*Communications in Applied Analysis* **13** (2009), no. 3, 305–316

# MULTIPLE SOLUTIONS FOR NONCOERCIVE RESONANT NEUMANN HEMIVARIATIONAL INEQUALITIES

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**ABSTRACT.** In this paper we consider semilinear Neumann problems with a nonsmooth potential. Using variational methods based on the nonsmooth critical point theory, we prove existence and multiplicity theorems. Our framework of analysis incorporates strongly resonant problems and in contrast to earlier works on the subject, the Euler functional of our problem need not be coercive.

**Keywords:** Second deformation theorem, nonsmooth critical point theory, resonant problems, multiple nontrivial solutions

**AMS (MOS) Subject Classification.** 35J20, 35J85.

## 1. INTRODUCTION

Let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a $C^2$-boundary $\partial Z$. In this paper, we study the following semilinear Neumann problem with a nonsmoth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\triangle x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\} \tag{1.1}$$

Here the potential function $j(z, x)$ is measurable in $z \in Z$ and only locally Lipschitz and in general nonsmooth in $x \in \mathbb{R}$. By $\partial j(z, x)$ we denote the generalized subdifferential of the locally Lipschitz function $x \to j(z, x)$ (see Section 2). Also, $n(z)$ denotes the outward unit normal on $\partial Z$ and $\frac{\partial x}{\partial n} = (Dx, n)_{\mathbb{R}^N}$ is the normal derivative of $x$ on $\partial Z$ in the sense of traces.

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[*]Researcher supported by a grant of the National Scholarship Foundation of Greece (I.K.Y.)

Received January 15, 2009 

The goal of this paper, is to prove a multiplicity theorem for problem (1.1), when the Euler functional of the problem is bounded below but need not be coercive. Our framework of analysis incorporates resonant and strongly resonant problems.

Previous works dealing with semilinear resonant Neumann problems, include those of Iannacci-Nkashama [5], Kuo [6], [7], Tang [9] and Tang-Wu [10]. In all the aforementioned works, the potential function $j(z,\cdot)$ is $C^1$ (smooth problems). In Iannacci-Nkashama [5] and Kuo [6],[7] the approach is degree theoretic and the authors employ Landesman-Lazer type conditions. They prove existence but not multiplicity results and their hypotheses imply that the Euler functional of the problem is coercive. Tang [9] proves existence theorems using a variational approach based on the saddle point theorem. Finally, Tang-Wu [10], is the only paper with a multiplicity result. They establish the existence of two nontrivial solutions using the local linking theorem (see, for example, Gasinski-Papageorgiou [3], p. 665). Their Euler functional is coercive. None of the above works incorporates in its framework of analysis strongly resonant problems. We recall that such problems, have the distinctive feature that the compactness property of the Euler functional (the Palais-Smale or the Cerami condition), is not valid at all levels. Our approach is variational based on nonsmooth critical point theory. In the next section, for the convenience of the reader, we recall some basic definitions and facts from this theory, which we shall need in the sequel.

## 2. MATHEMATICAL BACKGROUND

The nonsmooth critical point theeory, is based on the subdifferential theory for locally Lipschitz functions due to Clarke [1]. So, let us start with a quick review of the basics of this theory.

Let $X$ be a Banach space and $X^*$ its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets of the pair $(X^*,X)$. Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of $\varphi$ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x'\to x \\ \lambda\downarrow 0}} \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda}.$$

It is easy to check that the function $h \to \varphi^0(x;h)$ is sublinear and continuous. Therefore, by the Hahn-Banach theorem, we have that $\varphi^0(x,\cdot)$ is the support function of a nonempty, convex and $w^*$compact set $\partial\varphi(x)$, i.e.

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*,h\rangle \le \varphi^0(x;h) \ \text{ for all } \ h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the generalized subdifferential of $\varphi$. If $\varphi \in C^1(X)$, then $\varphi$ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$. If $\varphi : X \to \mathbb{R}$ is continuous convex, then $\varphi$ is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis $\partial_c\varphi(x)$, defined by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi(x+h) - \varphi(x) \ \text{ for all } \ h \in X\}.$$

If $\psi : X \to \mathbb{R}$ is another locally Lipschitz function and $\lambda \in \mathbb{R}$, then $\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$ for all $x \in X$.

In the first inclusion, equality holds if one of the functions is $C^1$.

We say that $x \in X$ is a critical point of the locally Lipschitz function $\varphi : X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. Then $c = \varphi(x)$ is a critical value of $\varphi$. Of course, when $\varphi \in C^1(X)$, this definition coincides with the usual one. If $x \in X$ is a local extremum of $\varphi$ (i.e., a local minimum or a local maximum), then $0 \in \partial\varphi(x)$, i.e., $x \in X$ is a critical point of $\varphi$.

In what follows,we set

$$m_\varphi(x) = \inf\{\|x^*\| : x^* \in \partial\varphi(x)\}.$$

From the smooth critical point theory, we know that some kind of compactness condition on $\varphi$ is necessary, in order to compensate for the lack of local compactness in the ambient space $X$ and derive minimax characterizations of the critical values of the function. In the present nonsmooth setting, this condition takes the following form:

"A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the Palais Smale condition at the level $c \in \mathbb{R}$ (the nonsmooth $PS_c$-condition, for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\varphi(x_n) \to c \ \text{ and } \ m(x_n) \to 0 \ \text{ in } \ X^* \ \text{ as } \ n \to \infty,$$

has a strongly convergent subsequence."

We say that $\varphi$ satisfies the nonsmooth Palais-Smale condition (the nonsmooth $PS$-condition for short), if it is satisfies the nonsmooth $PS_c$-condition for every $c \in \mathbb{R}$.

Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz function and $c \in \mathbb{R}$. We define

- $\dot{\varphi}^c = \{x \in X : \varphi(x) < c\}$ (the strict sublevel set of $\varphi$ at $c$),
- $K = \{x \in X : 0 \in \partial\varphi(x)\}$ (the critical set of $\varphi$ at $c$),
- $K_c = \{x \in X : \varphi(x) = c\}$ (the critical set of $\varphi$ at the level $c$).

The next result, is a nonsmooth version of the so-called second deformation theorem and it is due to Corvellec [2] (for the smooth version of the result see, for example, Gasinski-Papageorgiou [4], p. 628). In fact, the result of Corvellec [2], is formulated in the more general context of metric spaces and continuous functions, using the so-called weak slope. However, for our purposes, the following particular version of the result suffices.

**Theorem 2.1.** *If $X$ is a Banach space, $a \in \mathbb{R}$, $a < b \le \infty$, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmmoth $PS_c$-condition for every $c \in [a, b]$, $\varphi$ has no critical points in $\varphi^{-1}(a, b)$ and $K_a$ is a finite set consisting of only local minima, then there exists a continuous deformation $h : [0, 1] \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that*

(a) $h(t, \cdot)|_{K_a} = id|_{K_a}$ *for all* $t \in [0, 1]$;
(b) $h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a$;
(c) $\varphi(h(t, x) \le \varphi(x)$ *for all* $(t, x) \in [0, 1] \times \dot{\varphi}^b$.

**Remark 2.2.** In particular, the above theorem implies that the set $\dot{\varphi}^a \cup K_a$ is a weak deformation retract of $\dot{\varphi}^b$. In the smooth second deformation theorem, the conclusion is that $\varphi^b \backslash K_b$ is a strong deformation retract of $\varphi^b$, where $\varphi^b = \{x \in X : \varphi(x) \le b\}$.

More about the nonsmooth critical point theory, can be found in Gasinski-Papageorgiou [3] and Motreanu-Radulescu [8].

## 3. EXISTENCE THEOREM

In this section, we prove an existence theorem which is of independence interest and which will also be used in the multiplicity theorem in Section 4. The existence theorem concerns the following more general Neumann problem:

$$\left\{ \begin{array}{l} -\triangle x(z) \in \partial j(z, x(z)) + h(z) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z, h \in L^\infty(Z). \end{array} \right\} \tag{3.1}$$

For problem (3.1) we establish the existence of a nontrivial solution for all $h \in L^\infty(Z)$ such that $\int_Z h(z)dz = 0$. For this purpose, we shall need the following hypotheses (in what follows $\lambda_0 = 0 < \lambda_1$ are the first two eigenvalues of the negative Neumann Laplacian).

$\underline{H_1}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ for a.a. $z \in Z$ and

(i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(j, x)$, we have

$$|u| \le a(z) + c|x|^{r-1},$$

with $a \in L^\infty(Z)_+, c > 0$ and $2 \le r < 2^* = \left\{ \begin{array}{ll} \frac{2N}{N-2} & \text{if } N \ge 3 \\ +\infty & \text{if } N = 1, 2 \end{array} \right.$;

(iv) there exists $\xi \in L^\infty(Z)_+$ such that $j(z, x) \le \xi(z)$ for a.a. $z \in Z$ and all $x \in \mathbb{R}$;
(v) there exists a function $\eta \in L^\infty(Z)_+$, $\eta \ne 0$ such that

$$\eta(z) \le \liminf_{x \to 0} \frac{2j(z, x)}{x^2}$$

uniformly for a.a. $z \in Z$;

(vi) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $j(z,x) \leq \frac{\lambda_1}{2}x^2$;

(vii) $\int_Z \limsup_{|x|\to\infty} j(z,x)dz < +\infty$.

**Remark 3.1.** These hypotheses incorporate in our framework of analysis problems which at zero are resonant with respect to $\lambda_1 > 0$, the first nonzero eigenvalue of $(-\triangle, H^1(Z))$. Also, hypothesis $H_1(v)$ implies that at zero, we also have nonuniform nonresonance with respect to the principal eigenvalue $\lambda_0 = 0$. Note that hypothesis $H_1$ incorporate in our framework problems which at infinity are strongly resonant with respect to the principal eigenvalue $\lambda_0 = 0$ (i.e., problems in which $\lim_{x\to\pm\infty} j(z,x) = j_\pm(z)$ a.e. on $Z$, with $j_\pm \in L^\infty(Z)$).

**Example 3.2.** The following functions satisfy hypotheses $H_1$. For the sake of simplicity, we drop the $z$-dependence.

$$j_1(x) = \begin{cases} \frac{\lambda_1}{2}x^2 & \text{if } |x| \leq 1 \\ \frac{\lambda_1}{2x^2} + \frac{\xi \ln|x|}{\sqrt{|x|}} & \text{if } |x| > 1 \end{cases} \text{ with } \xi > 0,$$

and $j_2(x) = \frac{\lambda_1}{2} = \min\{x^2 - |x|^p, 1 - |x|^r\}$, $\;2 < p \leq 2^*$. Note that, if $\xi = 2\lambda_1$, then $j_1 \in C^1(\mathbb{R})$.

Let $\varphi_1 : H^1(Z) \to \mathbb{R}$ be the Euler functional for problem (3.1), defined by

$$\varphi_1(x) = \frac{1}{2}\|Dx\|_2^2 - \int_Z j(z,x(z))dz - \int_Z h(z)x(z) \text{ for all } x \in H^1(Z).$$

We know that $\varphi_1$ is Lipschitz continuous on bounded sets (see Clarke [1], p. 85), hence it is locally Lipschitz.

We consider the auxiliary Neumann problem

$$\left\{ \begin{array}{l} -\triangle x(z) = h(z) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\} \tag{3.2}$$

We also consider the following orthogonal direct sum decomposition

$$H^1(Z) = \mathbb{R} \oplus V, \tag{3.3}$$

with $V = \{x \in H^1(Z) : \int_Z x(z)dz = 0\}$.

**Proposition 3.3.** *If $h \in L^\infty(Z)$ and $\int_Z h(z)dz = 0$, then problem* (3.2) *admits a unique solution $x_0 \in C^1(\overline{Z}) \cap V$.*

*Proof.* Let $\psi : H^1(Z) \to \mathbb{R}$ be the $C^1$-functional, defined by

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z h(z)x(z)dz \text{ for all } x \in H^1(Z).$$

Let $\widehat{\psi} = \psi|_V$. By virtue of the Poincare-Wirtinger inequality (see, for example Gasinski-Papageorgiou [4], p. 224), we see that $\widehat{\psi} : V \to \mathbb{R}$ is coercive. Moreover, from the compact embedding of $H^1(Z)$ into $L^2(Z)$, it is clear that $\widehat{\psi}$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $x_0 \in V$ such that

$$-\infty < \widehat{m}_0 < \inf_V \widehat{\psi} = \widehat{\psi}(x_0),$$
$$\Rightarrow \widehat{\psi}'(x_0) = 0 \quad \text{in } V^*. \tag{3.4}$$

Every $x \in H^1(Z)$ has a unique decomposition

$$x = \overline{x} + \widehat{x} \quad \text{with } \overline{x} \in \mathbb{R},\ \widehat{x} \in V \ \ (\text{see } (3.3)).$$

Hence

$$\psi(x) = \psi(\overline{x} + \widehat{x}) = \frac{1}{2}\|D\widehat{x}\|_2^2 - \int_Z h\widehat{x}dz = \psi(\widehat{x}) \ \text{ for all } \ \text{for all } \ x \in H^1(Z) \tag{3.5}$$

(recall that by hypothesis $\int_Z h dz = 0$). So, if $p_V : H^1(Z) \to V$ denotes the orthogonal projection onto $V$, then from (3.5) we see that

$$\psi = \widehat{\psi} \circ p_V.$$

Therefore, by the chain rule, we have

$$\psi'(x) = p_V^* \widehat{\psi}'(p_V(x)) \ \text{ for all } \ x \in H^1(Z). \tag{3.6}$$

In what follows, by $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair $(H^1(Z)^*, H^1(Z))$ and by $\langle \cdot, \cdot \rangle_V$ the duality brackets for the pair $(V^*, V)$. For all $x, y, \in H^1(Z)$, we have

$$\langle \psi'(x), y \rangle = \langle p_V^* \widehat{\psi}'(p_V(x)), y \rangle \ \ (\text{see } (3.6)) \tag{3.7}$$
$$= \langle \widehat{\psi}'(p_V(x)), p_V(y) \rangle_V \tag{3.8}$$
$$= \langle \widehat{\psi}'(\widehat{x}), \widehat{y} \rangle_V. \tag{3.9}$$

Hence

$$\langle \psi'(x_0), y \rangle = \langle \widehat{\psi}'(x_0), \widehat{y} \rangle_V = 0 \ \ (\text{see } (3.7)). \tag{3.10}$$

Since $y \in H^1(Z)$ is arbitrary, from (3.10) it follows that

$$\psi'(x_0) = 0,$$
$$\Rightarrow A(x_0) = h, \tag{3.11}$$

where $A \in \mathcal{L}(H^1(Z), H^1(Z)^*)$ is defined by

$$\langle A(x), y \rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N} dz \ \text{ for all } \ x, y \in H^1(Z).$$

From (3.11) and using Green's identity, we obtain

$$-\triangle x_0(z) = h(z) \quad \text{a.e. on } Z, \quad \frac{\partial x}{\partial n} = 0 \ \text{ on } \ \partial Z.$$

Regularity theory, implies that $x_0 \in C_0^1(\overline{Z})$. Moreover, the strict monotonicity of the map $A|_V$ (equivalently the strict convexity of the function $\widehat{\psi}$), implies that the solution $x_0 \in C^1(\overline{Z}) \cap V$ is unique in $V$. $\square$

We shall use the solution $x_0 \in C^1(\overline{Z}) \cap V$, to produce a solution for the problem (3.1). In what follows, $\beta = \int_Z \limsup_{|x|\to\infty} j(z,x)dz$. By hypothesis $H_1(vii)$, we have $\beta < +\infty$. As we already mentioned in the Introduction, our setting includes problems which are strongly resonant at infinity with respect to the principal eigenvalue $\lambda_0 = 0$. A characteristic feature of such problems, is the failure of the global compactness condition. This is evident in the next proposition.

**Proposition 3.4.** *If hypotheses $H_1$ hold and $c < -\beta + \psi(x_0)$, then $\varphi_1$ satisfies the nonsmooth $PS_c$-condition.*

*Proof.* Let $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ be a sequence such that

$$\varphi_1(x_n) \to c \text{ and } m_{\varphi_1}(x_n) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Since $\partial\varphi_1(x_n) \subseteq H^1(Z)^*$ is weakly compact and the norm functional in a Banach space is sequentially weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi_1(x_n)$ such that $m_{\varphi_1}(x_n) = \|x_n^*\|$ for all $n \geq 1$. We know that

$$x_n^* = A(x_n) - u_n - h, \tag{3.13}$$

with $u_n \in L^{r'}(Z)$ ($\frac{1}{r} + \frac{1}{r'} = 1$) and $u_n(z) \in \partial j(z, x_n(z))$ for a.a. $z \in Z$, $n \geq 1$ (see Clarke [1], p. 83). Recall that

$$x_n = \overline{x}_n + \widehat{x}_n \text{ with } \overline{x}_n \in \mathbb{R},\ \widehat{x}_n \in V,\ n \geq 1 \text{ (see (3.3))}. \tag{3.14}$$

Because of (3.12), we can find $M_1 > 0$ such that for all $n \geq 1$, we have

$$\begin{aligned} M_1 \geq \varphi_1(x_n) &= \frac{1}{2}\|Dx_n\|_2^2 - \int_Z j(z,x_n)dz - \int_Z hx_n dz \\ &= \frac{1}{2}\|D\widehat{x_n}\|_2^2 - \int_Z j(z,x_n)dz - \int_Z h\widehat{x_n}dz \\ &\qquad \text{(see (3.14) and recall that } \textstyle\int_Z hdz = 0) \\ &= \psi(\widehat{x}_n) - \int_Z j(z,x_n)dz \\ &\geq \psi(\widehat{x}_n) - \|\xi\|_1 \text{ (see hypothesis } H_1(iv)). \end{aligned} \tag{3.15}$$

From (3.15) and the Poincare-Wirtinger inequality, it follows that $\{\widehat{x}_n\}_{n\geq 1} \subseteq H^1(Z)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$|\widehat{x}_n(z)| \leq k(z) \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \text{ with } k \in L^2(Z)_+. \tag{3.16}$$

Suppose that $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ is unbounded. We may assume that $\|x_n\| \to \infty$. Since $\{\widehat{x}_n\}_{n\geq} \subseteq H^1(Z)$ is bounded, we must have $|\overline{x}_n| \to \infty$. Then

$$\begin{aligned}&|x_n(z)| \geq |\overline{x}_n| - |\widehat{x}_n(z)| \geq |\overline{x}_n| - k(z) \text{ a.e. on } Z \text{ (see (3.16))},\\ \Rightarrow &|x_n(z)| \to +\infty \text{ for a.a. } z \in Z \text{ as } n \to \infty.\end{aligned}$$

From (3.15), we have

$$\begin{aligned}\varphi_1(x_n) &= \psi(\widehat{x}_n) - \int_Z j(z,x_n)dz\\ &\geq \psi(x_0) - \int_Z j(z,x_n)dz \text{ for all } n \geq 1 \quad (\text{recall } x_0 \text{ is the minimizer of } \widehat{\psi})\\ \Rightarrow c &\geq \psi(x_0) - \limsup_{n\to\infty} \int_Z j(z,x_n)dz \text{ (see (3.12))}\\ &\geq \psi(x_0) - \int_Z \limsup_{n\to\infty} j(z,x_n)dz \text{ (by Fatou's Lemma, see } H_1(iv))\\ &= \psi(x_0) - \beta,\end{aligned}$$

which contradicts the choice of $c$. This proves that $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ is bounded. So, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H^1(Z) \text{ and } x_n \to x \text{ in } L^r(Z). \tag{3.17}$$

From (3.12) and (3.13), we have

$$|\langle A(x_n), x_n - x\rangle - \int_Z u_n(x_n - x)dz - \int_Z h(x_n - x)dz| \leq \varepsilon_n \|x_n - x\| \text{ with } \varepsilon_n \downarrow 0. \tag{3.18}$$

Note that

$$\int_Z u_n(x_n - x)dz \to 0 \text{ and } \int_Z h(x_n - x)dz \to 0 \text{ as } n \to \infty \text{ (see (3.17))}. \tag{3.19}$$

So, if in (3.18) we pass to the limit as $n \to \infty$ and use (3.19), we obtain

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0. \tag{3.20}$$

Since $A(x_n) \xrightarrow{w} A(x)$ in $H^1(Z)$, from (3.20) we have

$$\|Dx_n\|_2^2 = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_2^2.$$

Recall that $Dx_n \xrightarrow{w} Dx$ in $L^2(Z, \mathbb{R}^{\mathbb{N}})$. Therefore, from the Kadec-Klee property of Hilbert spaces, we have $Dx_n \to Dx$ in $L^2(Z, \mathbb{R}^N)$, hence $x_n \to x$ in $H^1(Z)$ (see also (3.17)). This proves that $\varphi_1$ satisfies the nonsmooth $PS_c$-condition for all $c < -\beta + \psi(x_0)$. ☐

Using this proposition and a variational argument, we can establish the existence of a nontrivial solution for problem (3.1).

**Proposition 3.5.** *If hypotheses $H_1$ hold and $\beta < \int_Z j(z,x_0)dz$, then problem* (3.1) *has a nontrivial solution $y_0 \in C^1(\overline{Z})$.*

*Proof.* Recall (see the proof of Proposition 3.3) that

$$\varphi_1(x) = \psi(\widehat{x}) - \int_Z j(z,x)dz \geq \psi(x_0) - \|\xi\|_1,$$
$$\Rightarrow \varphi_1 \text{ is bounded below.}$$

Therefore, $\widehat{m}_1 = \inf_{H^1(Z)} \varphi_1 > -\infty$. We have

$$\begin{aligned} -\infty < \widehat{m}_1 \leq \varphi_1(x_0) &= \psi(x_0) - \int_Z j(z,x_0)dz \text{ (since } x_0 \in V) \\ &< \psi(x_0) - \beta \text{ (since by hypothesis } \beta < \textstyle\int_Z j(z,x_0)dz). \end{aligned}$$

Proposition 3.4 implies that $\varphi_1$ satisfies the nonsmooth $PS_{\widehat{m}_1}$-condition. Then from Gasinski-Papageorgiou [3], p. 144, we can find $y_0 \in H^1(Z)$ such that

$$\begin{aligned} \varphi_1(y_0) = \widehat{m}_1 &= \inf_{H^1(Z)} \varphi_1, \\ &\Rightarrow 0 \in \partial\varphi_1(y_0), \\ &\Rightarrow A(y_0) = u_0 + h, \end{aligned} \tag{3.21}$$

with $u_0 \in L^{r'}(Z)$, $u_0(z) \in \partial j(z, y_0(z))$ a.e. on $Z$. From (3.21), using Green's identity, we obtain

$$-\triangle y_0(z) = u_0(z) + h(z) \text{ a.e. on } Z,$$
$$\Rightarrow y_0 \in H^1(Z) \text{ is a solution of problem (3.1).}$$

Moreover, regularity theory implies that $y_0 \in C^1(\overline{Z})$. It remains to show that $y_0 \neq 0$. By virtue of hypothesis $H_1(v)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$j(z,x) \geq \frac{1}{2}(\eta(z) - \varepsilon)x^2 \text{ for a.a. } z \in Z, \text{ all } |x| \leq \delta. \tag{3.22}$$

Let $c \in (0, \delta]$. Then

$$\begin{aligned} \varphi_1(c) &= -\int_Z j(z,c)dz \;\; (\text{since } \textstyle\int_Z h dz = 0) \\ &\leq \frac{c^2}{2}[\varepsilon|Z|_N - \int_Z \eta dz] \;\; (\text{see (3.22)}). \end{aligned} \tag{3.23}$$

So, if we choose $\varepsilon \in (0, \frac{1}{|Z|_N}\|\eta\|_1)$, then from (3.23), we see that

$$\begin{aligned} &\varphi_1(c) < 0 \\ &\Rightarrow \widehat{m}_1 = \varphi_1(y_0) < 0 = \varphi_1(0), \\ &\Rightarrow y_0 \neq 0. \end{aligned}$$

□

## 4. MULTIPLICITY THEOREM

In this section, by strengthening the hypotheses on $j(z,\cdot)$, we prove a multiplicity theorem for problem (1.1), i.e. $h = 0$. Also, in this case the Euler functional $\varphi : H^1(Z) \to \mathbb{R}$ of the problem, is given by

$$\varphi(x) = \frac{1}{2}\|Dx\|_2^2 - \int_Z j(z, x(z))dz \text{ for all } x \in H^1(Z).$$

This too is locally Lipschitz.

The new hypotheses on the potential function $j(z,x)$, are the following:

$H_2$ : $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ a.e. on $Z$, hypotheses $H_2(i) \to (vi)$ are the same as the corresponding hypotheses $H_1(i) \to (vi)$ and

$(vii)$ $\beta = \int_Z \limsup_{|x|\to\infty} j(z,x)dz \le 0$.

**Example 4.1.** The following function satisfies hypotheses $H_2$. As before for the sake of simplicity we drop the $z$-dependence:

$$j_3(x) = \begin{cases} \frac{\lambda_1}{2}(x^2 - x^4) & \text{if } |x| \le 1 \\ c(\frac{\pi}{4} - \tan^{-1}|x|) & \text{if } |x| > 1 \end{cases}$$

with $c > 0$. If $c = 2\lambda_1$, then $j_3 \in C^1(\mathbb{R})$.

**Theorem 4.2.** *If hypotheses $H_2$ hold, then problem* (1.1) *has at least two nontrivial solutions $y_0, v_0 \in C^1(\overline{Z})$.*

*Proof.* From Proposition 3.5, we already have one trivial solution $y_0 \in C^1(\overline{Z})$. From the proof of Proposition 3.5, (see (3.23)), we see that we can find $\rho > 0$ small such that

$$\eta_\rho = \max_{\partial\overline{B}_\rho \cap \mathbb{R}} \varphi < 0, \tag{4.1}$$

where $\overline{B}_\rho = \{x \in H^1(Z) : \|x\| \le \rho\}$ and $\partial\overline{B}_\rho = \{x \in H^1(Z) : \|x\| = \rho\}$.

On the other hand, if $v \in V$, then

$$\begin{aligned}\varphi(v) &= \frac{1}{2}\|Dv\|_2^2 - \int_Z j(z, v(z))dz \\ &\ge \frac{1}{2}\|Dv\|_2^2 - \frac{\lambda_1}{2}\|v\|_2^2 \text{ (see hypothesis } H_2(vi)) \\ &\ge 0\end{aligned}$$

(by the variational characterization of $\lambda_1$, see Gasinski-Papageorgiou [4], p. 722),

$$\Rightarrow \inf_V \varphi = 0. \tag{4.2}$$

Now, let $\Gamma = \{\gamma \in C(\overline{B}_\rho \cap \mathbb{R}, H^1(Z)) : \gamma|_{\partial\overline{B}_\rho\cap\mathbb{R}} = id|_{\partial\overline{B}_\rho\cap\mathbb{R}}\}$ and define

$$\widehat{c}_\rho = \inf_{\gamma\in\Gamma} \sup_{x\in\overline{B}_\rho\cap\mathbb{R}} \varphi(\gamma(x)). \tag{4.3}$$

We know (see Gasinski-Papageorgiou [4], p. 642), that the pair $\{\partial\overline{B}_\rho \cap \mathbb{R}, \overline{B}_\rho \cap \mathbb{R}\}$ is linking with $V$ in $H^1(Z)$. Hence

$$\begin{aligned}\gamma(\overline{B}_\rho \cap \mathbb{R}) \cap V \neq \emptyset \quad &\text{for all} \quad \gamma \in \Gamma, \\ &\Rightarrow \widehat{c}_\rho \geq 0 \ \text{ (see (4.2) and (4.3)).}\end{aligned} \tag{4.4}$$

Suppose that $\{0, y_0\}$ are the only critical points of $\varphi$. Let

$$a = \inf \varphi = \varphi(y_0) < 0 = b \ \text{ (see the proof of Proposition 3.5).} \tag{4.5}$$

According to Proposition 3.4, $\varphi$ satisfies the nonsmooth $PS_c$-condition for all $c \in [a, b)$ (recall that since $h = 0, x_0 = 0$ and by hypothesis $H_2(vii)$, $\beta \leq 0$). Also, $K_a = \{y_0\}$ and $y_0 \in C^1(\overline{Z})$ is a minimizer of $\varphi$ (see the proof of Proposition 3.5 and recall that $h = 0$). Hence we can apply Theorem 2.1 and obtain a continuous deformation $h : [0,1] \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that $h(t, \cdot)|_{K_a} = id|_{K_a}$ for all $t \in [0,1]$ and

$$h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a = \{y_0\} \tag{4.6}$$

$$\text{and} \quad \varphi(h(t,x)) \leq \varphi(x) \ \text{ for all } \ t \in [0,1] \ \text{ and all } \ x \in \dot{\varphi}^b. \tag{4.7}$$

We introduce the map $\gamma_0 : \overline{B}_\rho \cap \mathbb{R} \to H^1(Z)$ defined by

$$\gamma_0(x) = \begin{cases} y_0 & \text{if } \ |x| \leq \frac{\rho}{2} \\ h(\frac{2(\rho - \|x\|)}{\rho}, \frac{\rho x}{\|x\|}) & \text{if } \ |x| > \frac{\rho}{2} \end{cases}. \tag{4.8}$$

Note that, if $\|x\| = \frac{\rho}{2}$, then $2(\rho - \|x\|) = \rho$ and $\rho \frac{x}{\|x\|} = 2x$. Thus

$$\begin{aligned} &h\left(\frac{2(\rho - \|x\|)}{\rho}, \frac{\rho x}{\|x\|}\right) = h(1, 2x) = y_0 \ \text{ (see (4.4) and (4.1)),} \\ &\Rightarrow \gamma_0 \ \text{ is continuous, i.e., } \ \gamma_0 \in C(\overline{B}_\rho \cap \mathbb{R}, H^1(Z)).\end{aligned}$$

Also, if $x \in \partial\overline{B}_\rho \cap \mathbb{R}$, then $\|x\| = \rho$ and so $\gamma_0(x) = h(0, x) = x$ (see (4.6) and recall that $h$ is a deformation). Therefore $\gamma_0 \in \Gamma$. Moreover,

$$\begin{aligned} &\varphi(\gamma_0(x)) \leq \varphi(x) \ \text{ (for all } \ x \in \overline{B}_\rho \cap \mathbb{R} \ \text{ (see (4.5), (4.7) and (4.8)),} \\ &\Rightarrow \varphi(\gamma_0(x)) \leq \eta_\rho < 0 \ \text{ for } x \in \overline{B}_\rho \cap \mathbb{R} \text{ (see (4.1)),} \\ &\Rightarrow \widehat{c}_\rho \leq \eta_\rho < 0 \ \text{ (see (4.3)).}\end{aligned} \tag{4.9}$$

Combining (4.4) and (4.9), we reach a contradiction. This means that $\varphi$ has a third critical point $v_0 \in H^1(Z)$ distinct from 0 and $y_0$. Then

$$\begin{aligned} &0 \in \partial\varphi(v_0), \\ &\Rightarrow A(v_0) = \overline{u}_0 \end{aligned}$$

with $\overline{u}_0 \in L^{r'}(Z), \overline{u}_0(z) \in \partial j(z, v_0(z))$ a.e. on $Z$,

$$\Rightarrow -\triangle v_0(z) = \overline{u}_0(z) \in \partial j(z, v_0(z)) \ \text{ a.e. on } \ Z, \ \frac{\partial v_0}{\partial n} = 0 \text{ on } \partial Z.$$

So, $v_0$ is a nontrivial solution of (1.1) and regularity theory implies $v_0 \in C(\overline{Z})$. Therefore, problem (1.1) has at least two nontrivial solutions $y_0, v_0 \in C^1(\overline{Z})$. $\square$

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