\varepsilon-\Delta)| + |B(n^\varepsilon)-B(n^\varepsilon-\Delta)| \tag{6.4}$$

where $\varepsilon \in (0,1)$. We now consider the first term on the right-hand side of the above inequality. By properties of Brownian motions,

$$\begin{aligned}\mathbb{P}[\sup_{n^\varepsilon \leq t \leq (n+1)^\varepsilon} |B(t)-B(n^\varepsilon)| > 1] &= 2\mathbb{P}[\sup_{0\leq t\leq \varepsilon \hat{n}^{\varepsilon-1}} B(t) > 1] \\ &= 2\mathbb{P}[B(\varepsilon\hat{n}^{\varepsilon-1}) > 1] = 2\left(1-\Phi(\frac{1}{\sqrt{\varepsilon\hat{n}^{\varepsilon-1}}})\right),\end{aligned}$$

where $\hat{n} \in [n, n+1]$. Again by Mill's estimate and the Borel-Cantelli lemma, we have

$$\limsup_{n\to\infty} \max_{t\in[n^\varepsilon,(n+1)^\varepsilon]} |B(t)-B(n^\varepsilon)| \leq 1 \quad \text{a.s., and} \tag{6.5}$$

$$\limsup_{n\to\infty} \max_{t\in[n^\varepsilon,(n+1)^\varepsilon]} |B(t-\Delta)-B(n^\varepsilon-\Delta)| \leq 1, \quad \text{a.s.} \tag{6.6}$$

Also by a similar argument as (6.2),

$$\limsup_{n\to\infty} \frac{|B(n^\varepsilon)-B(n^\varepsilon-\Delta)|}{\sqrt{2\log n}} = \sqrt{\Delta}, \quad \text{a.s.} \tag{6.7}$$

Therefore, combining the results from (6.4) to (6.7), for almost all $\omega \in \Omega$, if $n^\varepsilon \le t \le (n+1)^\varepsilon$ and $n > N(\omega)$, then

$$\frac{|B(t) - B(t-\Delta)|}{\sqrt{2\log t}} \le \frac{1}{\sqrt{2\varepsilon \log n}}[|B(t) - B(n^\varepsilon)| + |B(t-\Delta) - B(n^\varepsilon - \Delta)| + |B(n^\varepsilon) - B(n^\varepsilon - \Delta)|]$$

which implies $\limsup_{t\to\infty} |B(t)-B(t-\Delta)|/(\sqrt{2\log t}) \le \sqrt{\Delta}/\sqrt{\varepsilon}$ a.s. Finally, letting $\varepsilon \uparrow 1$ through the rational numbers, we obtain

$$\limsup_{t\to\infty} \frac{|B(t) - B(t-\Delta)|}{\sqrt{2\log t}} \le \sqrt{\Delta}, \quad \text{a.s.} \tag{6.8}$$

The proof is complete.

6.2. **Proof of Theorem 4.2.** To show the statements in part (i), we observe that

$$\log S(t) = \log S(0) + \mu t - \int_0^t \frac{1}{2}\gamma^2(Y(s))ds + X(t).$$

which implies

$$\frac{\log S(t)}{t} = \frac{\log S(0)}{t} + \mu - \frac{1}{2t}\int_0^t \gamma^2(Y(s))ds + \frac{X(t)}{t}.$$

Now by Corollary 3.3, we have $\lim_{t\to\infty} X(t)/t = 0$, a.s. while by the ergodic property of the Markov chain,

$$\lim_{t\to\infty} \frac{1}{2t}\int_0^t \gamma^2(Y(s))ds = \frac{\sigma_*^2}{2} \quad \text{a.s.}$$

Therefore by letting $t \to \infty$, the first assertion in part (i) is obtained. Since

$$\log S(t) - \Big(\mu t - \frac{1}{2}\int_0^t \gamma^2(Y(s))ds\Big) = \log S(0) + X(t),$$

also by Corollary 3.3, we get the second assertion in part (ii).

6.3. **Proof of Theorem 4.3.** Let $S_l(t) = \log S(t)$ and with R_δ as defined in (4.6), we have

$$S_l(t) = S_l(0) + \int_0^t \{\mu - \frac{1}{2}g^2(X(s), Y(s), s) + f(X(s), Y(s), s)\}\, ds + \int_0^t g(X(s), Y(s), s)\, dB(s), \tag{6.9}$$

$$R_\Delta(t) = \int_{t-\Delta}^t \{\mu - \frac{1}{2}g^2(X(s), Y(s), s) + f(X(s), Y(s), s)\}\, ds + \int_{t-\Delta}^t g(X(s), Y(s), s)\, dB(s). \tag{6.10}$$

Define $M_2(t) = \int_0^t g(X(s), Y(s), s)\, dB(s)$ and so $\langle M_2\rangle(t) = \int_0^t g^2(X(s), Y(s), s)\, ds$. As in the proof of Theorem 3.1, we invoke a time–change argument. We may define a new Brownian motion $\tilde{B}$ by $\tilde{B}(\langle M_2\rangle(t)) = M_2(t)$ where the stopping time $\theta(t)$

is defined by $\theta(t) = \inf\{s > 0 : \langle M_2\rangle(s) > t\}$. Since $g^2(x,y,t) \geq K_2$, we have $\theta(t) = \langle M_2\rangle^{-1}(t)$. Finally, $\tilde{B}(t) = M_2(\theta(t))$ is a $\mathcal{G}(t) := \mathcal{F}(\theta(t))$ Brownian motion. Set $\tilde{S}_l(t) = S_l(\theta(t))$, $\tilde{X}(t) = X(\theta(t))$, and $\tilde{Y}(t) = Y(\theta(t))$. Applying Problem 3.4.5 and Proposition 3.4.8 in [14] to (6.9), in the manner of the proof of Theorem 3.1, we arrive at

$$\tilde{S}_l(t) = \tilde{S}_l(0) + \int_0^t \frac{\mu - \frac{1}{2}g^2(\tilde{X}(u),\tilde{Y}(u),\tau(u)) + f(\tilde{X}(u),\tilde{Y}(u),\theta(u))}{g^2(\tilde{X}(s),\tilde{Y}(u),\theta(u))}\,du + \tilde{B}(t), \tag{6.11}$$

for all $t \geq 0$. By (3.1a),(3.1b) and the fact that $|f(x,y,t)| \leq \bar{f}$ for all $(x,y,t) \in \mathbb{R} \times \mathbb{S} \times [0,\infty)$, we have that the integrand in the Riemann integral in (6.11) is absolutely bounded by $C_1 > 0$. Therefore, there is a process c_Δ such that for $|c_\Delta(t)| \leq C_1\Delta$, we have $\tilde{S}_l(t) - \tilde{S}_l(t-\Delta) = c_\Delta(t) + \tilde{B}(t) - \tilde{B}(t-\Delta)$. Hence

$$\log(S(\theta(t))/S(\theta(t-\Delta))) = c_\Delta(t) + \tilde{B}(t) - \tilde{B}(t-\Delta).$$

Also, we have that $\Delta = \langle M_2\rangle(\theta(t)) - \langle M_2\rangle(\theta(t-\Delta)) = \int_{\theta(t-\Delta)}^{\theta(t)} g^2(X(s),Y(s),s)\,ds$, so that (3.1b) implies

$$\frac{\Delta}{K_1} \leq \theta(t) - \theta(t-\Delta) \leq \frac{\Delta}{K_2}. \tag{6.12}$$

This implies that θ_Δ defined by $\theta_\Delta(t) = \theta(t) - \theta(t-\Delta)$ obeys $\frac{\Delta}{K_1} \leq \theta_\Delta(t) \leq \frac{\Delta}{K_2}$. Using the definition of θ_Δ, we get

$$\frac{\log(S(\theta(t))/S(\theta(t)-\theta_\Delta(t))))}{\sqrt{2\log\theta(t)}} = \frac{c_\Delta(t)}{\sqrt{2\log t}}\cdot\frac{\sqrt{2\log t}}{\sqrt{2\log\theta(t)}} + \sqrt{\Delta}\frac{(\tilde{B}(t)-\tilde{B}(t-\Delta))/\sqrt{\Delta}}{\sqrt{2\log t}}\cdot\frac{\sqrt{2\log t}}{\sqrt{2\log\theta(t)}}.$$

Therefore, as $t/K_1 \leq \theta(t) \leq t/K_2$, and using (6.3) and (6.8), we have

$$\limsup_{t\to\infty}\frac{|\log(S(\theta(t))/S(\theta(t)-\theta_\Delta(t)))|}{\sqrt{2\log\theta(t)}} = \sqrt{\Delta}, \quad \text{a.s.}$$

Now, we have

$$\max_{\Delta/K_1\leq\delta\leq\Delta/K_2} R_\delta(\theta(t)) = \max_{\Delta/K_1\leq\delta\leq\Delta/K_2}\log\left(\frac{S(\theta(t))}{S(\theta(t)-\delta)}\right) \geq \log\left(\frac{S(\theta(t))}{S(\theta(t)-\theta_\Delta(t))}\right).$$

Combining the last two expressions, and using the fact that $\log\theta(t)/\log t \to 1$ as $t \to \infty$, we obtain

$$\limsup_{t\to\infty}\frac{\max_{0\leq\delta\leq\Delta/K_2}|R_\delta(t)|}{\sqrt{2\log t}} \geq \limsup_{t\to\infty}\frac{\max_{\Delta/K_1\leq\delta\leq\Delta/K_2}|R_\delta(t)|}{\sqrt{2\log t}} \geq \sqrt{\Delta}, \quad \text{a.s.} \tag{6.13}$$

To obtain an upper inequality, by (6.10), there exists a process $c_\Delta^{(2)}(t)$ such that $|c_\Delta^{(2)}(t)| \le C_2\Delta$ we have

$$\max_{0\le\delta\le\Delta} |R_\delta(t)| \le \max_{0\le\delta\le\Delta} c_\Delta^{(2)}(t) + \max_{0\le\delta\le\Delta} \left| \int_{t-\delta}^{t} g(X(s),Y(s),s)\,dB(s) \right|.$$

Therefore

$$\limsup_{t\to\infty} \frac{\max_{0\le\delta\le\Delta} |R_\delta(t)|}{\sqrt{2\log t}} \le \limsup_{t\to\infty} \frac{\max_{0\le\delta\le\Delta} \left| \int_{t-\delta}^{t} g(X(s),Y(s),s)\,dB(s) \right|}{\sqrt{2\log t}}.$$

Now, define $\bar\theta_\delta$ by $\bar\theta_\delta(t) = \langle M_2\rangle(t) - \langle M_2\rangle(t-\delta) \in [K_2\delta, K_1\delta]$. Then, as

$$\lim_{t\to\infty} \frac{\log\langle M_2\rangle(t)}{\log t} = 1,$$

we get

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\max_{0\le\delta\le\Delta} \left| \int_{t-\delta}^{t} g(X(s),Y(s),s)\,dB(s) \right|}{\sqrt{2\log t}} \\
&= \limsup_{t\to\infty} \frac{\max_{0\le\delta\le\Delta} \left| \tilde B(\langle M_2\rangle(t)) - \tilde B(\langle M_2\rangle(t) - \bar\theta_\delta(t)) \right|}{\sqrt{2\log \langle M_2\rangle(t)}} \\
&= \limsup_{T\to\infty} \frac{\max_{0\le\delta\le\Delta} \left| \tilde B(T) - \tilde B(T - \bar\theta_\delta(\theta(T))) \right|}{\sqrt{2\log T}},
\end{aligned}$$

where we used the substitution $\theta(T) = t$ at the last step. Now, as $\bar\theta_\delta(\theta(T)) \in [K_2\delta, K_1\delta]$, we have

$$\begin{aligned}
\max_{0\le\delta\le\Delta} |\tilde B(T) - \tilde B(T - \bar\theta_\delta(\theta(T)))| &\le \max_{0\le\delta\le\Delta} \max_{K_2\delta\le s\le K_1\delta} |\tilde B(T) - \tilde B(T-s)| \\
&\le \max_{0\le s\le K_1\Delta} |\tilde B(T) - \tilde B(T-s)|.
\end{aligned}$$

Hence, by defining $U_c(t) = \max_{0\le u\le c} |\tilde B(t) - \tilde B(t-u)|$, we get

$$\limsup_{t\to\infty} \frac{\max_{0\le\delta\le\Delta} |R_\delta(t)|}{\sqrt{2\log t}} \le \limsup_{t\to\infty} \frac{U_{K_1\Delta}(t)}{\sqrt{2\log t}}$$

We now determine the asymptotic behaviour of U_c. Since it is a result about the Brownian motion $\tilde B$, we can consider any Brownian motion W. First, fix $t \ge 0$, and let $m > 0$. Then

$$\begin{aligned}
&\mathbb{P}[U_c(t) \ge m] \\
&= \mathbb{P}[\{\max_{0\le u\le c} W(t) - W(t-u) \ge m\} \cup \{-\min_{0\le u\le c} W(t) - W(t-u) \ge m\}] \\
&\le \mathbb{P}[\max_{0\le u\le c} W(t) - W(t-u) \ge m] + \mathbb{P}[-\min_{0\le u\le c} W(t) - W(t-u) \ge m]. \qquad (6.14)
\end{aligned}$$

Now $\bar{W}$ defined by $\bar{W}(t) = W(t) - W(t-u)$, $0 \le u \le t$, is a standard Brownian motion. Thus, as $\max_{0\le u\le c} \bar{W}(u)$ has the same distribution as $|\bar{W}(c)|$,

$$\mathbb{P}[\max_{0\le u\le c} W(t) - W(t-u) \ge m] = \mathbb{P}[|\bar{W}(c)| \ge m] = 2\mathbb{P}\left[\frac{\bar{W}(c)}{\sqrt{c}} \ge \frac{m}{\sqrt{c}}\right] = 2\left(1 - \Phi\left(\frac{m}{\sqrt{c}}\right)\right). \quad (6.15)$$

Define $W^* = -\bar{W}$. Then $-\min_{0\le u\le c} \bar{W}(u) = \max_{0\le u\le c} W^*(u)$. But as W^* is also a standard Brownian motion, this has the same distribution as $|W^*(c)|$. Therefore

$$\mathbb{P}[-\min_{0\le u\le c} W(t) - W(t-u) \ge m] = \mathbb{P}[|W^*(c)| \ge m] = 2\left(1 - \Phi\left(\frac{m}{\sqrt{c}}\right)\right). \quad (6.16)$$

Combining (6.14), (6.15), (6.16) gives $\mathbb{P}[U_c(t) \ge m] \le 4\left(1 - \Phi\left(\frac{m}{\sqrt{c}}\right)\right)$. By (2.6), we have

$$\mathbb{P}[U_c(a_n) \ge \sqrt{2\log a_n}\sqrt{c(1+\eta)}] \le 4\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{1+\eta}\sqrt{2\log a_n}} \cdot \frac{1}{a_n^{1+\eta}}.$$

Hence, if $a_n \to \infty$ and if, for all $\eta > 0$ we have $\sum_{n=1}^{\infty} a_n^{-(1+\eta)} < \infty$, then

$$\limsup_{n\to\infty} \frac{U_c(a_n)}{\sqrt{2\log a_n}} \le \sqrt{c}, \quad \text{a.s.}$$

Next, let $a_n \le t \le a_{n+1}$. Then

$$\begin{aligned} |U_c(t) - U_c(a_n)| &\le \max_{0\le u\le c} ||W(t) - W(t-u)| - |W(a_n) - W(a_n - u)|| \\ &\le \max_{0\le u\le c} |W(t) - W(a_n) - (W(t-u) - W(a_n - u))| \\ &\le |W(t) - W(a_n)| + \max_{0\le u\le c} |W(t-u) - W(a_n - u))|. \end{aligned}$$

Therefore

$$\begin{aligned} &\max_{a_n\le t\le a_{n+1}} |U_c(t) - U_c(a_n)| \\ &\le \max_{a_n\le t\le a_{n+1}} |W(t) - W(a_n)| + \max_{a_n\le t\le a_{n+1}} \max_{0\le u\le c} |W(t-u) - W(a_n - u))| \\ &\le 2 \max_{\substack{a_n - c\le v<u\le a_{n+1} \\ 0\le u-v\le a_{n+1}-a_n}} |W(u) - W(v)|. \end{aligned}$$

We next notice that W' defined by $W'(t) = tW(1/t)$ for $t > 0$ and $W'(0) = 0$ is a standard Brownian motion. Now, suppose that $a_n - c \le v < u \le a_{n+1}$, $0 \le u - v \le a_{n+1} - a_n$, and define $u' = 1/u$, $v' = 1/v$. Then we have $W(u) - W(v) = u\left(W'(u') - W'(v')\right) + (u-v)W'(v')$. Therefore $v' - u' = (u-v)/uv \le (a_{n+1} -$

$a_n)/(a_{n+1}(a_n - c)) =: \delta_n > 0$, and we have

$$\begin{aligned}
&\max_{\substack{a_n - c \le v < u \le a_{n+1} \\ 0 \le u - v \le a_{n+1} - a_n}} |W(u) - W(v)| \\
&\le \max_{\substack{a_n - c \le v < u \le a_{n+1}, v' = 1/v, u' = 1/u \\ 0 \le u - v \le a_{n+1} - a_n}} u|W'(u') - W'(v')| + (u - v)|W'(v')| \\
&\le a_{n+1} \max_{\substack{1/a_{n+1} \le u' < v' \le (a_n - c)^{-1} \\ 0 \le v' - u' \le \delta_n}} |W'(u') - W'(v')| \\
&\qquad + (a_{n+1} - a_n) \max_{1/a_{n+1} \le v \le 1/(a_n - c)} |W'(v')|.
\end{aligned}$$

Next, fix $\varepsilon > 0$ and let $a_n = n\varepsilon$. Then $a_{n+1}/a_n \to 1$ as $n \to \infty$, so $\delta_n \to 0$ as $n \to \infty$. Now, as $1/a_{n+1} \to 0$ as $n \to \infty$ and $a_n - c > 0$ for all $n > N$, for $n > N$, we have

$$\begin{aligned}
&\frac{\max_{\substack{a_n - c \le v < u \le a_{n+1} \\ 0 \le u - v \le a_{n+1} - a_n}} |W(u) - W(v)|}{\sqrt{2 \log a_n}} \\
&\quad \le \frac{\max_{\substack{0 \le u' < v' \le 1 \\ 0 \le v' - u' \le \delta_n}} |W'(u') - W'(v')|}{\sqrt{2\delta_n \log(1/\delta_n)}} \cdot \sqrt{\frac{a_{n+1}^2 \delta_n \log(1/\delta_n)}{\log a_n}} \\
&\qquad + \frac{(a_{n+1} - a_n)}{\sqrt{2 \log a_n}} \max_{1/a_{n+1} \le v \le 1/(a_n - c)} |W'(v')|.
\end{aligned}$$

Since $a_{n+1} - a_n = \varepsilon$ and $a_{n+1}/a_n \to 1$ as $n \to \infty$, we have

$$\begin{aligned}
&\frac{a_{n+1}^2 \delta_n \log(1/\delta_n)}{\log a_n} \\
&\quad = \frac{a_{n+1}}{a_n - c}(a_{n+1} - a_n) \left(\frac{\log a_{n+1}}{\log a_n} + \frac{\log(a_n - c)}{\log a_n} - \frac{\log(a_{n+1} - a_n)}{\log a_n} \right).
\end{aligned}$$

Hence

$$\lim_{n \to \infty} \frac{a_{n+1}^2 \delta_n \log(1/\delta_n)}{\log a_n} = 2\varepsilon.$$

Since $\delta_n \to 0$ as $n \to \infty$, Lévy's result on the modulus of continuity of Brownian motion (see e.g., [14, Theorem 2.9.25]) implies

$$\lim_{n \to \infty} \frac{\max_{\substack{0 \le u' < v' \le 1 \\ 0 \le v' - u' \le \delta_n}} |W'(u') - W'(v')|}{\sqrt{2\delta_n \log(1/\delta_n)}} = 1, \quad \text{a.s.},$$

and so

$$\limsup_{n \to \infty} \frac{\max_{\substack{a_n - c \le v < u \le a_{n+1} \\ 0 \le u - v \le a_{n+1} - a_n}} |W(u) - W(v)|}{\sqrt{2 \log a_n}} \le \sqrt{2\varepsilon}, \quad \text{a.s.}$$

Hence

$$\limsup_{n \to \infty} \frac{\max_{a_n \le t \le a_{n+1}} |U_c(t) - U_c(a_n)|}{\sqrt{2 \log a_n}} \le 2\sqrt{2\varepsilon}, \quad \text{a.s.}$$

Now, for each $t > 0$ there exists $n(t) \in \mathbb{N}$ such that $a_{n(t)} \le t < a_{n(t)+1}$. Therefore, as $a_{n(t)} \le t$, we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{U_c(t)}{\sqrt{2\log t}} &\le \limsup_{t\to\infty} \frac{|U_c(t) - U_c(a_{n(t)})|}{\sqrt{2\log t}} + \frac{U(a_{n(t)})}{\sqrt{2\log t}} \\
&\le \limsup_{t\to\infty} \Bigg\{ \frac{\max_{a_{n(t)} \le s \le a_{n(t)+1}} |U_c(s) - U_c(a_{n(t)})|}{\sqrt{2\log a_{n(t)}}} \cdot \frac{\sqrt{\log a_{n(t)}}}{\sqrt{\log t}} \\
&\qquad + \frac{U(a_{n(t)})}{\sqrt{2\log a_{n(t)}}} \cdot \frac{\sqrt{\log a_{n(t)}}}{\sqrt{\log t}} \Bigg\} \\
&\le 2\sqrt{2\varepsilon} + \sqrt{c}, \quad \text{a.s.}
\end{aligned}$$

Letting $\varepsilon \to 0^+$ through the rational numbers gives $\limsup_{t\to\infty} U_c(t)/\sqrt{2\log t} \le \sqrt{c}$, a.s. Therefore

$$\limsup_{t\to\infty} \frac{\max_{0\le\delta\le\Delta} |R_\delta(t)|}{\sqrt{2\log t}} \le \limsup_{t\to\infty} \frac{U_{K_1\Delta}(t)}{\sqrt{2\log t}} \le \sqrt{K_1}\sqrt{\Delta}, \quad \text{a.s.}$$

as required.

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