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ON THE NEUMANN PROBLEM WITH SINGULAR AND SUPERLINEAR NONLINEARITIES

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ABSTRACT. We establish the existence of two distinct solutions for problem (1.1) for small values of a parameter $\lambda > 0$ in a subcritical case. This is obtained as a combination of approximation and variational methods. In a critical case we show the existence of at least one solution.

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1. INTRODUCTION

In this paper we investigate the solvability of the Neumann problem

$$\begin{cases} -\Delta u &= Q(x)u^p + \lambda P(x)u^{-\gamma} \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega, \, u > 0 \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and $\lambda > 0$ is a parameter. The exponent p is subcritical, that is $1 < p < \frac{N+2}{N-2}$. The exponent γ of the singular term satisfies $0 < \gamma < 1$. It is assumed that the coefficients P and Q are continuous on $\bar{\Omega}$, $P > 0$ on $\bar{\Omega}$, Q changes sign on Ω, that is, $Q^+ \not\equiv 0$ and $Q^- \not\equiv 0$ on Ω and moreover

$$\int_\Omega Q(x)\, dx < 0. \tag{1.2}$$

Solutions to this problem are sought in the Sobolev space $H^1(\Omega)$. We recall that $H^1(\Omega)$ is the Sobolev space equipped with norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx.$$

We say that $u \in H^1(\Omega)$, with $u > 0$ on Ω, is a solution to problem (1.1) if

$$\int_\Omega \left(\nabla u \nabla v - Q(x)u^p v - \lambda P(x)u^{-\gamma} v\right) dx = 0 \tag{1.3}$$

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for every $v \in H^1(\Omega)$. If both coefficients Q and P are positive, then problem (1.1) does not have a solution in $H^1(\Omega)$. Indeed, testing (1.3) with $v = 1$, we get

$$\int_\Omega \big(Q(x)u^p + \lambda P(x)u^{-\gamma}\big)\, dx = 0.$$

Since this integral must positive, we have arrived at a contradiction. This remark justifies our assumption on the coefficient Q. Equation (1.1) with the Dirichlet boundary conditions has quite an extensive literature (see [2], [3], [4], [9], [8], [6], [7], [14]). Further bibliographical references can be found in [5]. It seems that the corresponding Neumann problem has attracted less attention. In particular, for the Dirichlet problem

$$\begin{cases} -\Delta u &= u^p + \lambda u^{-\gamma} \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega,\ u > 0 \ \text{ on } \ \Omega, \end{cases}$$

it was shown in [2] that there exists a constant $\lambda_* > 0$ such that there exists at least one solution for $\lambda \in (0, \lambda_*)$ and no solution for $\lambda > \lambda_*$. This has been extended in [4] [8], [6], [7] to $p = \frac{N+2}{N-2}$. The authors of these papers also proved the existence of at least two solutions for $0 < \lambda < \lambda^*$ and at least one solution for $\lambda = \lambda_*$ and no solution for $\lambda > \lambda_*$. Similar results have also been obtained for the problem (see [8])

$$\begin{cases} -\Delta u &= \lambda W(x)u^p + h(x)u^{-\gamma} \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega,\ u > 0 \ \text{ on } \ \Omega, \end{cases}$$

where the coefficient W is allowed to change sign.

The variational functional associated with (1.1) has the form

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p+1}\int_\Omega Q(x)|u|^{p+1}\, dx - \frac{\lambda}{1-\gamma}\int_\Omega P(x)|u|^{1-\gamma}\, dx.$$

This functional is not C^1. To obtain solutions to problem (1.1), we consider the approximating problem

$$\begin{cases} -\Delta u &= Q(x)u^p + \lambda P(x)u\big(u^2+\epsilon\big)^{-\frac{1+\gamma}{2}} \ \text{ in } \ \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \ \text{ on } \ \partial\Omega,\ u > 0 \ \text{ on } \ \Omega, \end{cases} \tag{1.4}$$

where $\epsilon > 0$ is small. The corresponding variational functional is given by

$$J_{\lambda,\epsilon}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p+1}\int_\Omega Q(x)|u|^{p+1}\, dx - \frac{\lambda}{1-\gamma}\int_\Omega P(x)\big(u^2+\epsilon\big)^{\frac{1-\gamma}{2}}\, dx.$$

If $\epsilon = 0$ we write $J_{\lambda,0} = J_\lambda$. The functional $J_{\lambda,\epsilon}$ is C^1 for $\epsilon > 0$. For small $\lambda > 0$ and $\epsilon > 0$ this functional has two critical points: a local minimizer and a critical point of the mountain-pass type. Two distinct solutions to problem (1.1) are obtained as limits of these two critical points as $\epsilon \to 0$. In the critical case we only prove the existence of at least one solution of problem (1.1).

The paper is organized as follows. In Section 2 we show that the functionals $J_{\lambda,\epsilon}$ have a mountain-pass structure. Existence of at least two solutions in the subcritical case is given in Sections 3 and 4. The regularity of solutions is discussed in Section 5.

The existence of at least one solution for problem (1.1) in the critical case is given in Section 6.

Throughout this paper, in a given Banach space we denote strong convergence by "$\to$" and a weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, are denoted by $\|\cdot\|_p$.

2. MOUNTAIN-PASS GEOMETRY OF $J_{\lambda,\epsilon}$ AND PALAIS-SMALE CONDITION

In this section and Sections 3 and 4 we assume that $1 < p < \frac{N+2}{N-2}$. First we show that the functional $J_{\lambda,\epsilon}$ has a mountain-pass geometry. The space $H^1(\Omega)$ admits the following decomposition

$$H^1(\Omega) = V \oplus \mathbb{R},$$

where $V = \{v \in H^1(\Omega); \int_\Omega v\,dx = 0\}$. This decomposition allows us to define an equivalent norm on $H^1(\Omega)$:

$$\|u\|_V^2 = \int_\Omega |\nabla v|^2\,dx + t^2.$$

We use the following quantitative statement from [1]: there exists $\eta > 0$ such that for every $t \in \mathbb{R}$ and every $v \in V$ the inequality $\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{1}{2}} \le \eta|t|$ yields

$$\int_\Omega Q(x)|t + v(x)|^{p+1}\,dx \le \frac{|t|^{p+1}}{2}\int_\Omega Q(x)\,dx. \tag{2.1}$$

Lemma 2.1. *There exist positive numbers $\epsilon_\circ$, $\lambda_\circ$, ρ and β such that*

$$J_{\lambda,\epsilon}(u) \ge \beta \;\; \textit{for} \;\; \|u\| = \rho \tag{2.2}$$

and for all $0 < \lambda \le \lambda_\circ$, $0 < \epsilon \le \epsilon_\circ$.

Proof Let $\rho^2 = \|u\|_V^2 = \|\nabla v\|_2^2 + t^2$. We distinguish two cases: (i) $\|\nabla v\|_2 \le \eta|t|$ and (ii) $\|\nabla v\|_2 \ge \eta|t|$. If $\|\nabla v\|_2 \le \eta|t|$ and $\|\nabla v\|_2^2 + t^2 = \rho^2$, then $t^2 \ge \frac{\rho^2}{1+\eta^2}$. By (2.1) we have

$$\int_\Omega Q(x)|v + t|^{p+1}\,dx \le -|t|^{p+1}\alpha$$

with $\alpha = -\frac{1}{2}\int_\Omega Q(x)\,dx > 0$. Using this we obtain the following estimate of $J_{\lambda,\epsilon}$

$$J_{\lambda,\epsilon}(u) \ge \frac{\alpha}{p+1}\left(\frac{\rho^2}{1+\eta^2}\right)^{\frac{p+1}{2}} - \frac{\lambda}{1-\gamma}\int_\Omega P(x)\big(u^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx. \tag{2.3}$$

In case (ii), we first observe that $\|u\|_V \le \|\nabla v\|_2\big(1+\frac{1}{\eta^2}\big)^{\frac{1}{2}}$. Thus applying the Sobolev inequality we get

$$\int_\Omega Q(x)|u|^{2^*}\,dx \le C\|u\|_V^{2^*} \le C_1\big(1+\frac{1}{\eta^2}\big)^{\frac{2^*}{2}}\|\nabla v\|_2^{2^*}$$

for some constants $C, C_1 > 0$. Hence, if ρ is sufficiently small we get

$$\begin{aligned} J_{\lambda,\epsilon}(u) &\geq \frac{1}{2}\|\nabla v\|_2^2 - C_1\big(1+\frac{1}{\eta^2}\big)^{\frac{2^*}{2}}\|\nabla v\|_2^{2^*} - \frac{\lambda}{1-\gamma}\int_\Omega P(x)\big(u^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx \\ &\geq \frac{1}{4}\|\nabla v\|_2^2 - \frac{\lambda}{1-\gamma}\int_\Omega P(x)\big(u^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx. \end{aligned}$$

Since $\|\nabla v\|_2 \geq \frac{\eta\rho}{\left(1+\eta^2\right)^{\frac{1}{2}}}$, we deduce from the above estimate that

$$J_{\lambda,\epsilon}(u) \geq \frac{\rho^2\eta^2}{4\big(1+\eta^2\big)} - \frac{\lambda}{1-\gamma}\int_\Omega P(x)\big(|u|^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx. \tag{2.4}$$

We put $\kappa = \min\left(\frac{\alpha}{p+1}\left(\frac{\rho^2}{1+\eta^2}\right)^{\frac{p+1}{2}}, \frac{\rho^2\eta^2}{4\left(1+\eta^2\right)}\right)$. It follows from (2.3) and (2.4) that

$$J_{\lambda,\epsilon}(u) \geq \kappa - \frac{\lambda}{1-\gamma}\int_\Omega P(x)\big(u^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx \tag{2.5}$$

for $\|u\|_V = \rho$. By the Hölder and Sobolev inequalities, we deduce from (2.5) that

$$J_{\lambda,\epsilon}(u) \geq \kappa - \frac{\lambda M}{1-\gamma}|\Omega|^{\frac{p+\gamma}{p+1}}\left(\|u\|_V^{1-\gamma} + \epsilon^{\frac{1-\gamma}{2}}\right)$$

for some constant $M > 0$ independent of λ and ϵ. We now fix $\epsilon_\circ > 0$, suitably small and select $\lambda_\circ > 0$ so that

$$J_{\lambda,\epsilon}(u) \geq \frac{\kappa}{2} \quad \text{for} \quad \|u\|_V = \rho \tag{2.6}$$

and for all $0 < \lambda < \lambda_\circ$ and $0 < \epsilon < \epsilon_\circ$. Since the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent estimate (2.1) follows. □

Let $\varphi \in H^1(\Omega)$ be such that supp $\varphi \subset$ supp Q^+, $\varphi \not\equiv 0$. Then for $t > 0$ we have

$$J_{\lambda,\epsilon}(t\varphi) \leq \frac{t^2}{2}\int_\Omega |\nabla\varphi|^2\,dx - \frac{t^{p+1}}{p+1}\int_\Omega Q^+|\varphi|^{p+1}\,dx.$$

We choose $t_\circ > 0$ so that $J_{\lambda,\epsilon}(t_\circ\varphi) < 0$ and $\|t_\circ\varphi\| > \rho$. This choice of $t_\circ\varphi$ is independent of λ and ϵ. We set $v_1 = t_\circ\varphi$ and put

$$\Gamma = \{\gamma \in C\big([0,1], H^1(\Omega)\big);\ \gamma(0) = 0,\ \gamma(1) = v_1\}.$$

We now define the mountain-pass level for $J_{\lambda,\epsilon}$

$$c_{\lambda,\epsilon} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_{\lambda,\epsilon}(\gamma(t)) \tag{2.7}$$

for $0 < \lambda \leq \lambda_\circ$ and $0 < \epsilon \leq \epsilon_\circ$. It is easy to check that $c_{\lambda,\epsilon_2} \leq c_{\lambda,\epsilon_1}$ if $\epsilon_1 \leq \epsilon_2$. Since for every $0 < \lambda \leq \lambda_\circ$, $0 < \epsilon \leq \epsilon_\circ$ and $u \in H^1(\Omega)$

$$J_{\lambda,\epsilon}(u) \leq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega Q(x)|u|^{p+1}\,dx,$$

we deduce from this that $c_{\lambda,\epsilon}$ is uniformly bounded in $\epsilon \in (0, \epsilon_\circ]$.

Proposition 2.2. *Let $0 < \lambda \leq \lambda_\circ$ and $0 < \epsilon \leq \epsilon_\circ$. Then every $(PS)_{c_{\lambda,\epsilon}}$ sequence is relatively compact in $H^1(\Omega)$.*

Proof Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_{c_{\lambda,\epsilon}}$ sequence, that is,

$$J_{\lambda,\epsilon}(u_n) \to c_{\lambda,\epsilon} \text{ and } J'_{\lambda,\epsilon}(u_n) \to 0 \text{ in } H^{-1}(\Omega).$$

First we show that $\{u_n\}$ is bounded in $H^1(\Omega)$. Arguing by contradiction, assume $\|u_n\| \to \infty$. We put $v_n = \frac{u_n}{\|u_n\|}$. Since $\|v_n\| = 1$ for every n, we may assume that $v_n \rightharpoonup v$ in $H^1(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$ for every $1 \leq q < 2^*$, where $2^* = \frac{2N}{N-2}$. We then have

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx - \frac{\|u_n\|^{p-1}}{p+1}\int_\Omega Q|v_n|^{p+1}\,dx - \frac{\|u_n\|^{-1-\gamma}}{1-\gamma}\int_\Omega P\Big(\frac{\epsilon}{\|u_n\|^2} + v_n^2\Big)^{\frac{1-\gamma}{2}}\,dx = o(1)$$

and

$$\int_\Omega |\nabla v_n|^2\,dx - \|u_n\|^{p-1}\int_\Omega Q|v_n|^{p+1}\,dx - \|u_n\|^{-1-\gamma}\int_\Omega Pv_n^2\Big(\frac{\epsilon}{\|u_n\|^2} + v_n^2\Big)^{\frac{-1-\gamma}{2}}\,dx = 0.$$

The third term in the first relation tends to 0 as $n \to \infty$. For the third term in the second relation, we have the following estimate

$$\|u_n\|^{-1-\gamma}\int_\Omega Pv_n^2\Big(\frac{\epsilon}{\|u_n\|^2} + v_n^2\Big)^{\frac{-1-\gamma}{2}}\,dx \leq \|u_n\|^{-1-\gamma}\int_\Omega P|v_n|^{1-\gamma}\,dx,$$

which shows that this term tends to 0 as $n \to \infty$. Therefore we can rewrite the last two relations as

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx - \frac{\|u_n\|^{p-1}}{p+1}\int_\Omega Q|v_n|^{p+1}\,dx = o(1)$$

and

$$\int_\Omega |\nabla v_n|^2\,dx - \|u_n\|^{p-1}\int_\Omega Q|v_n|^{p+1}\,dx = o(1).$$

This is only possible when $\int_\Omega |\nabla v_n|^2\,dx \to 0$ and $\|u_n\|^{p-1}\int_\Omega Q|v_n|^{p+1}\,dx \to 0$ as $n \to \infty$. Hence $v = l$ (constant) and $\int_\Omega Q|l|^{p+1}\,dx = 0$. By (1.2) $l = 0$ and $v_n \to 0$ in $H^1(\Omega)$. This contradicts the fact that $\|v_n\| = 1$ for every n. Therefore $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $u_n \to u$ in $L^q(\Omega)$ for $1 \leq q < 2^*$. Since the functional $J_{\lambda,\epsilon}$ contains subcritical nonlinearities, it is easy to show that $\{u_n\}$ is relatively compact in $H^1(\Omega)$. □

Proposition 2.3. *Problem* (1.4) *admits a solution* u_ϵ *for every* $0 < \epsilon \leq \epsilon_\circ$ *and* $0 < \lambda \leq \lambda_\circ$.

Proof This is a consequence of the mountain-pass principle and Proposition 2.2. This solution u_ϵ can be taken to be nonnegative. The Harnack inequality implies that $u_\epsilon > 0$. □

3. EXISTENCE OF A FIRST SOLUTION OF PROBLEM (1.1)

A solution to problem (1.1) will be obtained as a limit point of a family of solutions $\{u_\epsilon\}$, $0<\epsilon\le\epsilon_\circ$, of problem (1.4).

Proposition 3.1. *A family of solutions $\{u_\epsilon\}$, $0<\epsilon\le\epsilon_\circ$, of problem* (1.4) *is relatively compact in $H^1(\Omega)$.*

Proof We commence by showing that family $\{u_\epsilon\}$, $0<\epsilon\le\epsilon_\circ$, is bounded in $H^1(\Omega)$. In the contrary case we can find a sequence $\epsilon_n\to 0$ such that $\|u_{\epsilon_n}\|\to\infty$. We put $u_n=u_{\epsilon_n}$ and $v_n=\frac{u_n}{\|u_n\|}$. We then have

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx-\frac{\|u_n\|^{p-1}}{p+1}\int_\Omega Q|v_n|^{p+1}\,dx - \frac{\lambda}{1-\gamma}\|u_n\|^{-1-\gamma}\int_\Omega P\Big(\frac{\epsilon_n}{\|u_n\|^2}+v_n^2\Big)^{\frac{1-\gamma}{2}}\,dx=o(1) \tag{3.1}$$

and

$$\int_\Omega |\nabla v_n|^2\,dx-\|u_n\|^{p-1}\int_\Omega Q|v_n|^{p+1}\,dx-\lambda\|u_n\|^{-1-\gamma}\int_\Omega P\frac{v_n^2}{\big(\frac{\epsilon_n}{\|u_n\|^2}+v_n^2\big)^{\frac{1+\gamma}{2}}}\,dx=0. \tag{3.2}$$

Since the third terms in (3.1) and (3.2) tend to 0 as $n\to\infty$, we can rewrite these relations as

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2\,dx-\frac{\|u_n\|^{p-1}}{p+1}\int_\Omega Q|v_n|^{p+1}\,dx=o(1)$$

and

$$\int_\Omega |\nabla v_n|^2\,dx-\|u_n\|^{p-1}\int_\Omega Q|v_n|^{p+1}\,dx=o(1).$$

From this we derive a contradiction, as in the proof of Proposition 2.2. Since $\{u_\epsilon\}$, $0<\epsilon\le\epsilon_\circ$, is bounded we can choose a sequence $\{u_{\epsilon_n}\}$ with $\epsilon_n\to 0$, denoted again by $\{u_n\}$, such that $u_n\rightharpoonup u$ in $H^1(\Omega)$ and $u_n\to u$ in $L^q(\Omega)$ for $1\le q<2^*$. To proceed further we need the following estimate: there exists a constant C_1, independent of ϵ, such that

$$\int_\Omega \frac{u_\epsilon}{\big(u_\epsilon^2+\epsilon\big)^{\gamma+\frac{1}{2}}}\,dx\le C_1 \tag{3.3}$$

for $0<\epsilon\le\epsilon_\circ$. To show this, we observe that u_ϵ satisfies

$$\int_\Omega \nabla u_\epsilon\nabla v\,dx-\int_\Omega Qu_\epsilon^p v\,dx-\lambda\int_\Omega P\frac{u_\epsilon v}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma+1}{2}}}\,dx=0 \tag{3.4}$$

for every $v\in H^1(\Omega)$. Testing (3.4) with $v=\big(u_\epsilon^2+\epsilon\big)^{-\frac{\gamma}{2}}$ we get

$$-\gamma\int_\Omega \frac{u_\epsilon|\nabla u_\epsilon|^2}{\big(u_\epsilon^2+\epsilon\big)^{1+\frac{\gamma}{2}}}\,dx-\int_\Omega Q\frac{u_\epsilon^p}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma}{2}}}\,dx=\lambda\int_\Omega P\frac{u_\epsilon}{\big(u_\epsilon^2+\epsilon\big)^{\gamma+\frac{1}{2}}}\,dx.$$

From this, we derive the following inequality

$$\lambda\int_\Omega P\frac{u_\epsilon}{\big(u_\epsilon^2+\epsilon\big)^{\gamma+\frac12}}\,dx \le \int_\Omega Q^-\frac{u_\epsilon^p}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma}{2}}}\,dx \le \int_\Omega Q^- u_\epsilon^{p-\gamma}\,dx.$$

Since $\{\|u_\epsilon\|_{p+1-\gamma}\}$ is bounded independently of ϵ and $\min_{x\in\bar\Omega}P(x)>0$, the estimate (3.3) follows. For $n<m$, we have

$$\begin{aligned}\int_\Omega|\nabla(u_n-u_m)|^2\,dx \;&-\; \int_\Omega Q\big(u_n^p-u_m^p\big)(u_n-u_m)\,dx \qquad (3.5)\\ &=\; \lambda\int_\Omega P\bigg(\frac{u_n}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma+1}{2}}}-\frac{u_m}{\big(u_m^2+\epsilon_m\big)^{\frac{\gamma+1}{2}}}\bigg)(u_n-u_m)\,dx.\end{aligned}$$

It is clear that

$$\lim_{n,m\to\infty}\int_\Omega Q\big(u_n^p-u_m^p\big)(u_n-u_m)\,dx=0 \qquad (3.6)$$

Let us denote the integral on the right-hand side of equation (3.5) by $I_{n,m}$. With the aid of (3.3) and the Hölder inequality, we obtain

$$\begin{aligned}|I_{n,m}| \;&\le\; \int_\Omega P\frac{u_n^{\frac12}}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma}{2}+\frac14}}|u_n-u_m|\,dx+\int_\Omega P\frac{u_m^{\frac12}}{\big(u_m^2+\epsilon_m\big)^{\frac{\gamma}{2}+\frac14}}|u_n-u_m|\,dx\\ &\le\; 2\|P\|_\infty C_1^{\frac12}\bigg(\int_\Omega|u_n-u_m|^2\,dx\bigg)^{\frac12}.\end{aligned}$$

Hence $I_{n,m}\to0$ as $n,m\to\infty$ and so

$$\lim_{n,m\to\infty}\int_\Omega|\nabla(u_n-u_m)|^2\,dx=0$$

and this completes the proof. □

We are now in a position to formulate the existence result for problem (1.1).

Theorem 3.2. *Suppose that $\gamma<\min(p-1,1)$. For every $0<\lambda\le\lambda_\circ$ problem* (1.1) *admits a solution u_λ such that $J_\lambda(u_\lambda)>0$.*

Proof Let $\{u_\epsilon\}$, $0<\epsilon<\epsilon_\circ$, be a family of solutions of problem (1.4). By Proposition 3.1 there exists a sequence ϵ_n such that $\epsilon_n\to0$ and $u_n:=u_{\epsilon_n}\to u$ in $H^1(\Omega)$. First we show that $u\not\equiv0$. Testing (3.4) with $v=u_n$ we get

$$\int_\Omega|\nabla u_n|^2\,dx-\int_\Omega Qu_n^{p+1}\,dx-\lambda\int_\Omega P\frac{u_n^2}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma+1}{2}}}\,dx=0.$$

Combining this with $J_{\lambda,\epsilon_n}(u_n)=c_{\lambda,\epsilon_n}$ we get

$$\Big(\frac12-\frac1{p+1}\Big)\int_\Omega Qu_n^{p+1}\,dx+\frac{\lambda}{2}\int_\Omega P\frac{u_n^2}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma+1}{2}}}\,dx-\frac{\lambda}{1-\gamma}\int_\Omega P\big(u_n^2+\epsilon_n\big)^{\frac{1-\gamma}{2}}\,dx=c_{\lambda,\epsilon_n}.$$

We rewrite this as

$$\begin{aligned}(\frac{1}{2}-\frac{1}{p+1})\int_\Omega Qu_n^{p+1}\,dx &+ (\frac{1}{2}-\frac{1}{1-\gamma})\lambda\int_\Omega P\big(u_n^2+\epsilon_n\big)^{\frac{1-\gamma}{2}}\,dx \\ &= c_{\lambda,\epsilon_n}+\frac{\epsilon_n\lambda}{2}\int_\Omega P\big(u_n^2+\epsilon_n\big)^{\frac{-1-\gamma}{2}}\,dx.\end{aligned} \tag{3.7}$$

Assuming that $u_n \rightharpoonup 0$ in $H^1(\Omega)$, the left-hand side of (3.7) tends to 0 as $n\to\infty$, while for the right-hand side we have

$$\liminf_{n\to\infty}\Big(c_{\lambda,\epsilon_n}+\frac{\epsilon_n\lambda}{2}\int_\Omega P\big(u_n^2+\epsilon_m\big)^{\frac{-1-\gamma}{2}}\,dx\Big)\geq\beta>0.$$

and we have arrived at a contradiction. In the final step we show that u is a weak solution of (1.1). Let $v\in H^1(\Omega)$. For every n we have

$$\int_\Omega \nabla u_n\nabla v\,dx-\int_\Omega Qu_n^p v\,dx=\lambda\int_\Omega P\frac{u_n v}{\big(u_n^2+\epsilon_n\big)^{\frac{1+\gamma}{2}}}\,dx.$$

The left-hand side has a limit

$$\lim_{n\to\infty}\Big(\int_\Omega \nabla u_n\nabla v\,dx-\int_\Omega Qu_n^p v\,dx\Big)=\int_\Omega \nabla u\nabla v\,dx-\int_\Omega Qu^p v\,dx.$$

Hence the right-hand side also has a limit. We now evaluate this limit. For a small $\delta>0$ we write

$$\begin{aligned}\int_\Omega P\frac{u_n v}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma+1}{2}}}\,dx &= \int_{u_n\leq\delta} P\frac{u_n v}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma+1}{2}}}\,dx \\ &\quad+\int_{u_n>\delta} P\frac{u_n v}{\big(u_n^2+\epsilon_n\big)^{\frac{\gamma+1}{2}}}\,dx = I_{1,n}+I_{2,n}.\end{aligned}$$

We need the following estimate: there exists a constant $C_2>0$, independent of ϵ, such that

$$\int_\Omega \frac{u_\epsilon}{\big(u_\epsilon^2+\epsilon\big)^{\gamma+1}}\,dx\leq C_2 \tag{3.8}$$

for all $0<\epsilon\leq\epsilon_\circ$. To show this, we test (3.4) with $v=\frac{1}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma+1}{2}}}$ and get

$$-(\gamma+1)\int_\Omega \frac{u_\epsilon|\nabla u_\epsilon|^2}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma+3}{2}}}\,dx-\int_\Omega Q\frac{u_\epsilon^p}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma+1}{2}}}\,dx=\lambda\int_\Omega P\frac{u_\epsilon}{\big(u_\epsilon^2+\epsilon\big)^{\gamma+1}}\,dx.$$

From this we deduce

$$\int_\Omega Q^-\frac{u_\epsilon^p}{\big(u_\epsilon^2+\epsilon\big)^{\frac{\gamma+1}{2}}}\,dx\geq\lambda\int_\Omega P\frac{u_\epsilon}{\big(u_\epsilon^2+\epsilon\big)^{\gamma+1}}\,dx.$$

Since $\|u_\epsilon\|_{p-\gamma-1}$ is bounded independently of ϵ, estimate (3.8) follows. By (3.8) and the Hölder inequality we have

$$\begin{aligned} I_{1,n} &\leq \|P\|_\infty \delta^{\frac{1}{2}} \int_{u_n \leq \delta} \frac{u_n^{\frac{1}{2}}|v|}{\left(u_n^2+\epsilon_n\right)^{\frac{\gamma+1}{2}}}\,dx \\ &\leq \|P\|_\infty \delta^{\frac{1}{2}} \left(\int_\Omega \frac{u_n}{\left(u_n^2+\epsilon_n\right)^{\gamma+1}}\,dx\right)^{\frac{1}{2}} \|v\|_2 \leq \|P\|_\infty \delta^{\frac{1}{2}} C_2^{\frac{1}{2}} \|v\|_2. \end{aligned}$$

For $I_{2,n}$, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} I_{2,n} = \int_{u\geq\delta} P u^{-\gamma}\,dx.$$

Since δ is arbitrary, we get

$$\lim_{n\to\infty} \int_\Omega P \frac{u_n v}{\left(u_n^2+\epsilon_n\right)^{\frac{1+\gamma}{2}}}\,dx = \int_\Omega P u^{-\gamma} v\,dx.$$

The fact that u is strictly positive on Ω follows from the Harnack inequality. Details of this are given in Section 5. Since $\lim_{\epsilon\to 0} c_{\lambda,\epsilon} > 0$, we see that $J_\lambda(u_\lambda) > 0$. □

4. EXISTENCE OF A SECOND SOLUTION OF PROBLEM (1.1)

We begin by showing that there exists a second solution of problem (1.4) for every $0 < \epsilon \leq \epsilon_\circ$. These solutions are local minimizers of functionals $J_{\lambda,\epsilon}$. Let $\psi \in H^1(\Omega)$ with $\psi \not\equiv 0$ and let $0 < \lambda \leq \lambda_\circ$. Then there exist $t > 0$ small and $a > 0$ such that

$$J_{\lambda,\epsilon}(t\psi) \leq \frac{t^2}{2}\int_\Omega |\nabla\psi|^2\,dx - \frac{t^{p+1}}{p+1}\int_\Omega Q|\psi|^{p+1}\,dx - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega P\left(\psi^2+\frac{\epsilon}{t^2}\right)^{\frac{1-\gamma}{2}}\,dx \leq -a$$

and $\|t\psi\| < \rho$. It is clear that the choice of t and a can be made independently of $0 < \epsilon \leq \epsilon_\circ$. Hence, we have

$$d_{\lambda,\epsilon} = \inf_{\|u\|\leq\rho} J_{\lambda,\epsilon}(u) \leq -a. \tag{4.1}$$

Since problem (4.1) is subcritical, for every $0 < \epsilon \leq \epsilon_\circ$ there exists a minimizer w_ϵ which is a solution problem (1.4). We aim to show that the family of minimizers $\{w_\epsilon\}$, $0 < \epsilon \leq \epsilon_\circ$, has a limit point in $H^1(\Omega)$ which is a solution to problem (1.1).

Theorem 4.1. *Let $\gamma < \min(p-1, 1)$. For every $0 < \lambda \leq \lambda_\circ$ problem (1.1) has a solution w_λ such that $J_\lambda(w_\lambda) < 0$.*

Proof We follow the argument used in Section 3. As in Proposition 3.1 we show that $\{w_\epsilon\}$, $0 < \epsilon \leq \epsilon_\circ$, is relatively compact in $H^1(\Omega)$.Then there exists a sequence $\epsilon_n \to 0$ such that $w_{\epsilon_n} \to w$ in $H^1(\Omega)$. We put $w_n = w_{\epsilon_n}$. It is easy to show that

$$\lim_{n\to\infty} \int_\Omega P\left(w_n^2+\epsilon\right)^{\frac{1-\gamma}{2}}\,dx = \int_\Omega P w^{1-\gamma}\,dx.$$

Hence $J_\lambda(w) \leq \lim_{n\to\infty} J_{\lambda,\epsilon_n}(w_n) \leq -a$. This shows that $w \not\equiv 0$. To show that w satisfies (1.1) we repeat the final part of the proof of Theorem 3.2. □

5. REMARKS ON THE REGULARITY OF SOLUTIONS OF PROBLEM (1.1)

We use the following results from [13]:

Lemma 5.1. *Suppose that $\partial\Omega \in C^1$ and $a(x) \in L^{\frac{N}{2}}(\Omega)$. If u is a weak solution of problem*

$$\begin{cases} -\Delta u & = a(x)u \quad in \quad \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad on \quad \partial\Omega, \end{cases}$$

then $u \in L^s(\Omega)$ for every $s \geq 1$.

Lemma 5.2. *Suppose that $\partial\Omega \in C^2$, $f \in L^s(\Omega)$, $s > 1$. If u is a weak solution of problem*

$$\begin{cases} -\Delta u & = f(x) \quad in \quad \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad on \quad \partial\Omega, \end{cases}$$

then $u \in H^{2,s}(\Omega)$.

These two lemmas yield the following regularity result for solutions of problem (1.1).

Theorem 5.3. *If $\partial\Omega \in C^2$ then a weak solution to problem* (1.1) *belongs to $C^{1+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$.*

Proof First we observe that testing (1.3) with $v = \frac{1}{\left(u^2+\epsilon\right)^{\frac{p}{2}}}$ gives

$$-p\int_\Omega \frac{u|\nabla u|^2}{\left(u^2+\epsilon\right)^{\frac{p}{2}+1}}\,dx - \int_\Omega Q\frac{u^p}{\left(u^2+\epsilon\right)^{\frac{p}{2}}}\,dx = \lambda\int_\Omega P\frac{u^{-\gamma}}{\left(u^2+\epsilon\right)^{\frac{p}{2}}}\,dx.$$

From this we deduce, using the Fatou lemma, that

$$\int_\Omega Q^-\,dx \geq \lambda m_P \int_\Omega u^{-(\gamma+p)}\,dx,$$

where $m_P = \min_{x\in\bar{\Omega}} P(x)$. Using this estimate and a test function $v = \frac{1}{\left(u^2+\epsilon\right)^{\frac{2p+\gamma}{2}}}$ we get

$$\frac{\|Q^-\|_\infty}{\left(\lambda m_P\right)^2}\int_\Omega Q^-\,dx \geq \int_\Omega \frac{dx}{u^{2p+2\gamma}}.$$

By iteration we obtain

$$\frac{\|Q^-\|_\infty^{k-1}}{\left(\lambda m_P\right)^k}\int_\Omega Q^-\,dx \geq \int_\Omega \frac{dx}{u^{kp+k\gamma}}$$

for every integer $k \geq 1$. This shows that $u^{-1} \in L^q(\Omega)$ for every $q \geq 1$. Applying Lemma 5.1 with $a(x) = Q(x)u(x)^{p-1} + P(x)u^{-1-\gamma} \in L^{\frac{N}{2}}(\Omega)$, we see that $u \in L^s(\Omega)$ for every $s \geq 1$. Since by Lemma 5.2 $u \in H^{2,s}(\Omega)$, obviously $u \in C^{1+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. $\square$

Remark 5.4. A solution u to problem (1.1) satisfies following equation

$$-\Delta u + c(x)u = 0 \text{ in } \Omega$$

where $c(x) = -\big(Q(x)u(x)^{p-1} + P(x)u^{-1-\gamma}\big) \in L^N(\Omega)$. Therefore we can apply the Harnack inequality (see Théorème 7.1 in [10]) to obtain $u > 0$ on Ω.

6. CRITICAL CASE

In this section we consider the case $p = 2^* - 1$. We establish the existence of at least one solution of problem (1.1) through a local minimization and approximation. It is clear that Lemma 2.1 continues to hold for $p = 2^* - 1$. As in Section 4 we can show that there exists $a > 0$ such that (4.1) is valid.

Proposition 6.1. *For every $0 < \epsilon \leq \epsilon_\circ$ and $0 < \lambda \leq \lambda_\circ$ problem* (1.4) *has a solution u_ϵ.*

Proof Let $\{u_n\}$ be a minimizing sequence for $d_{\lambda,\epsilon}$. Since $\|u_n\| \leq \rho$ we may assume that $u_n \rightharpoonup u_\epsilon$ in $H^1(\Omega)$ and $L^{2^*}(\Omega)$ and moreover $u_n \to u_\epsilon$ in $L^q(\Omega)$ for $1 \leq q < 2^*$. Also, we may assume that $\|u_n\| \leq \frac{\rho}{2}$ because $J_{\lambda,\epsilon}(u) \geq \beta_2 > 0$ for $\|u_n\| = \rho$. By the Ekeland variational principle $J_{\lambda,\epsilon}(u_n) \to 0$ in $H^{-1}(\Omega)$. First we show that $u_\epsilon \not\equiv 0$ on Ω. Indeed, we have

$$\begin{aligned} -a \;&\geq\; d_{\lambda,\epsilon} \geq \liminf_{n\to\infty}\left[J_{\lambda,\epsilon}(u_n) - \frac{1}{2^*}\langle J'_{\lambda,\epsilon}(u_n), u_n\rangle\right] \qquad (6.1)\\ &=\; \liminf_{n\to\infty}\left[\frac{1}{N}\int_\Omega |\nabla u_n|^2\,dx + \frac{\lambda}{2^*}\int_\Omega P u_n^2\big(u_n^2+\epsilon\big)^{-\frac{1+\gamma}{2}}\,dx\right.\\ &-\; \left.\frac{\lambda}{1-\gamma}\int_\Omega P\big(u_n^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx\right]. \end{aligned}$$

Assume that $u_\epsilon = 0$ on Ω. It is easy to show that (6.1) cannot be satisfied for $\epsilon > 0$ sufficiently small and we get a contradiction. Obviously u_ϵ satisfies (1.4) in a weak sense. By the Harnack inequality $u_\epsilon > 0$ on $\bar\Omega$. Letting $n \to \infty$ in (6.1) we get

$$\begin{aligned} d_{\lambda,\epsilon} \;&\geq\; \frac{1}{N}\int_\Omega |\nabla u_\epsilon|^2\,dx + \frac{\lambda}{2^*}\int_\Omega P u_\epsilon^2\big(u_\epsilon+\epsilon\big)^{-\frac{1+\gamma}{2}}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega P\big(u_\epsilon+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx\\ &=\; J_{\lambda,\epsilon}(u_\epsilon) - \frac{1}{2^*}\langle J'_{\lambda,\epsilon}(u_\epsilon), u_\epsilon\rangle = J_{\lambda,\epsilon}(u_\epsilon) \geq d_{\lambda,\epsilon} \end{aligned}$$

as $\|u_\epsilon\| \leq \rho$. Hence we have $J_{\lambda,\epsilon}(u_\epsilon) = d_{\lambda,\epsilon}$. □

Theorem 6.2. *Suppose that $0 < \gamma < 1$. For every $0 < \lambda \leq \lambda_\circ$ problem* (1.1) *has a solution u such that $J_\lambda(u) < 0$.*

Proof Let $\{u_\epsilon\}$, $0 < \epsilon \leq \epsilon_\circ$, be a family of solution of problem (1.4) obtained in Proposition 6.1. Since u_ϵ is bounded in $H^1(\Omega)$ there exists a sequence $\epsilon_n \to 0$

such that $u_n := u_{\epsilon_n} \rightharpoonup u$ in $H^1(\Omega)$ and $L^{2^*}(\Omega)$ and moreover $u_n \to u$ in $L^q(\Omega)$ for $1 \leq q < 2^*$. We now observe that for every n we have

$$\begin{aligned} -a \geq d_{\lambda,\epsilon_n} &= J_{\lambda,\epsilon_n}(u_n) = J_{\lambda,\epsilon_n}(u_n) - \frac{1}{2^*}\langle J'_{\lambda,\epsilon_n}(u_n), u_n\rangle \\ &= \frac{1}{N}\int_\Omega |\nabla u_n|^2\,dx + \frac{\lambda}{2^*}\int_\Omega P u_n^2\big(u_n^2+\epsilon\big)^{-\frac{1+\gamma}{2}}\,dx \\ &- \frac{\lambda}{1-\gamma}\int_\Omega P\big(u_n^2+\epsilon\big)^{\frac{1-\gamma}{2}}\,dx. \end{aligned} \tag{6.2}$$

From this we easily deduce that $u \not\equiv 0$. It is clear that estimate (3.8) remains true in the critical case. This estimate allows us to show that u satisfies (1.1) in the weak sense (see the final part of the proof of Theorem 3.2). Finally, we show that $J_\lambda(u) < 0$. Letting $n \to \infty$ in (6.2) and using the lower semi-continuity of norm with respect to weak convergence, we obtain

$$0 > -a \geq \frac{1}{N}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2^*}\int_\Omega P u^{1-\gamma}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega P u^{1-\gamma}\,dx.$$

Since u satisfies (1.1) we get

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega Q u^{p+1}\,dx + \lambda\int_\Omega P u^{1-\gamma}\,dx.$$

Combining the last two relations we obtain $J_\lambda(u) \leq a < 0$. □

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