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ABSTRACT STOCHASTIC PROBLEMS WITH GENERATORS OF REGULARIZED SEMIGROUPS

IRINA V. MELNIKOVA[1] AND ALEXEI FILINKOV[2]

[1]Department of Mathematics, Ural State University, Ekaterinburg 620083, Russia

[2]School of Mathematical Sciences, University of Adelaide
Adelaide SA 5005, Australia

ABSTRACT. We study the stochastic Cauchy problem

$$dX(t) = AX(t)dt + BdW(t),\ t \in [0, T], \quad X(0) = \xi,$$

where A is the generator of a regularized semigroup in a separable Hilbert space H, B is a bounded linear operator and W is an H-valued Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. We construct regularized solutions to this problem in $L^2(\Omega; H)$ and in spaces of abstract stochastic distributions. We also study the semi-linear problem

$$dX(t) = \big[AX(t) + F(t, X)\big]dt + B(t, X)dW(t),\ X(0) = \xi,$$

where F and B satisfy some appropriate growth and Lipschitz conditions.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t,\ t \geq 0\}$ and H be a separable Hilbert space. Let Q be a linear symmetric nonnegative trace-class operator on H, then there is an orthonormal basis $\{e_j\}$ in H such that $Qe_j = \lambda_j e_j$.

We consider the stochastic Cauchy problem

$$dX(t) = AX(t)dt + BdW(t),\ t \in [0, T], \quad X(0) = \xi, \tag{1.1}$$

where A is the generator of a regularized semigroup $\{S(t), t \in [0, \tau)\}$, $T < \tau \leq \infty$, in H, B is a bounded linear operator in H, and $W = \{W(t), t \geq 0\}$ is an H-valued Wiener process.

Problem (1.1) with the generator of a C_0-semigroup was studied by semigroup methods by many authors, see [4, 13] and references therein. In this paper we consider a much wider class of operators A that do not necessarily generate C_0-semigroups (that is semigroups of class C_0). Typical examples of regularized semigroups include integrated semigroups, convoluted semigroups and R-semigroups (see, for example,

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[2, 5, 17, 12, 16]). Problem (1.1) with the generator of an integrated semigroup was also studied in [6, 7, 13, 14].

Section 2 of this paper is devoted to the basic facts related to the deterministic Cauchy problem

$$u'(t) = Au(t), \quad t \in [0,T], \qquad u(0) = \xi, \tag{1.2}$$

with the generator of a regularized semigroup. In Section 2.1 we give definitions and discuss properties of regularized semigroups, k-convoluted semigroups and R-semigroups. The main result of this section is devoted to adjoint regularized semigroups. This result is used in Section 3 for studying the stochastic problem (1.1). In Section 2.2 we give some examples of k-convoluted semigroups and R-semigroups and their generators, in particular, semigroups generated by differential operators.

In Section 3 we construct regularized solutions to problem (1.1) both in $L^2(\Omega; H)$ and in spaces of abstract stochastic distributions. Firstly, we introduce the notion of a weak regularized solution to problem (1.1), where A is the generator of a regularized semigroup $\{S(t),\ t \in [0,\tau)\}$ in H, $\{W(t),\ t \geq 0\}$ is an H-valued Q-Wiener process, B is a bounded linear operator in H and ξ is a $\mathcal{F}_0$-measurable H-valued random variable. The main result of Section 3.1 is devoted to the existence and uniqueness of such solutions in $L^2(\Omega; H)$. We use the stochastic version of the variation of constants formula and the notion of a stochastic convolution to construct these solutions.

In Section 3.2 we extend our discussion to the case of a cylindrical H-valued Wiener process. We discuss the existence and uniqueness of regularized solutions to problem (1.1) in $L^2(\Omega; H)$ and in spaces of abstract stochastic distributions. For basic facts about white noise analysis and H-valued stochastic distributions we refer readers to [9, 10, 11, 7, 14].

Section 3.3 is devoted to the semi-linear problem

$$dX(t) = \big[AX(t) + F(t,X)\big]dt + B(t,X)dW(t),\ X(0) = \xi,$$

where A is the generator of a regularized semigroup $\{S(t), t \in [0,\tau)\}$ in H and W is an H-valued Q-Wiener process. We suppose that $F : [0,T]\times\Omega\times H \to H$ and $B : [0,T]\times\Omega\times H \to \mathcal{L}_2$ are measurable mappings from $([0,T]\times\Omega\times H,\ \mathcal{P}_T\times\mathcal{B}(H))$ to $(H,\ \mathcal{B}(H))$ and $(\mathcal{L}_2,\ \mathcal{B}(\mathcal{L}_2))$ respectively, and that they satisfy the Lipschitz conditions:

$$\|F(t,\omega;x) - F(t,\omega;y)\| + \|B(t,\omega;x) - B(t,\omega;y)\|_{\mathcal{L}_2} \leq C\,|x-y|$$

for $x, y \in H,\ t \in [0,T],\ \omega \in \Omega$, and the growth conditions:

$$\|F(t,\omega;x)\|^2 + \|B(t,\omega;x)\|^2_{\mathcal{L}_2} \leq C(1+|x|^2)$$

for $x \in H,\ t \in [0,T],\ \omega \in \Omega$. Here $\mathcal{P}_T$ is a predictable σ-field on $[0,T]\times\Omega$ and $\mathcal{L}_2 = \mathcal{L}_2(H_1, H)$ is the space of all Hilbert-Schmidt operators from $H_1 = Q^{\frac{1}{2}}H$ into H, where inner product in H_1 is defined by $\langle Q^{\frac{1}{2}}u, Q^{\frac{1}{2}}v\rangle_{H_1} = \langle u, v\rangle_H$.

We introduce the notion of a mild regularized solution to the semi-linear problem and investigate the existence and uniqueness of such solutions.

2. REGULARIZED SEMIGROUPS

2.1. Definitions and properties of regularized semigroups, k-convoluted semigroups and R-semigroups.

Definition 2.1. Let A be a closed linear operator and $R(t),\ t \geq 0$, be bounded linear operators on a Banach space H.

A strongly continuous family of bounded linear operators $S := \{S(t),\ t \in [0,\tau)\}$, $\tau \leq \infty$, is called a *regularized (R-regularized) semigroup* with generator A if

$$S(t)A\xi = AS(t)\xi \quad \text{for} \quad \xi \in \operatorname{dom} A$$

and

$$S(t)\xi = A\int_0^t S(s)\xi ds + R(t)\xi \quad \text{for} \quad \xi \in H\,. \tag{2.1}$$

Semigroup S is called *exponentially bounded* if $\|S(t)\| \leq Me^{\omega t}, t \geq 0$, for some $M > 0$ and $\omega \in \mathbb{R}$. We say that S is *local* if $\tau < \infty$.

Let k be a continuous real valued function. If $R(t) = I\int_0^t k(s)ds$, then S is called a *k-convoluted semigroup* (see, for example, [2, 3]).

If operator A is densely defined and $R(t) \equiv R$, where R is an invertible bounded linear operator with dense range, then S is called an *R-semigroup*.

Note that if $k(t) = t^{n-1}/(n-1)!$, then k-convoluted semigroup is an *n-times integrated semigroup*. If $R = I$, then R-semigroup is a *semigroup of class C_0*.

Usually (see [5, 17]) R-semigroups are introduced as a strongly continuous family of bounded operators satisfying the R-semigroup relation:

(R1) $S(t+s)R = S(t)S(s),\ \ s,t,s+t \in [0,\tau), \quad S(0) = R,$

with infinitesimal generators:

$$Gf := \lim_{t\to 0}\frac{S(t)R^{-1} - I}{t}f\,, \quad \operatorname{dom} G = \left\{f \in \operatorname{ran} R:\ \exists \lim_{t\to 0}\frac{S(t)R^{-1} - I}{t}f\right\},$$

$$Zf := R^{-1}\lim_{t\to 0}\frac{S(t) - R}{t}f\,, \quad \operatorname{dom} Z = \left\{f \in H:\ \lim_{t\to 0}\frac{S(t) - R}{t}f \in \operatorname{ran} R\right\}.$$

(In [5, 17] such semigroups are called C-semigroups. In order to avoid confusion with C_0-semigroups we use the term R-semigroups.)

The following result is due to the connection between the R-semigroup relation (R1) and the homogeneous Cauchy problem (1.2).

Proposition 2.2. *Let A be a densely defined closed linear operator on a Banach space H. Then a strongly continuous operator-family $\{S(t) \in \mathcal{L}(H),\ t \in [0, \tau)\}$ is an R-semigroup with generator A if and only if it satisfies relation* (R1). *In this case $A = \overline{G}$.*

This proposition clearly gives us an equivalent definition of an R-semigroup. The corresponding semigroup relation for k-convoluted semigroups [3]:

(k1) $S(t)S(s) = \int_s^{t+s} k(t+s-r)S(r)\,dr - \int_0^t k(t+s-r)S(r)\,dr, \quad t, s, t+s \in [0;\tau),$

generally is not used as their definition.

We note that defining semigroups via relations (R1), (k1) emphasizes the structural properties of semigroups. On the other hand, Definition 2.1 shows the connection between a regularized semigroup and a Cauchy problem with operator A being the generator of this semigroup. For example, if S is an R-semigroup, then $u(\cdot) = S(\cdot)\xi,\ \xi \in \operatorname{dom} A$, is a solution to (1.2) with the initial value $u(0) = R\xi$.

The following result on adjoint regularized semigroups will be useful for studying stochastic Cauchy problems.

Theorem 2.3. *Let A be the generator of an R-regularized semigroup $\{S(t), t \in [0, \tau)\}$ on a Hilbert space H. Suppose that family $\{R(t)\}$ is strongly differentiable and $\overline{\operatorname{dom} A} = H$. Then the $\{S^*(t), t \in [0, \tau)\}$ is an R^*-regularized semigroup on H with generator A^*. If operators $R(t)$ are invertible and with dense ranges, then adjoint operators $R^*(t)$ have the same properties.*

Proof. Firstly, we note that since operators $R(t)$ and $S(t)$ are bounded for each t, then their adjoint operators $R^*(t)$ and $S^*(t)$ are bounded too. Secondly, since A a closed densely defined operator, then it is well-known (see [1], for example) that operator A^* is closed and $\overline{\operatorname{dom} A^*} = H$.

Next we show that the family $\{S^*(t),\ t \in [0, \tau)\}$ forms an R^*-regularized semigroup with the generator A^*. The commutativity of operators $S^*(t)$ with A^* on $\operatorname{dom} A^*$ follows from the commutativity of $S(t)$ with A.

We need to prove that family $\{S^*(t),\ t \in [0, \tau)\}$ is strongly continuous in t and the following equality

$$S(t)^*y - R^*(t)y = \int_0^t S^*(s)A^*y\,ds, \quad y \in \operatorname{dom} A^*\,. \tag{2.2}$$

Due to continuity of the scalar product, equation (2.1) implies

$$\begin{aligned} \langle S(t)f - R(t)f, y\rangle &= \langle f, S(t)^*y - R^*(t)y\rangle = \langle \int_0^t S(s)Af\,ds, y\rangle \qquad (2.3) \\ &= \int_0^t \langle AS(s)f, y\rangle ds = \int_0^t \langle S(s)f, A^*y\rangle ds \end{aligned}$$

for each $y \in \mathrm{dom}\ A^*$ and $f \in H = \overline{\mathrm{dom}\ A}$. Then we have

$$\begin{aligned} \frac{d}{dt}\langle f, S^*(t)y\rangle &= \lim_{\Delta t\to 0}\left\langle f, \frac{S^*(t+\Delta t)-S^*(t)}{\Delta t}y\right\rangle \\ &= \langle f, S^*(t)A^*y\rangle + \langle R'(t)f, y\rangle \\ &= \langle S(t)f, A^*y\rangle + \langle R'(t)f, y\rangle, \quad f \in H,\ y \in \mathrm{dom}\, A^*, \end{aligned} \tag{2.4}$$

which implies weak convergence of

$$\frac{S^*(t+\Delta t)-S^*(t)}{\Delta t}y$$

as $\Delta t \to 0$. Hence for any $t \in [0,\tau)$ and $y \in \mathrm{dom}A^*$

$$\left\|\frac{S^*(t+\Delta t)-S^*(t)}{\Delta t}y\right\|$$

is uniformly bounded for all Δt such that $t+\Delta t \in [0,\tau_1],\ \tau_1 < \tau$. Therefore

$$\|S^*(t+\Delta t)y - S^*(t)y\| \to 0 \quad \text{for} \quad y \in \mathrm{dom}\, A^* \quad \text{as} \quad \Delta t \to 0\,.$$

Since norms $\|S^*(t+\Delta t)\| = \|S(t+\Delta t)\|$ are uniformly bounded for $t+\Delta t \in [0,\tau_1]$, $\tau_1 < \tau$, then by the Banach-Steinhaus theorem, $S^*(t)y$ is continuous in t for any $y \in H = \overline{\mathrm{dom}\, A^*}$.

Now strong continuity of $S^*(s), s \in [0,\tau)$, and equality (2.3) imply

$$\begin{aligned} \langle f, S(t)^*y - R^*(t)y\rangle &= \int_0^t \langle f, A^*S^*(s)y\rangle \\ &= \left\langle f, \int_0^t S^*(s)A^*y\right\rangle, \quad f \in H,\ y \in \mathrm{dom}\, A^*, \end{aligned} \tag{2.5}$$

which proves (2.2).

Finally, we show that operators $R^*(t)$, $t \in [0,\tau)$, are invertible and have dense ranges if the operators $R(t)$ are invertible with dense ranges. For any bounded operator R we have the equality $(\ker R)^\perp = \overline{\mathrm{ran}\, R^*}$. If $R(t)$ is invertible, then $\ker R(t) = \{0\}$, and therefore $\overline{\mathrm{ran}\, R^*(t)} = H$. In addition,

$$(\ker R^*(t))^\perp = \overline{\mathrm{ran}\, R^{**}(t)} = H\,,$$

and hence $R^*(t)$ is invertible. □

We finish this section with a remark that although k-convoluted semigroups and R-semigroups share a lot of common properties as special cases of R-regularized semigroups, they have different spectral properties. The generator of a k-convoluted semigroup has a resolvent $\mathcal{R}(\lambda),\ \lambda \in \Lambda$. In the local case the resolvent exists in the region $\Lambda = \Lambda^M_{\alpha,\gamma,\beta} = \{\lambda \in \mathbb{C} :\ \Re\lambda > \alpha M(\gamma|\lambda|) + \beta\}$ and satisfies the estimate:

$$\|\mathcal{R}(\lambda)\| \le Ce^{\beta M(\gamma|\lambda|)}, \qquad \lambda \in \Lambda^M_{\alpha,\gamma,\beta}, \tag{2.6}$$

where function M and parameters α, γ, β depend on k and τ. The inverse result is also true: this resolvent estimate implies the existence of a local k-convoluted semigroup $\{S(t), t \in [0,\tau)\}$ with k and τ depending on the estimate parameters.

Theorem 2.4. [3, 12] *Let $M(s)$, $s \geq 0$, be a positive function increasing as $s \to \infty$ and its growth doesn't exceed s^p, $p < 1$. Let the resolvent of A satisfies* (2.6) *with some parameters γ, α, β. Then A generates a local k–convoluted semigroup $\{S(t),\, t \in [0, \frac{\delta-\beta}{\alpha})\}$ with k whose Laplace transform satisfies the condition*

$$|\widetilde{k}(\lambda)| = O_{|\lambda|\to\infty}\left(e^{-\delta M(\gamma|\lambda|)}\right), \quad \delta > \beta. \tag{2.7}$$

In contrast to k-convoluted semigroups, the generator of an R-semigroup generally has no resolvent. The notion of a regularized resolvent is usually used in this case.

2.2. **Examples of regularized semigroups and their generators.** We now give some examples of k-convoluted semigroups and R-semigroups and their generators, in particular, semigroups generated by differential operators. More examples can be found in [1, 12, 16].

Example 2.5. (A differential operator-matrix that can generate a semigroup of class C_0, a convoluted (integrated) semigroup or an R-semigroup)

Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Consider the following system of differential equations

$$\left\{ \begin{array}{lcl} \dfrac{\partial u_1(x;t)}{\partial t} & = & \dfrac{\partial^2 u_1(x;t)}{\partial x^2} \\[2ex] \dfrac{\partial u_2(x;t)}{\partial t} & = & i^m \, \dfrac{\partial^m u_1(x;t)}{\partial x^m} + \dfrac{\partial^2 u_2(x;t)}{\partial x^2} \end{array} \right. \qquad x \in \mathbb{R},\ t \geq 0\,, \tag{2.8}$$

with the initial conditions $u_1(x;0) = \xi_1(x),\ u_2(x;0) = \xi_2(x),\ x \in \mathbb{R}$. The problem can be written in the abstract form (1.2):

$$u'(t) = Au(t), \quad t \geq 0, \qquad u(0) = \xi,$$

in the Hilbert space $H = L_2(\mathbb{R}) \times L_2(\mathbb{R})$, where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad A = \begin{pmatrix} \dfrac{d^2}{dx^2} & 0 \\[2ex] i^m \dfrac{d^m}{dx^m} & \dfrac{d^2}{dx^2} \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Applying Fourier transform to the equation and initial data we obtain the Cauchy problem

$$\begin{cases} \dfrac{d\widetilde{u}_1(s;t)}{dt} = -s^2\,\widetilde{u}_1(s;t) \\ \\ \dfrac{d\widetilde{u}_2(s;t)}{dt} = s^m\,\widetilde{u}_1(s;t) - s^2\,\widetilde{u}_2(s;t) \end{cases},\ s\in\mathbb{C},\ t\geq 0, \quad \begin{cases} \widetilde{u}_1(s;0)=\widetilde{\xi}_1(s) \\ \widetilde{u}_2(s;0)=\widetilde{\xi}_2(s) \end{cases}. \tag{2.9}$$

We are looking for a solution of (2.9) of the form $\widetilde{u}(s;t)=e^{tA(s)}\widetilde{\xi}(s)$, where

$$e^{tA(s)}=\sum_{k=1}^{\infty}\frac{t^kA^k(s)}{k!} \quad \text{and} \quad A(s):=\begin{pmatrix} -s^2 & 0 \\ s^m & -s^2 \end{pmatrix}.$$

We obtain

$$\widetilde{u}(s;t)=e^{tA(s)}\,\widetilde{\xi}(s)=\begin{pmatrix} e^{-ts^2} & 0 \\ ts^m\,e^{-ts^2} & e^{-ts^2} \end{pmatrix}\begin{pmatrix} \widetilde{\xi}_1(s) \\ \widetilde{\xi}_2(s) \end{pmatrix}=e^{-ts^2}\begin{pmatrix} 1 & 0 \\ ts^m & 1 \end{pmatrix}\begin{pmatrix} \widetilde{\xi}_1(s) \\ \widetilde{\xi}_2(s) \end{pmatrix}.$$

The solution of the original problem (2.8) is given by the following convolution:

$$u(x;t)=G(x;t)*\xi(x)=:U(t)\xi, \qquad x\in\mathbb{R},\ t\geq 0, \tag{2.10}$$

where

$$G(x;t)=\begin{pmatrix} \dfrac{1}{2\sqrt{\pi t}}\,e^{-\frac{x^2}{4t}} & 0 \\ \\ \dfrac{i^m}{2}\sqrt{\dfrac{t}{\pi}}\dfrac{\partial^m}{\partial x^m}\left(e^{-\frac{x^2}{4t}}\right) & \dfrac{1}{2\sqrt{\pi t}}\,e^{-\frac{x^2}{4t}} \end{pmatrix}$$

is the inverse Fourier transform of $e^{A(s)t}$ and operators $U(t)$ act from $\operatorname{dom} U(t)\subset H$ to H.

Due to the Plancherel theorem we have $\|f\|=\|\widetilde{f}\|$ for all $f\in H$. Hence boundedness of operators $U(t)$, $t\geq 0$, and $(\lambda I-A)^{-n}$, $\lambda\in\Lambda\subset\mathbb{C}$, can be expressed in terms of estimates for $e^{A(\cdot)t}$ and $(\lambda I-A(\cdot))^{-n}$. In [12, 16] the following estimates were obtained for the operator-matrix

$$\left\|e^{-t(1+x^2)}\begin{pmatrix} 1 & 0 \\ -tx^{2\gamma} & 1 \end{pmatrix}\right\|\leq \max\{e^{-t};\gamma^\gamma t^{1-\gamma}e^{-t-\gamma}\}$$

in spaces $L_p(\mathbb{R})\times L_p(\mathbb{R})$, $p\geq 1$. These imply that for any $m\in\mathbb{N}_0$ solution operators of problem (2.8) form a strongly continuous family $\{U(t),\ t>0\}$ of bounded operators on $H=L_2(\mathbb{R})\times L_2(\mathbb{R})$.

Further,

(a): if $0\leq m\leq 2$, then operators $U(t)$ are bounded for all $t\geq 0$ and the following estimates

$$\left\|\frac{1}{n!}\cdot\frac{d^n}{d\lambda^n}(\lambda I-A)^{-1}f\right\|\leq\frac{2}{\lambda^{n+1}}\|f\|, \quad n\in\mathbb{N}_0,\ f\in H,$$

guarantee that $\{U(t),\ t \geq 0\}$ is a semigroup of class C_0;

(b): if $m > 2$, then $\|U(t)\| \leq Ct^{1-m/2}$, i.e. U is a semigroup of growth order $\alpha = m/2 - 1$ and hence is an R-semigroup with $R = (\lambda I - A)^{-n},\ n = [\alpha] + 1$; in particular:

(c): for $m = 3$ the singularity of $U(t)$ at $t = 0$ is integrable, and A generates an (exponentially bounded) 1-time integrated semigroup or, in other words, a k-convoluted semigroup with $k(t) = t$. In this case the operators $(\lambda I - A)^{-1}$ are bounded for $\lambda > 0$, and, therefore, the resolvent of A is defined;

(d): for $m \geq 4$ the operators $U(t)$ may have a non-integrable singularity at $t = 0$ and operators $(\lambda I - A)^{-n}$, in general, are not powers of the resolvent.

Example 2.6. (A local R-semigroup)

Let $H = L_2(\mathcal{O}),\ \mathcal{O} = \left\{x \in \mathbb{R}^N\ ; 0 < x_k < a_k\ ,\ k = 1, \dots, N\right\}$. Define

$$Au = \triangle u, \quad u \in \operatorname{dom} A := H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}), \tag{2.11}$$

where the Laplace operator $\triangle$ is understood in the sense of distributions. The spectrum of A consists of its eigenvalues

$$Sp(A) = \left\{ -\sum_{i=1}^{N} \frac{k_i^2 \pi^2}{a_i^2};\ k_i \in \mathbb{N} \right\}$$

with eigenfunctions

$$w_{k_1,k_2,\dots,k_N} = \prod_{i=1}^{N} \left(\frac{2}{a_i}\right)^{1/2} \left(\sin \frac{k_i \pi x_i}{a_i}\right) .$$

To simplify notation, we denote by $\{-\mu_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ an ordering of the eigenvalues and eigenbasis of A, respectively. Operator A generates a semigroup $\{U(t), t \geq 0\}$ of class C_0 on $L_2(\mathcal{O})$ given by

$$U(t)\xi = \sum_{k=1}^{\infty} \langle \xi, e_k \rangle_{L_2(\mathcal{O})} e^{-\mu_k t} e_k. \tag{2.12}$$

Hence $U(\cdot)\xi,\ \xi \in \operatorname{dom} A$, is a solution of the well-posed Cauchy problem (1.2) for the heat equation. The ill-posed Cauchy problem with operator $-A$:

$$u'(t) = -Au(t), \quad t \in [0, T], \qquad u(0) = f,$$

corresponds to the ill-posed backward Cauchy problem for the heat equation, and $-A$ is the generator of the following local R-semigroups :

$$S_1(t)f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e^{\mu_k t - \alpha \mu_k^n T} e_k, \quad S_2(t)f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e^{\mu_k t} \left(\gamma + e^{\mu_k T}\right)^{-1} e_k,\ t \leq T, \tag{2.13}$$

with the corresponding bounded and invertible operators

$$R_1 f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e^{-\alpha \mu_k^n T} e_k, \quad \text{and} \quad R_2 f = \sum_{k=1}^{\infty} \langle f, e_k \rangle \left(\gamma + e^{\mu_k T}\right)^{-1} e_k,$$

$n \in \mathbb{N}$, $\alpha, \gamma > 0$, $f \in H$. Note that these semigroups are defined on $[0, T]$. Typically, they are used for regularization of ill-posed differential-operator problems (see [12, 16] for details).

Example 2.7. (An exponentially bounded convoluted (integrated) semigroup)

Consider a Cauchy problem for the second order equation

$$w''(t) = Aw(t), \qquad t \geq 0, \quad w(0) = \xi_1, \; w'(0) = \xi_2, \tag{2.14}$$

in a Banach space Y. Suppose that A generates a family of cosine and sine operator-functions $\{\mathtt{C}(t), \mathtt{S}(t), \; t \in \mathbb{R}\}$ (see, for example, [12]). The Laplace operator from Example 2.6 is one of the typical examples of such an operator. In this case problem (2.14) is well-posed and its solution has the form

$$w(t) = \mathtt{C}(t)\xi_1 + \mathtt{S}(t)\xi_2, \quad \xi_1, \xi_2 \in \operatorname{dom} B, \; t \geq 0 \,.$$

The change of variables

$$u(t) = \begin{pmatrix} w(t) \\ w'(t) \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

reduces problem (2.14) to the first order abstract Cauchy problem (1.2) in the space $X = Y \times Y$. Its solution has the form

$$u(t) = \begin{pmatrix} \mathtt{C}(t)\xi_1 + \mathtt{S}(t)\xi_2 \\ \mathtt{C}'(t)\xi_1 + \mathtt{C}(t)\xi_2 \end{pmatrix} =: U(t)\xi, \qquad t \geq 0.$$

The introduced operators

$$U(t) = \begin{pmatrix} \mathtt{C}(t) & \mathtt{S}(t) \\ \mathtt{C}'(t) & \mathtt{C}(t) \end{pmatrix}, \qquad t \geq 0,$$

are not defined on the whole of X, since function $\mathtt{C}(\cdot)$ is not differentiable on the whole of Y. However, the integrated operators

$$S(t) = \begin{pmatrix} \mathtt{S}(t) & \int_0^t \mathtt{S}(\tau) d\tau \\ \mathtt{C}(t) - I & \mathtt{S}(t) \end{pmatrix}, \qquad t \geq 0,$$

are bounded on X. Using properties of $\mathtt{C}, \mathtt{S}$-functions, it is not difficult to verify that family $\{S(t), \; t \geq 0\}$ is a k-convoluted semigroup with $k(t) = t$ and generator $\mathcal{A}$, i.e. S is an integrated semigroup.

Example 2.8. (A k-convoluted semigroup)

Consider the differential equation

$$\frac{\partial u(x;t)}{\partial t} = -\frac{\partial^2 u(x;t)}{\partial x^2} + i\frac{\partial^4 u(x;t)}{\partial x^4}, \qquad x \in \mathbb{R},\ t \geq 0,$$

with the initial condition $u(x;0) = \xi(x)$. The problem may be written in the abstract form (1.2) with $A = -\frac{\partial^2}{\partial x^2} + i\frac{\partial^4}{\partial x^4}$ in space $L_2(\mathbb{R})$. Applying the Fourier transform we obtain the Cauchy problem

$$\frac{d\widetilde{u}(s;t)}{dt} = s^2\widetilde{u}(s;t) + is^4\ \widetilde{u}(s;t), \qquad s \in \mathbb{R},\ t \geq 0, \qquad \widetilde{u}(s;0) = \widetilde{\xi}(s),$$

and its solution $\widetilde{u}(s;t) = e^{t(s^2+is^4)}\widetilde{\xi}(s)$.

We have $Sp(A) = \{\lambda = s^2 + is^4, s \in \mathbb{R}\}$. The resolvent of A exists for $\lambda \notin Sp(A)$ and satisfies (see [3] for details) the estimate $\|\mathcal{R}(\lambda)\| = O_{\lambda\to\infty}(|\lambda|/\Re\lambda)$. Hence by Theorem 2.4, A is the generator of a k-convoluted (but not an integrated!) semigroup $S(t){=}U(t){*}k(t)$, where as in (2.10), $U(t)\xi = G(\cdot,t){*}\xi(\cdot)$. Here G is the inverse Fourier transform of $e^{t(s^2+is^4)}$, $s \in \mathbb{R}$, and k is a continuous function such that its Laplace transform $\widetilde{k}$ satisfies the growth condition (2.7). The convolution $G{*}\xi$ and the Fourier transform are well-defined in an appropriate space of generalized functions (see [8]).

3. REGULARIZED SOLUTIONS OF ABSTRACT STOCHASTIC CAUCHY PROBLEMS

3.1. **Construction of regularized solutions to the linear stochastic Cauchy problem with Q-Wiener process.** Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t,\ t \geq 0\}$, and let H be a separable Hilbert space. Let Q be a linear symmetric nonnegative trace-class operator on H, then there is an orthonormal basis $\{e_j\}$ in H such that $Qe_j = \lambda_j e_j$.

Definition 3.1. [4] An H-valued stochastic process $W = \{W(t), t \geq 0\}$ is called a *Q-Wiener process*, if

- **(W1):** $W(0) = 0$;
- **(W2):** W has continuous trajectories;
- **(W3):** W has independent increments;
- **(W4):** the distribution law of $[W(t) - W(s)]$ is $\mathcal{N}(0, (t-s)Q)$, $0 \leq s \leq t$.

For each t, the Q-Wiener process W has the following expansion in H, (see [4] for details):

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j\,,$$

where $\beta_j = \frac{1}{\sqrt{\lambda_j}}\langle W, e_j\rangle$ are independent Brownian motions.

Consider a stochastic Cauchy problem in the setting that extends the Itô approach to the infinitely-dimensional case:

$$dX(t) = AX(t)dt + BdW(t),\ t \in [0, T], \quad X(0) = \xi\,. \tag{3.1}$$

Here A is the generator of a regularized semigroup $\{S(t),\ t \in [0, \tau)\}$ in H, $\{W(t),\ t \geq 0\}$ is an H-valued Q-Wiener process, B is a bounded linear operator in H, and ξ is a $\mathcal{F}_0$-measurable H-valued random variable.

Definition 3.2. Let A be the generator of a regularized semigroup and W be a Q-Wiener process in H. Then predictable H-valued process $X = \{X(t), t \in [0, T]\}$, is a *weak solution* of (3.1), if

- **(a):** $\int_0^t \|X(s)\|_H \, ds < \infty$ P-a.s.;
- **(b):** for each $y \in \text{dom }(A^*)$ the following equation holds P-a.s.:

$$\langle X(t),\, y\rangle = \langle \xi,\, y\rangle + \int_0^t \langle X(s), A^*y\rangle ds + \langle BW(t),\, y\rangle\,, \quad t \in [0, T]\,.$$

We say that process X is a *weak R-regularized solution* of (3.1) if

$$\langle X(t), y\rangle = \langle R(t)\xi,\, y\rangle + \int_0^t \langle X(s),\, A^*y\rangle ds + \int_0^t \langle R(t-s)BdW(s),\, y\rangle\,, \tag{3.2}$$

$t \in [0, T]$.

Let $H_1 := Q^{\frac{1}{2}}H$ endowed with the inner product

$$\langle Q^{\frac{1}{2}}u, Q^{\frac{1}{2}}v\rangle_{H_1} = \langle u, v\rangle_H\,,$$

and let $\mathcal{L}_2 := \mathcal{L}_2(H_1, H)$ be the space of all Hilbert-Schmidt operators from H_1 into H. Note that $\mathcal{L}_2$ is a separable Hilbert space with the norm

$$\|\Psi\|^2_{\mathcal{L}_2} := \text{tr}\Psi Q^{\frac{1}{2}} Q^{*\frac{1}{2}} \Psi^* = \sum_{j=1}^{\infty} \|\Psi Q^{\frac{1}{2}} e_j\|^2\,.$$

Consider an $\mathcal{L}_2$-valued process $\Psi(t),\ t \in [0, T]$. Then its stochastic integral

$$\int_0^t \Psi(s)\ dW(s), \quad t \in [0, T]$$

is defined under the condition

$$E\int_0^t \|\Psi(r)\|^2_{\mathcal{L}_2} dr < \infty\,. \tag{3.3}$$

As in the case of semigroups of class C_0 ([4]), we can define a stochastic convolution for a regularized semigroup.

Definition 3.3. Let $\{S(t),\ t \in [0, \tau)\}$ be a regularized semigroup such that

$$\int_0^t \|S(r)B\|^2_{\mathcal{L}_2} dr < \infty, \tag{3.4}$$

then the process $W_A = \{\int_0^t S(t-s)B\ dW(s),\ t \in [0,\tau)\}$ is called a *stochastic convolution.*

Theorem 3.4. *Let A be the generator of an R-regularized semigroup $\{S(t), t \in [0,\tau)\}$ and $\{W(t), t \geq 0\}$ be a Q-Wiener process. Suppose that operator A is densely defined and that condition* (3.4) *is fulfilled. Then for each $\mathcal{F}_0$-measurable $\xi \in H$,*

$$X(t) = S(t)\xi + W_A(t)\,, \quad t \in [0,T]\,,$$

is a weak R-regularized solution of (3.1)*. If S is a k-convoluted or R-semigroup, the solution is unique.*

Proof. Firstly we show that the process $S\xi = \{S(t)\xi, t \in [0,T]\}$ is a weak R-regularized solution of the corresponding homogeneous equation. Process $S\xi$ is clearly $\mathcal{F}_t$-measurable, integrable and predictable. Let $y \in \operatorname{dom} A^*$, then

$$\begin{aligned}\int_0^t \langle S(s)\xi, A^*y\rangle ds &= \langle \int_0^t S(s)\xi\ ds, A^*y\rangle = \langle A\int_0^t S(s)\xi\ ds, y\rangle \\ &= \langle S(t)\xi - R(t)\xi, y\rangle\,, \quad t \in [0,T]\,.\end{aligned}$$

Now consider the stochastic convolution W_A. It is not difficult to show that the process W_A is predictable (see [4] for details). Due to condition (3.4), the function $\int_0^t \|S(t-s)B\|^2_{\mathcal{L}_2} ds$ is continuous in $t \in [0,T]$ and, therefore, integrable

$$\int_0^r \int_0^t \|S(t-s)B\|^2_{\mathcal{L}_2} ds\, dt = \int_0^r \int_0^t \|S(s)B\|^2_{\mathcal{L}_2}\, dsdt = \int_0^r E\int_0^t \|S(s)B\|^2_{\mathcal{L}_2}\, dsdt < \infty\,,$$

$r \in [0,T]$. Thus $\int_0^T \|W_A(t)\|^2_H\ dt < \infty$ P-a.s.

Let $y \in \mathit{dom}\, A^*$, then taking into account properties of $\{S^*(t), t \in [0,\tau)\}$ from Theorem 2.3 and continuity of the scalar product, we have

$$\begin{aligned}&\int_0^t \langle \int_0^s S(s-r)B\ dW(r), A^*y\rangle ds = \int_0^t \int_0^s \langle S(s-r)B\ dW(r), A^*y\rangle ds \\ &= \int_0^t \int_0^s \langle B\ dW(r), S^*(s-r)A^*y\rangle ds = \int_0^t \langle B\ dW(r), \int_r^t S^*(s-r)A^*y\ ds\rangle \\ &= \int_0^t \langle B\ dW(r), \int_0^{t-r} S^*(\sigma)A^*y\ d\sigma\rangle = \int_0^t \langle B\ dW(r), S^*(t-r)y - R^*(t-r)y\rangle \\ &= \int_0^t \langle S(t-r)B\ dW(r), y\rangle - \int_0^t \langle B\ dW(r), R^*(t-r)y\rangle \\ &= \langle \int_0^t S(t-r)B\ dW(r), y\rangle - \langle \int_0^t R(t-r)B\ dW(r), y\rangle\,, \quad t \in [0,T].\end{aligned}$$

So $W_A(t)$ satisfies (3.2) with $\xi = 0$ and hence $X(t) = S(t)\xi + W_A(t),\ t \in [0,T]$, is a weak R-regularized solution of (3.1).

Similarly to the case of strongly continuous semigroups (see [4, 18]), the proof of uniqueness is based on an auxiliary equality. Suppose that X is a weak R-regularized solution of (3.1) with $\xi = 0$ and $y(\cdot) \in C^1([0,T]; \operatorname{dom} A^*)$, then the following equality holds for X

$$\langle X(t), y(t)\rangle = \int_0^t \langle X(s), y'(s) + A^* y(s)\rangle ds + \int_0^t \left\langle \int_0^s R'(s-r)\ B\ dW(r), y(s) \right\rangle ds + \int_0^t \langle B\, dW(s), R^*(0) y(s)\rangle\,.$$

Let $y(s) = S^*(t-s)y_0,\ y_0 \in \operatorname{dom} A^*$, then due to properties of adjoint regularized semigroups,

$$\langle X(t), R^*(0)y_0\rangle = \int_0^t \langle X(s), -R^{*'}(t-s)y_0\rangle ds + \int_0^t \langle S(t-s) \int_0^s R'(s-r)\ B\ dW(r)\, ds\, , y_0\rangle + \int_0^t \langle S(t-s)\ B\ dW(s), R^*(0)y_0\rangle\,. \tag{3.5}$$

If semigroup S in (3.5) is an R-semigroup (i.e. $R(t) = R$ and R is invertible), we obtain

$$\langle RX(t), y_0\rangle = \left\langle \int_0^t RS(t-r)B\ dW(r), y_0 \right\rangle,$$

and since $\overline{\operatorname{dom} A^*} = H$,

$$X(t) = \int_0^t S(t-r)B\ dW(r)\,.$$

If S is a k-convoluted semigroup (i.e. $R'(t) = k(t), R(0) = 0$), we have

$$\left\langle \int_0^t k(t-s)X(s)ds, y_0 \right\rangle = \left\langle \int_0^t S(t-s) \int_0^s k(s-r)\ B\ dW(r)\, ds\, , y_0 \right\rangle$$

for any solution of (3.2) with $\xi = 0$. In particular, for $X = W_A$, taking into consideration $\overline{\operatorname{dom} A^*} = H$, we obtain the equality

$$\int_0^t k(t-s) \int_0^s S(s-r)\ B\ dW(r)\, ds = \int_0^t S(t-s) \int_0^s k(s-r)\ B\ dW(r)\, ds\,.$$

Hence

$$\int_0^t k(t-s)X(s)ds = \int_0^t k(t-s) \int_0^s S(s-r)\ B\ dW(r)\, ds$$

and therefore

$$X(t) = \int_0^t S(t-r)B\ dW(r) + \eta(t)\,,$$

where η is a solution of $k * \eta = 0$. Since $\widetilde{k} \neq 0$, we have $\eta = 0$. $\square$

Remark 3.5. We also have the following equalities for the mathematical expectation and the covariation operator

$$\mathbb{E}[X(t)] = R(t)\xi, \quad \mathbb{C}ov[X(t)] = S(t)\mathbb{C}ov[\xi]S^*(t) + \int_0^t [S(t-s)B]Q[S(t-s)B]^* ds\,,$$

$t \in [0,T]$.

Remark 3.6. If the regularized semigroup in Theorem 3.4 is an R-semigroup and we additionally assume that

$$\int_0^t \|S(r)R^{-1}B\|^2_{\mathcal{L}_2} dr < \infty\,, \tag{3.6}$$

then the process

$$S(t)R^{-1}\xi + \int_0^t S(t-s)R^{-1}BdW(s), \quad t \in [0,T],$$

is a weak solution of (3.1). Condition (3.6) is clearly more restrictive than (3.4) since the solution operators $U(t) = S(t)R^{-1}$ are not bounded in this case.

3.2. **Regularized solutions to the linear stochastic Cauchy problem with cylindrical Wiener process.** A *cylindrical* H-valued Wiener process is usually defined by the following formal expansion in H (see, for example, [4]):

$$W(t) = \sum_{j=1}^{\infty} \beta_j(t)e_j\,.$$

This series is not convergent in $L^2(\Omega; H)$. However $\langle BW(t), y\rangle$, $y \in H$, is a well defined process and the stochastic convolution

$$W_A(t) = \sum_{j=1}^{\infty} \int_0^t S(t-s)Be_j d\beta_j(t)$$

is convergent in $L^2(\Omega; H)$ if condition (3.4) is satisfied for $Q = I$. We have the following result for the Cauchy problem with a cylindrical Wiener process.

Theorem 3.7. *Let A be the generator of an R-regularized semigroup $\{S(t), t \in [0,\tau)\}$ and $\{W(t), t \geq 0\}$ be a cylindrical Wiener process. Suppose that operator A is densely defined and that condition* (3.4) *with $Q = I$ is fulfilled. Then for each $\mathcal{F}_0$-measurable $\xi \in H$,*

$$X(t) = S(t)\xi + W_A(t)\,, \quad t \in [0,T]\,,$$

is a weak R-regularized solution of (3.1)*. If S is a k-convoluted or R-semigroup, the solution is unique.*

In this case condition (3.4) is naturally more restrictive. For example, the stochastic convolution related to the stochastic heat equation

$$\begin{aligned}
&d_t X(t,x) = \triangle_x X(t,x)dt + dW(t,x)\,, \quad t \in [0,T]\,,\ x \in \mathcal{O} \subseteq \mathbb{R}^N\,,\\
&X(t,x) = 0\,, \quad t \in [0,T]\,,\ x \in \partial\mathcal{O}\,,\\
&X(0,x) = 0\,, \quad x \in \mathcal{O}\,,
\end{aligned}$$

exists in $L^2(\Omega; H)$ only if $N = 1$ (see Example 5.7 in [4]).

Now we consider the probability space $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)), \mu)$, where $S'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$, μ is the (unique) probability measure on $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)))$ satisfying the condition

$$\int_{S'(\mathbb{R}^d)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L_2(\mathbb{R}^d)}}$$

and $\langle\omega, \phi\rangle$ denotes the action of $\omega \in S'(\mathbb{R}^d)$ on $\phi \in S(\mathbb{R}^d)$.

Consider spaces of H-valued stochastic distributions $\mathbb{S}(H)_{-\rho}$, $\rho \in [0, 1]$:

$$\mathbb{S}(H)_1 \subset \mathbb{S}(H)_\rho \subset \mathbb{S}(H)_0 \subset L_2(S'; H) \subset \mathbb{S}(H)_{-0} \subset \mathbb{S}(H)_{-\rho} \subset \mathbb{S}(H)_{-1}\,.$$

As illustrated in [7, 14], the cylindrical H-valued Wiener process W and all its derivatives belong to the space $\mathbb{S}(H)_{-0}$.

The *generalized stochastic convolution* is defined in $\mathbb{S}(H)_{-0}$ as a Hitsuda-Skorohod integral

$$\int_0^t S(t-s)B\delta W(s) = \sum_{j=1}^{\infty} \int_0^t S(t-s)Be_j\delta\beta_j(s)\,.$$

Note that the condition (3.4) is not needed for existence of the generalized stochastic convolution. Similarly to the case of the semigroups of class C_0 (see [7, 14]), we arrive at the following result.

Theorem 3.8. *Suppose that a densely defined operator A is the generator of an R-regularized semigroup $\{S(t), t \in [0, \tau)\}$ on H, operator-function $R(t)$ is strongly continuously differentiable in t and $R(0) = 0$ or $R(t) \equiv R$. If $\xi \in \mathbb{S}(H)_{-1}$, then the process*

$$X(t) = S(t)\xi + \int_0^t S(t-s)B\delta W(s)\,, \quad t \in [0, T]\,,$$

is a unique continuously differentiable $\mathbb{S}(H)_{-1}$-valued solution to the equation

$$X(t) = R(t)\xi + A\int_0^t X(s)ds + \int_0^t R(t-s)B\delta W(s), \quad t \in [0, T]\,.$$

3.3. **Semi-linear stochastic Cauchy problems in Hilbert spaces.** Consider a semi-linear stochastic Cauchy problem

$$dX(t) = \big[AX(t) + F(t, X)\big]dt + B(t, X)dW(t), \; X(0) = \xi, \tag{3.7}$$

where A is the generator of a regularized semigroup $\{S(t), t \in [0, \tau)\}$ in a Hilbert space H and W is an H-valued Q-Wiener process.

Suppose that $F : [0, T] \times \Omega \times H \to H$ and $B : [0, T] \times \Omega \times H \to \mathcal{L}_2$ are measurable mappings from $([0, T] \times \Omega \times H,\ \mathcal{P}_T \times \mathcal{B}(H))$ to $(H,\ \mathcal{B}(H))$ and $(\mathcal{L}_2,\ \mathcal{B}(\mathcal{L}_2))$ respectively, and that they satisfy the Lipschitz conditions:

$$\|F(t, \omega; x) - F(t, \omega; y)\| + \|B(t, \omega; x) - B(t, \omega; y)\|_{\mathcal{L}_2} \le C\,|x - y| \tag{3.8}$$

for $x, y \in H,\ t \in [0,T],\ \omega \in \Omega$, and the growth conditions:

$$\|F(t,\omega;x)\|^2 + \|B(t,\omega;x)\|^2_{\mathcal{L}_2} \le C(1+|x|^2) \tag{3.9}$$

for $x \in H,\ t \in [0,T],\ \omega \in \Omega$. Here $\mathcal{P}_T$ is a predictable σ-field on $[0,T] \times \Omega$.

We now introduce the notion of a mild regularized solution to a semi-linear problem.

Definition 3.9. An H-valued predictable process $X = \{X(t), t \in [0,T]\}$, is a *mild solution* of (3.7), if $\int_0^t \|X(\tau\|^2 d\tau < \infty$ P-a.s. and

$$X(t) = U(t)\xi + \int_0^t U(t-s)F(s,X)ds + \int_0^t U(t-s)B(s,X)dW(s),$$

where $U(t), t \in [0,T]$, are solving operators for the homogeneous Cauchy problem.

We say that process X is a *mild regularized solution* of (3.7) if

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s,X)ds + \int_0^t S(t-s)B(s,X)dW(s). \tag{3.10}$$

Using the design of the proof of Theorem 3.4 we can show that (3.10) is a weak solution to the following regularized Cauchy problem:

$$\begin{aligned} X(t) &= R(t)\xi + A\int_0^t X(s)ds + \\ &+ \int_0^t R(t-s)\, F(s,X(s))ds + \int_0^t R(t-s)\, B(s,X(s))dW(s). \end{aligned}$$

Theorem 3.10. *Let operator A be the generator of a regularized semigroup $\{S(t), t \in [0,\tau)\}$ in a Hilbert space H, W be an H-valued Q-Wiener process, $F : [0,T]\times\Omega\times H \to H$ and $B : [0,T] \times \Omega \times H \to H_{\mathcal{L}_2}$ satisfy the Lipschitz and growth conditions* (3.8)–(3.9). *Then a mild regularized solution to* (3.7) *exists and is unique.*

Proof. Firstly we note that both integrals in (3.10) are well defined and similarly to the case of semigroups of class C_0 [4], $X(t)$ is a predictable process. Let $\mathbb{H}_p,\ p \ge 2$, be the Banach space of all H-valued predictable processes Y with the norm

$$\|Y\|_p = \Big(\sup_{t\in[0,T]} \mathbb{E}\|Y(t)\|^p\Big)^{1/p}.$$

Define the following mappings on $\mathbb{H}_p$:

$$K_1(Y) := \int_0^t S(t-s)F(s,Y(s))ds, \quad K_2(Y) := \int_0^t S(t-s)B(s,Y(s))dW(s),$$

and $K(Y) := S(t)Y(0) + K_1(Y) + K_2(Y)$.

Due to the growth condition (3.9) we have the following estimates for K_1:

$$\begin{aligned} \|K_1(Y)\|_p^p &\leq M^p\mathbb{E}\Big[\int_0^T \|F(s,Y(s))\|ds\Big]^p \leq T^{p-1}M^p\mathbb{E}\Big[\int_0^T \|F(s,Y(s))\|^p ds\Big] \\ &\leq 2^{p/2-1}T^{p-1}M^pC^p\mathbb{E}\Big[\int_0^T \big(1+\|Y(s)\|^p\big)ds\Big] \\ &\leq 2^{p/2-1}(TMC)^p\Big(1+\|Y\|_p^p\Big), \end{aligned}$$

where $M = \sup_{t\in[0,T]}\|S(t)\|$. Hence K_1 acts from $\mathbb{H}_p$ to $\mathbb{H}_p$. The corresponding estimates for K_2

$$\|K_2(Y)\|_p^p \leq \mathbf{C}\left(1+\|Y\|_p^p\right)$$

hold due to the equality

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\int_0^t \Psi(\tau)dW(\tau)|^{2r}\Big] \leq C\,\mathbb{E}\Big[\int_0^t \|\Psi(\tau)\|_{\mathcal{L}_2}^2 ds\Big]^r, \quad t\in[0,T],$$

that holds for any $r \geq 1$ and for arbitrary $\mathcal{L}_2$-valued predictable process Ψ (see [4]). Hence K_2 acts from $\mathbb{H}_p$ to $\mathbb{H}_p$ as well. Due to the Lipschitz condition (3.8), we obtain the following estimates:

$$\begin{aligned} \|K(Y_1)-K(Y_2)\|_p &\leq M\|Y_1-Y_2\|_p + \|K_1(Y_1)-K_1(Y_2)\|_p + \|K_2(Y_1)-K_2(Y_2)\|_p \\ &\leq C_{M,T}\|Y_1-Y_2\|_p\,. \end{aligned}$$

Application of the fixed point theorem for contraction mappings proves the existence of a mild regularized solution to (3.7). The proof of uniqueness is similar to one in [4, Theorem 7.4] . □

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