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EXISTENCE RESULTS TO $-\Delta_p u = V(x)f(u)$ IN A BOUNDED DOMAIN OF $\mathbb{R}^d$

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1. INTRODUCTION

In this paper we shall investigate the existence of weak solutions of the Dirichlet p-Laplacian problem:

$$\begin{cases} -\Delta_p u &= V(x)f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded smooth open set of $\mathbb{R}^d$ and $p > 1$. We will exhibit a sufficient condition on the norm of the positive function V which belong to a certain space depending on the position of p and d. The map f considered here is supposed continuous, positive and non decreasing. The case $p = 2$ was treated in [12]. Along this investigation we will use some regularity results (see [10] or [19]) and estimations (see [14]) for the potential to replace the lack of linearity. The proof in the case $p = 2$ can be seen as a direct approach using estimations for the Green function. We will show also a kind of sharpness of our estimations for certain class of functions V.

There are two different cases to distinguish for the study of this problem:

• The case where $f(0) > 0$, the trivial solution is eliminated. We will show the existence of a positive solution using the topological degree theory applied to the operator $T_V : C_0\left(\overline{\Omega}\right) \longrightarrow C_0\left(\overline{\Omega}\right)$. This operator is defined by $T_V(v) = u$ if and only if u satisfies

$$\begin{cases} -\Delta_p u &= V(x)f(v) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{cases}$$

• The second case where $f(0) = 0$, we shall find a way to eliminate the trivial solution. Therefore we will use a classical minimization technic and we will show that the energy of the solution is negative, so the problem have a non trivial solution.

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2. MAIN RESULTS

Let us consider the Dirichlet problem

$$\begin{cases} -\Delta_p u &= V(x)f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{P}$$

where Ω is a bounded smooth open set of $\mathbb{R}^d$ and V is a positive function which belong to a certain space depending on the value of $1 < p < \infty$. The non linearity f is a nondecreasing continuous positive function.

We will denote

$$E = \begin{cases} L^q(\Omega),\ q > \frac{d}{p}, \text{ if } p < d \\ L^1(Log(L)^\beta)(\Omega),\ \beta > d-1, \text{ if } p = d \\ L^q(\Omega), q \geq 1 \text{ if } p > d \end{cases}$$

and $\| \ \|_E$ the corresponding norm.

The main result of this paper can be stated as follows.

Theorem 2.1. *Let $1 < p < \infty$, if $V \in E$, then there exists a constant $c = c(d, p, \Omega)$ such that, if $f(0) > 0$ and V satisfies*

$$\|V\|_E < c \sup_{\alpha > 0} \frac{\alpha^{p-1}}{f(\alpha)} \tag{2.1}$$

then (P) have at least one positive bounded solution u. Furthermore, this solution is stable i.e. $d(I - T_V, \mathcal{M}, 0) = 1$ where $\mathcal{M}$ is a set to be pointed out later in the proof.

Theorem 2.2. *Under the same assumptions of Theorem* **2.1***, if we suppose furthermore that $\left(\lambda^{-\gamma} V(\frac{x}{\lambda})\right)_{\lambda>0}$ converges weakly in measure to a positive Radon measure μ when $\lambda \longrightarrow \infty$ for a constant $\gamma > p$ and that 0 is an isolated zero of f, then (P) have at least one positive solution.*

As an example for such function V one can think about a $-\gamma$-homogeneous function, or a combination of an approximation of the identity and an homogeneous function such that $\left(\lambda^{-\gamma} V(\frac{x}{\lambda})\right)_{\lambda>0}$ converges to a Dirac.

Now if Ω' is an open set contained in Ω, we denote $\lambda_1(\Omega', \psi)$ the first eigenvalue of the p-Laplace operator in the set Ω' with weight function ψ, i.e

$$\lambda_1(\Omega', \psi) = \inf\left\{ \int_{\Omega'} |\nabla u|^p \ ; \int_{\Omega'} |u|^p \psi = 1,\ u \in H_0^1(\Omega') \right\}.$$

So we can state the following theorem.

Theorem 2.3. *Assume that $\widetilde{V} = \frac{V}{\|V\|_E} \in \mathcal{H}_\psi(\Omega') = \{v \in E;\ v > \psi \text{ on } \Omega'\}$, where $\psi \in E$ is a positive function, then there exist $\lambda^* \in [0, +\infty]$ such that:*

i) if $\|V\|_E < \lambda^*$, problem (P) have at least one positive solution.

ii) if $\|V\|_E > \lambda^*$, problem (P) have no positive solution.

Moreover, we have the following estimation :

$$c(p,d,\Omega)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)}\leq\lambda^*\leq\lambda_1(\Omega',\psi)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)}.$$

Corollary 2.4. *Assume that* $\widetilde{V}=\frac{V}{\|V\|_E}\in\mathcal{H}_\delta(\Omega)=\{v\in E;\ v>\delta\}$*, then there exist* $\lambda^*\in[0,+\infty]$ *such that :*

i) if $\|V\|_E<\lambda^*$*, problem (P) have at least one positive solution.*

ii) if $\|V\|_E>\lambda^*$*, problem (P) have no positive solution.*

Moreover, we have the following estimation :

$$c(p,d,\Omega)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)}\leq\lambda^*\leq\frac{\lambda_1}{\delta}\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)}.$$

3. PRELIMINARY RESULTS

Lemma 3.1. *Let* $u_1,\ u_2\in W_0^{1,p}(\Omega)$*, there exist a constant* c_p *such that*

$$\langle-\Delta_p u_1-(-\Delta_p u_2),u_1-u_2\rangle\geq\begin{cases} c_p\left|\nabla u_1\text{-}\nabla u_2\right|^p, & \text{if}\quad p\geq 2\\ c_p\dfrac{\left|\nabla u_1\text{-}\nabla u_2\right|^2}{\left(|\nabla u_1|+|\nabla u_2|\right)^{2-p}}, & \text{if}\quad 1<p<2\end{cases}$$

Consider the problem

$$\begin{cases}-\Delta_p u &= f(x,u)\text{ in }\Omega\\ u &= 0\text{ on }\partial\Omega,\end{cases}\tag{3.1}$$

where $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function.

Definition 3.2. We say that $\overline{U}$ is a super-solution of (3.1) if $\overline{U}\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and

$$\begin{cases}-\Delta_p\overline{U} &\geq f(x,\overline{U})\text{ in }\Omega\\ \overline{U} &\geq 0\text{ on }\partial\Omega.\end{cases}$$

Respectively, we say that $\underline{U}$ is a sub-solution of (3.1) if $\underline{U}\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and

$$\begin{cases}-\Delta_p\underline{U} &\leq f(x,\underline{U})\text{ in }\Omega\\ \underline{U} &\leq 0\text{ on }\partial\Omega.\end{cases}$$

Theorem 3.3 ([7])**.** *Let us assume the following conditions :*

i) The problem (3.1) *have a sub-solution* $\underline{U}$ *and a super-solution* $\overline{U}$ *such that* $\underline{U}\leq\overline{U}$.

ii) There exist $K\in L^q(\Omega)$*,* $q>(p^*)'$*, such that*

$$|f(x,s)|\leq K(x),\ a.e,\ x\in\Omega,\ \forall s:\underline{U}(x)\leq s\leq\overline{U}(x).$$

Then (3.1) have at least one solution between $\underline{U}$ *and* $\overline{U}$.

Let us define the Orlicz-Zygmund space $L^sLog^\beta L$, $1\leq s$, $\beta\in\mathbb{R}$.

Definition 3.4. Let f be a measurable function in Ω, we say that

$$f \in L^s Log^\beta L(\Omega) \text{ if } \int_\Omega |f|^s \log^\beta (e + |f|) < +\infty.$$

This space is equipped with the Luxemburg norm defined by

$$\|f\|_{L^s Log^\beta L} = \inf \left\{ \lambda > 0; \int_\Omega \frac{|f|^s}{|\lambda|^s} \log^\beta \left(e + \frac{|f|}{|s|} \right) \leq 1 \right\}$$

Remark that for $\beta \geq 0$, $L^s Log^\beta L(\Omega) \subset L^s(\Omega)$ and if we note

$$[f]_{\alpha,s} = \int_\Omega |f|^s \log^\beta \left(e + \frac{|f|}{\|f\|_{L^s}} \right),$$

then

$$\|f\|_{L^s Log^\beta L} \leq [f]_{\alpha,s} \leq 2 \|f\|_{L^s Log^\beta L}.$$

4. PROOF OF THEOREMS

4.1. **Proof of Theorem 2.1.** STEP 1: Boundedness results.

We start with the case $p < d$.

Lemma 4.1 ([1])**.** *Let $u \in W_0^1(\Omega)$ and $f \in W^{-1,r}(\Omega)$ where $r > \frac{d}{p-1}$ and $p < d$, such that*

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{4.1a}$$

then $u \in L^\infty(\Omega)$, furthermore if we take F such that $f = \mathrm{div} F$ then

$$\|u\|_{L^\infty(\Omega)} \leq c(p, d, \Omega) \|F\|_{L^r}^{\frac{1}{p-1}}.$$

Before proving this lemma we will use the following result, introduced by Stampacchia for the study of the regularity of elliptic equation in [17].

Lemma 4.2 ([12])**.** *Let $\varphi : [k_0, +\infty[\longrightarrow \mathbb{R}_+$ be non decreasing function, such that if $k_0 \leq k < h$ then $\varphi(h) \leq \frac{c}{(h-k)^\alpha} \varphi(k)^\beta$, where c, α and β are given positive constants. If $\beta > 1$, then $\varphi(k_0 + l) = 0$ where*

$$l^\alpha = c [\varphi(k_0)]^{\beta - 1} 2^{\frac{\alpha\beta}{\beta - \alpha}}.$$

Proof. To proof these theorem we will use the classical Stampacchia approach.

Since $f \in W^{-1,r}(\Omega)$, there exist $F \in L^r(\Omega)$ such that $f = \mathrm{div} F$ and then we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \int_\Omega F \cdot \nabla v, \ \forall v \in W_0^{1,p}(\Omega). \tag{4.2}$$

So if we take for $k > 0$ the following test function

$$v = \mathrm{sign}(u - k)(|u| - k)^+ = \begin{cases} u - k & \text{if} \quad u > k \\ 0 & \text{if} \quad -k \leq u \leq k \\ u + k & \text{if} \quad -k > u. \end{cases}$$

(4.2) become

$$\int_{A(k)} |\nabla v|^p = \int_\Omega F\cdot\nabla v,$$

where $A(k)=\{|u|>k\}$. Using Hôlder inequality, we have

$$\int_\Omega F\cdot\nabla v \le \operatorname{mes}(A(k))^{1-\frac{1}{p}-\frac{1}{r}}\left(\int_{A(k)} |F|^q\right)^{\frac{1}{r}}\left(\int_{A(k)} |\nabla v|^p\right)^{\frac{1}{r}}.$$

So that

$$\left(\int_{A(k)} |\nabla v|^p\right)^{1-\frac{1}{p}} \le \operatorname{mes}(A(k))^{1-\frac{1}{p}-\frac{1}{r}}\left(\int_{A(k)} |F|^r\right)^{\frac{1}{r}}.$$

Using Sobolev embeddings, there exist $C>0$ such that

$$C\left(\int_{A(k)} |\nabla v|^{p^*}\right)^{\frac{p}{p^*}} \le \operatorname{mes}(A(k))^{1-\frac{1}{p}-\frac{1}{r}}\left(\int_{A(k)} |F|^r\right)^{\frac{1}{r}}$$

note that if $0<k<h$ then $A(h)\subset A(k)$ and that implies

$$\operatorname{mes}(A(h))^{\frac{1}{p^*}}(h-k) = \left(\int_{A(h)} (h-k)^{p^*}\right)^{\frac{1}{p^*}} \le \left(\int_{A(h)} |v|^{p^*}\right)^{\frac{1}{p^*}} \le \left(\int_{A(k)} |v|^{p^*}\right)^{\frac{1}{p^*}}$$

finally we have

$$\operatorname{mes}(A(h)) \le \frac{\|F\|_{L^r}^{\frac{p^*}{p-1}}}{C^{\frac{p^*}{p}}(h-k)^{p^*}}\operatorname{mes}(A(k))^{p^*(\frac{1}{p}-\frac{1}{(p-1)r})}$$

and using the fact that $r>\frac{d}{p-1}$, we have $p^*(\frac{1}{p}-\frac{1}{(p-1)r})>1$ so we apply the Stampacchia's lemma to $\varphi(h)=\operatorname{mes}(A(h))$ to obtain

$$\|u\|_\infty \le c\frac{\|F\|_{L^r}^{\frac{1}{p-1}}}{C^{\frac{1}{p}}}\operatorname{mes}(\Omega)^{(\frac{1}{p}-\frac{1}{(p-1)r})-\frac{1}{p^*}}.$$

□

The case $d=p$ is more different than the previous.

The Orlicz-Zygmund spaces are defined using the function $P_\beta:\mathbb{R}^+\longrightarrow\mathbb{R}$ such that $P_\beta(t)=t\log(e+s^\beta)$.

Lemma 4.3. *If $c>1$ then for s small enough,*

$$sP_\beta^{-1}\left(\frac{1}{s}\right) \le \frac{c}{\left(\log(e+\frac{1}{s})\right)^\beta}.$$

Proof. It is easy to show that

$$\lim_{x\longrightarrow+\infty}\frac{1}{x}P_\beta(\frac{cx}{(\log(e+x))^\beta})=c>1,$$

and so the lemma is proved. ☐

Now we can state a result similar to Lemma **4.1**

Lemma 4.4 ([4]). *Let $u\in W_0^{1,d}(\Omega)$ satisfying $-\Delta_d u=f$, assume that $f\in L^1(\log L)^\beta(\Omega)$, where $\beta>d-1$ then $u\in L^\infty(\Omega)$. More precisely*

$$\|u\|_{L^\infty}\leq C(\Omega,d)\left(\|f\|_{L^1(\log L)^\beta(\Omega)}\right)^{\frac{1}{d-1}}.$$

Proof. We know that

$$\int_\Omega|\nabla u|^{d-2}\nabla u\nabla v=\int_\Omega fv$$

for every test function $v\in W^{1,d}(\Omega)$. We define for $k>0$,

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|<k\\ k\frac{s}{|s|} & \text{if} \quad |s|\geq k\end{cases}$$

and for $\varepsilon>0$ we take $v_\varepsilon=\frac{1}{\varepsilon}(T_{k+\varepsilon}(u)-T_k(u))$. Since Ω is bounded, $v_\varepsilon\in W_0^{1,1}(\Omega)$ and by the Sobolev imbedding we obtain

$$C_d\|v_\varepsilon\|_{L^{\frac{d}{d-1}}}\leq\|\nabla v_\varepsilon\|_{L^1}=\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}}|\nabla u|.$$

We define $F_\varepsilon(t)=\min(t^+,1)$, so

$$C_d\|F_\varepsilon(|u|-k)\|_{L^{\frac{d}{d-1}}}\leq\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}}|\nabla u|.$$

In the other side, we have for $s>1$,

$$\varphi(k+\varepsilon)=\int_{\{|u>k+\varepsilon|\}}\leq\int_\Omega|F_\varepsilon(|u|-k)|^s$$

where $\varphi(k)=\text{mes}\,\{|u|>k\}$. And for $s=\frac{d}{d-1}$ we have

$$\varphi(k+\varepsilon)\leq\left(\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}}|\nabla u|\right)^{\frac{d}{d-1}}.$$

By Hölder's inequality we have

$$\begin{aligned}\varphi(k+\varepsilon) &\leq \left(\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}}|\nabla u|^d\right)^{\frac{1}{d-1}}\left(\frac{\text{mes}\,(\{k<|u|<k+\varepsilon\})}{\varepsilon}\right)\\ &\leq \left(\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}}|\nabla u|^d\right)^{\frac{1}{d-1}}\left(\frac{\varphi(k)-\varphi(k+\varepsilon)}{\varepsilon}\right).\end{aligned}$$

Finally we note

$$\Phi_{k,\varepsilon}(s)=\begin{cases}0, & \text{if} \quad 0\le s\le k\\ s-k, & \text{if} \quad k\le s\le k+\varepsilon\\ \varepsilon, & \text{if} \quad s\ge k+\varepsilon\,,\end{cases}$$

so we have

$$\int_{\{k<|u|<k+\varepsilon\}}|\nabla u|^n=\langle-\Delta_n u,\Phi_{k,\varepsilon}(u)\rangle\le\int_\Omega f\Phi_{k,\varepsilon}(u)\le\varepsilon\int_{\{|u>k|\}}|f|\,.$$

But we know that in the dual space we have

$$\|\chi_E\|_{\left(L^1(\log L)^\beta(\Omega)\right)^*}\le \operatorname{mes}(E)P_\beta^{-1}(\frac{1}{\operatorname{mes}(E)}),$$

then if we use Lemma **4.2** we obtain

$$\int_{\{|u>k|\}}|f|\le\|f\|_{L^1(\log L)^\beta(\Omega)}\frac{c}{\left(\log\left(1+\frac{1}{\varphi(k)}\right)\right)^\beta}.$$

Thus

$$\varphi(k+\varepsilon)\le C\,\|f\|_{L^1(\log L)^\beta(\Omega)}^{\frac{1}{d-1}}\frac{1}{\left(\log\left(1+\frac{1}{\varphi(k)}\right)\right)^{\frac{\beta}{d-1}}}\left(\frac{\varphi(k)-\varphi(k+\varepsilon)}{\varepsilon}\right),$$

then

$$\varphi(k)\le -C\,\|f\|_{L^1(\log L)^\beta(\Omega)}^{\frac{1}{d-1}}\frac{1}{\left(\log\left(1+\frac{1}{\varphi(k)}\right)\right)^{\frac{\beta}{d-1}}}\varphi'(k),$$

so

$$1\le -C\,\|f\|_{L^1(\log L)^\beta(\Omega)}^{\frac{1}{d-1}}\frac{1}{\left(\log\left(1+\frac{1}{\varphi(k)}\right)\right)^{\frac{\beta}{d-1}}}\frac{\varphi'(k)}{\varphi(k)}.$$

After integration we obtain

$$\frac{1}{\gamma}\left[\frac{1}{\left(\log(\frac{1}{\operatorname{mes}\Omega})\right)^\gamma}-\frac{1}{\left(\log(\frac{1}{\varphi(t)})\right)^\gamma}\right]\ge t\left[C\,\|f\|_{L^1(\log L)^\beta(\Omega)}^{\frac{1}{d-1}}\right]^{-1},$$

where $\gamma=\frac{\beta}{d-1}-1$. Then

$$\frac{1}{\left(\log(\frac{1}{\operatorname{mes}\Omega})\right)^\gamma}-\gamma t\left[C\,\|f\|_{L^1(\log L)^\beta(\Omega)}^{\frac{1}{d-1}}\right]^{-1}\ge\frac{1}{\left(\log(\frac{1}{\varphi(t)})\right)^\gamma}$$

and consequently we have the existence of t_0 such that $\varphi(t_0)=0$ and moreover

$$\|u\|_{L^\infty}\le C(\Omega,d)\left(\|f\|_{L^1(\log L)^\beta(\Omega)}\right)^{\frac{1}{n-1}}.$$

□

The case $p>d$ is a trivial case. It is a direct consequence from the Sobolev embedding.

Lemma 4.5. *Let $u \in W_0^{1,p}(\Omega)$ and $f \in L^1(\Omega)$ such that*

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{cases} \tag{4.3a}$$

Then $u \in L^\infty(\Omega)$, furthermore there exists $c = c(d, \Omega, p)$ such that

$$\|u\|_{L^\infty} \leq c\,\|f\|_{L^1}^{\frac{1}{p-1}}\,.$$

Remark 4.6. i) The spaces used in this proof are in some way optimal if $d > 2$, in fact we can show that the problem

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{4.4}$$

have a non bounded solution if $f \in L^1(\log L)^{d-1}(\Omega)$. We consider the function

$$f(r) = r^{-d}\,|\log(r)|^{-d}\,|\log|\log r||^{-\alpha}\,.$$

We have, if $\alpha > 1$, $f \in L^1(\log L)^{d-1}(\Omega)$ where Ω is a small ball, and $f \notin L^1(\log L)^{\beta}(\Omega)$ for all $\beta > d-1$. And we can show that the corresponding solution of (4.4) is not bounded if $\alpha \leq d-1$.

ii) In the case $p = d = 2$, we can use the Hardy spaces and we can show the optimality of these spaces (see [12]).

The coming two steps are common for all cases of p.

STEP 2: The super-solution.

Let $\alpha > 0$ and $\overline{U}$ the solution of

$$\begin{cases} -\Delta_p u &= V(x) f(\alpha) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{cases}$$

Since $V \in E$, we have, using Lemma **4.1**, **4.4** or **4.5**,

$$\left\|\overline{U}\right\|_{L^\infty(\Omega)} \leq c(p,d,\Omega)\,\|V\|_E^{\frac{1}{p-1}}\,f(\alpha)^{\frac{1}{p-1}}.$$

So if

$$\|V\|_{L^q} < c \sup_{\alpha>0} \frac{\alpha^{p-1}}{f(\alpha)},$$

then there exist $\alpha > 0$ such that $\left\|\overline{U}\right\|_{L^\infty(\Omega)} \leq \alpha$, and consequently, $\overline{U}$ is a super-solution of (P).

STEP 3: Existence and stability.

Let us define

$$\widetilde{f}(s) = \begin{cases} f(\overline{U}) & \text{if} \quad s \geq \overline{U} \\ f(s) & \text{if} \quad 0 \leq s \leq \overline{U} \\ f(0) & \text{if} \quad 0 \geq s\,. \end{cases}$$

We note $\widetilde{T}_V : C_0(\Omega) \longrightarrow C_0(\Omega)$ the operator defined by $\widetilde{T}_V(v) = u$ if and only if

$$\begin{cases} -\Delta_p u &= V(x)\widetilde{f}(v) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{cases}$$

We know that $\widetilde{T}_V$ is a compact operator and if $v \in C_0(\Omega)$ then

$$\widetilde{T}_V(v) \in B\left(0, c(p,n,\Omega)\, \|V\|_E^{\frac{1}{p-1}}\, \widetilde{f}(\left\|\overline{U}\right\|_\infty)^{\frac{1}{p-1}}\right).$$

So $\widetilde{T}_V$ is uniformly bounded. This imply the existence of an $R_0 > 0$ such that

$$\forall R > R_0, d(I - \widetilde{T}_V, B(0,R), 0) = 1,$$

where d denote the topological degree in $C_0(\Omega)$. In fact, let us consider the homotopy

$$H(\cdot, t) = I - t\widetilde{T}_V,\ \forall t \in [0,1]\,.$$

This homotopy is admissible for R sufficiently large. Because if there exists $u \in C_0(\Omega)$ such that $\|u\|_\infty = R$ and $u = t\widetilde{T}_V(u)$, then $R = \|u\|_\infty = t\left\|\widetilde{T}_V(u)\right\| \leq R_0$ which is impossible. So it follow that

$$d(H(\cdot,1), B(0,R), 0) = d(H(\cdot,0), B(0,R), 0) = d(I, B(0,R), 0) = 1.$$

Consider now, the operator T_V defined by $T_V(v) = u$ iff

$$\begin{cases} -\Delta_p u &= V(x)f(v) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{cases}$$

We have the fact that $T_V = \widetilde{T}_V$ in $M_R = B(0,R) \cap \left\{v \in C_0(\Omega);\ 0 < v < \overline{U}\right\}$. It is easy to show that $0 \notin (I - T_V)\,(\partial M_R)$. We conclude that

$$\forall R > R_0, \quad d(I - T_V, B(0,R) \cap M_R, 0) = d(I - \widetilde{T}_V, B(0,R), 0) = 1.$$

Remark 4.7. We can also obtain the existence of a minimal solution using the monotone iteration method of Theorem **3.3**, but we lose the relative result of stability obtained by the degree theory (for further results concerning the degree theory we can see for example [11]).

4.2. **Proof of Theorem 2.** Here we have $f(0) = 0$, so we must find a solution different from the trivial one.

We consider then the function f_K defined by

$$f_K(s) = \begin{cases} f(s) & \text{if} \quad 0 \leq s \leq K \\ f(K) & \text{if} \quad s \geq K. \end{cases}$$

We expand V by zero outside Ω. In this case, f_K is a continuous bounded function. Since there exists an $0 < s_0 < K$ such that $f(s_0) > 0$, we have $F(s_0) > 0$ where $F(s) = \int_0^s f_K(t)dt$. So we consider the energy functional E defined on $W_0^{1,p}(\Omega)$ by

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega VF(u).$$

Without loss of generality, we can suppose that $0 \in \Omega$. Let $\varphi \in D(\Omega)$ such that: $\varphi \geq 0$, $\varphi = s_0$ in $B(0, \frac{R}{2})$ and 0 outside $B(0, R)$. We define also the scaled function $\varphi_\lambda(x) = \varphi(\lambda x)$. After all these definitions we can start the proof.

It is easy to check that E is bounded from below on $W_0^{1,p}(\Omega)$ and $E(0) = 0$.

$$\begin{aligned}
E(\varphi_\lambda) &= \frac{1}{p}\int_\Omega |\nabla \varphi_\lambda|^p - \int_\Omega F(\varphi(\lambda x))V(x)dx \\
&= \frac{\lambda^{p-d}}{p}\int_\Omega |\nabla \varphi|^p - \int_{B(0,\frac{R}{\lambda})} F(\varphi(\lambda x))V(x)dx \\
&= \frac{\lambda^{p-d}}{p}\int_\Omega |\nabla \varphi|^p - \lambda^{-d}\int_{B(0,R)} F(\varphi(x))V(\frac{x}{\lambda})dx \\
&= \lambda^{-d}\left(\frac{\lambda^p}{p}\int_\Omega |\nabla \varphi|^p - \lambda^\gamma \int_{B(0,R)} F(\varphi(x))\lambda^{-\gamma}V(\frac{x}{\lambda})dx\right).
\end{aligned}$$

Since $\left(\lambda^{-\gamma}V(\frac{x}{\lambda})\right)_\lambda$ converge weakly in measure to $\mu > 0$. We have

$$E(\varphi_\lambda) \underset{\infty}{\sim} -\lambda^{\gamma-d}\int_{B(0,R)} F(\varphi(x))d\mu(x).$$

We deduce that $\inf_{u \in W_0^{1,p}(\Omega)} E(u) < 0$. It follows by usual minimization technics that E have a critical point u_1 with negative energy, which correspond to a non trivial solution of

$$\begin{cases} -\Delta_p u &= V(x) f_K(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \,. \end{cases}$$

Using the same a priori estimates as in the previous proof, we have

$$\|u\|_{L^\infty} \leq c(\Omega, d, p) f(K)^{\frac{1}{1-p}} \|V\|_E^{\frac{1}{1-p}}.$$

Since V satisfies (2.1), there exists K such that

$$\|u\|_{L^\infty} \leq c(\Omega, d, p) f(K)^{\frac{1}{1-p}} \|V\|_E^{\frac{1}{1-p}} \leq K.$$

It follows that u is a solution of (P).

5. NONEXISTENCE RESULTS

In this section we will proof Theorem **2.3** and Corollary **2.4**. So we suppose that Ω is a bounded domain of $\mathbb{R}^d$.

Let us remind an interesting result concerning the first eigenvalue of the p-Laplace operator defined by:

$$\lambda_1(\Omega) = \inf\left\{\int_\Omega |\nabla u|^p ;\ u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1\right\}.$$

We can define also the first eigenvalue with weight $V \geq 0$ and $V \in L^q(\Omega)$, where $q > \frac{d}{p}$ if $p \leq d$ and $q = 1$ if $p > d$, by

$$\lambda_1(\Omega, V) = \inf\left\{\int_\Omega |\nabla u|^p;\ u \in W_0^{1,p}(\Omega), \int_\Omega V|u|^p = 1\right\}.$$

We know, (see [8] or [4]), that this minimum is achieved in a function φ_1 that satisfies the Euler equation:

$$\begin{cases} -\Delta_p \varphi_1 &= \lambda_1 V |\varphi_1|^{p-2} \varphi_1 \text{ in } \Omega \\ \varphi_1 &= 0 \text{ in } \partial\Omega \end{cases}$$

and have the following result:

Theorem 5.1 ([6]). *Let $V : \Omega \longrightarrow \mathbb{R}$ be a given function such that $V^+ \neq 0$. Assume that $V \in L^q(\Omega)$, where $q > \frac{d}{p}$ if $p \leq d$ and $q = 1$ if $p > d$, then $\lambda_1(\Omega, V)$ is simple, isolated, and the corresponding eigenfunction is positive and belong to $C^{1,\alpha}(\Omega)$ for some $\alpha \in]0,1[$.*

Remark 5.2. The signification of "$\lambda_1(\Omega, V)$ is simple" is that any other eigenfunction ψ with fixed sign is of the form $\psi = \beta\varphi_1$.

We consider for $0 \leq h \in L^q(\Omega)$, where $q > \frac{d}{p}$ if $p \leq d$ and $q = 1$ if $p > n$, the set

$$\mathcal{H}_h(\Omega) = \{V \in E;\ V > h\}.$$

Using this result we will show that for $\|V\|_E$ sufficiently large, the problem (P) have no solution if $\widetilde{V} = \frac{V}{\|V\|_E} \in \mathcal{H}_\psi(\Omega)$. In fact suppose that (P) has a positive solution u for every V. Let $\widetilde{V} = \frac{V}{\|V\|_E}$ and φ_1 the first eigenfunction associated to

$$\begin{cases} -\Delta_p \varphi_1 &= \lambda_1 h |\varphi_1|^{p-2} \varphi_1 \text{ in } \Omega \\ \varphi_1 &= 0 \text{ on } \partial\Omega\,. \end{cases}$$

Since $\varphi_1 > 0$ and $\frac{\partial\varphi_1}{\partial\nu} < 0$, there exist $t > 0$ such that $\Psi = t\varphi_1 < u$ in Ω. Assume that

$$\lambda = \|V\|_E > \lambda_1(\Omega, h) \sup_{\alpha>0} \frac{\alpha^{p-1}}{f(\alpha)},$$

then for $\varepsilon > 0$ sufficiently small and $\lambda_\varepsilon = \lambda_1 + \varepsilon$ we have

$$-\Delta_p \Psi = \lambda_1 h |\Psi|^{p-2} \Psi \leq \lambda_\varepsilon h |\Psi|^{p-2} \Psi \leq \lambda_\varepsilon \widetilde{V} |u|^{p-2} u \leq \lambda \widetilde{V} f(u) = -\Delta_p u.$$

So the problem

$$\begin{cases} -\Delta_p v &= \lambda_\varepsilon |v|^{p-2} v \text{ on } \Omega \\ v &= 0 \text{ in } \partial\Omega \end{cases} \tag{5.1}$$

have a sub and super-solution. Using Theorem **3.3** the problem (5.1) have a positive solution and this is a contradiction because λ_1 is isolated. Then we have the existence of a constant $\lambda^* = \lambda^*(p, d, \Omega) > 0$ such that: If $\|V\|_E < \lambda^*$, the problem (P) have at

least one positive solution; If $\|V\|_E > \lambda^*$, the problem (P) have no positive solution. Moreover
$$c(p,d,\Omega)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)} \le \lambda^* \le \lambda_1(\Omega,h)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)},$$
and that shows the sharpness of our estimation.

We can also define for a subdomain $\widetilde{\Omega} \subset \Omega$ the set $\mathcal{H}_h(\widetilde{\Omega}) = \left\{V \in E;\ V > h \text{ in } \widetilde{\Omega}\right\}$. Then we have
$$c(p,d,\Omega)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)} \le \lambda^* \le \lambda_1(\widetilde{\Omega},h)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)}.$$
The proof is the same, since we will consider the restriction of our problem to $\widetilde{\Omega}$.

Remark 5.3. We can see also that if we assume that $\widetilde{V} = \frac{V}{\|V\|_E} \in \mathcal{H}_\delta(\Omega') = \{v \in E;\ v > \delta \text{ on } \Omega'\}$, then we get the existence of $\lambda^* \in [0,+\infty]$ such that:

i) if $\|V\|_E < \lambda^*$, problem (P) have at least one positive solution.

ii) if $\|V\|_E > \lambda^*$, problem (P) have no positive solution.

Moreover, we have the following estimation:
$$c(p,d,\Omega)\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)} \le \lambda^* \le \frac{\lambda_1(\Omega')}{\delta}\sup_{\alpha>0}\frac{\alpha^{p-1}}{f(\alpha)}.$$

6. OTHER EXISTENCE RESULTS

Now we are going to weaken the hypothesis of monotonicity of f.

Proposition 6.1. *Assume that f is an absolutely continuous function, then $f = \overline{f} + \underline{f}$, where $\overline{f}$ is a continuous nondecreasing function and $\underline{f}$ is a nonincreasing function.*

Let us define the set
$$D_f^+(I) = \left\{ \begin{array}{c} g : I \longrightarrow \mathbb{R};\ g \text{ is nondecreasing, } g(0) = f(0) \text{ and} \\ \text{there exists a nonincreasing function } h : I \longrightarrow \mathbb{R} \text{ such that } f = g + h \end{array} \right\}.$$

Theorem 6.2. *Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ is an absolutely continuous function such that $f(0) > 0$. If $V \in E$ and*
$$\|V\|_E < c(p,d,\Omega)\sup_{g\in D_f^+(\mathbb{R}^+)}\sup_{\alpha>0}\frac{\alpha^{p-1}}{g(\alpha)},$$
then (P) have at least one positive solution.

Proof. We know that 0 is a subsolution of (P), so our propose is to find a supersolution. But since f is absolutely continuous, $f \le g$ for every $g \in D_f^+(\mathbb{R}^+)$, and using Theorem 2.1 the problem
$$\left\{ \begin{array}{rcl} -\Delta_p u &=& V(x)g(u) \text{ in } \Omega \\ u &=& 0 \text{ on } \partial\Omega \end{array} \right.$$
have at least one positive solution. So this solution is a supersolution of (P) and that achieve the proof using Theorem **3.3**. □

Theorem 6.3. *Assume that* $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ *is a continuous function such that* $f(0) > 0$. *If* $V \in E$ *and*

$$\|V\|_E < c(p,d,\Omega) \sup_{\alpha>0} \inf_{\gamma\in[0,\alpha]} \frac{\alpha^{p-1}}{f(\gamma)},$$

then (P) have at least one positive solution.

Proof. Let $T_V : C_0(\Omega) \longrightarrow C_0(\Omega)$ the operator defined by $T_V(v) = u$, if

$$\begin{cases} -\Delta_p u &= V(x)f(v) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega\,. \end{cases}$$

If $v \in B\,(0,\alpha)$, for some $\alpha > 0$, using the previous a priori estimates, we get

$$\|T_V(v)\|_\infty \le C(p,d,\Omega)\,\|Vf(v)\|_E^{\frac{1}{p-1}} \le C(p,d,\Omega)\left(\|V\|_E \sup_{\gamma\in[0,\alpha]} f(\gamma)\right)^{\frac{1}{p-1}}.$$

So if $\|V\|_E < c(p,d,\Omega)\inf_{\gamma\in[0,\alpha]} \frac{\alpha^{p-1}}{f(\gamma)}$, then T_V applies the ball $B\,(0,\alpha)$ to it self. Now using the fact that if $v \in C_0(\Omega)$ then $T_V(v) \in C^{1,\nu}(\Omega)$ (Di Benedetto, Tolksdorf). We obtain the compacity of the operator T_V. So by the Schauder fixed point theorem, T_V have a fixed point u, which correspond to a solution of (P). □

Remark 6.4. The previous result is a generalization of Theorem **2.1**. But remark that we loose the result of stability obtained by the degree theory.

7. AN EXTENSION TO THE WHOLE SPACE

Let us define the set

$$\mathcal{H} = \Big\{\psi : \mathbb{R}^+ \longrightarrow \mathbb{R};\ \psi \text{ is measurable}, \\ (N(\psi))^{\frac{1}{p-1}} = \int_0^{+\infty} \frac{1}{s^{d-1}}\left(\int_0^s t^{d-1}\,|\psi(t)|\,dt\right)^{\frac{1}{p-1}} ds < \infty\Big\},$$

and consider the problem

$$\begin{cases} -\Delta_p u = \psi \text{ in } \mathbb{R}^d \\ \lim\limits_{|x|\longrightarrow+\infty} u(x) = 0. \end{cases} \tag{7.1a}$$

Proposition 7.1. *If* f *is a positive radial function such that* $\psi(x) = \widetilde{\psi}(|x|)$ *where* $\widetilde{\psi} \in \mathcal{H}$, *then problem* (7.1a) *has a positive bounded solution* u_ψ *given by*

$$u_\psi(x) = \int_{|x|}^{+\infty} \frac{1}{s^{d-1}}\left(\int_0^s t^{d-1}\widetilde{\psi}(t)dt\right)^{\frac{1}{p-1}} ds.$$

In particular $\|u\|_\infty \le \left(N(\widetilde{\psi})\right)^{\frac{1}{p-1}}$.

Proof. Simple calculation. □

If V is a measurable function, we define

$$\mathcal{L}_V = \left\{\psi \in \mathcal{H};\ \ |V(x)| \le \psi\left(|x|\right),\ a.e \text{ in } \mathbb{R}^d\right\},$$

and

$$h(V) = \inf_{\psi \in \mathcal{L}_V} N(\psi).$$

Theorem 7.2. *Let us consider a positive measurable function $V \in L^1\left(\mathbb{R}^d\right)$ such that $h(V) < \infty$. If*

$$h\left(V\right) < \sup_{\alpha > 0} \frac{\alpha^{p-1}}{f(\alpha)},$$

the problem

$$\begin{cases} -\Delta_p u = V(x) f(u) \ \ in\ \mathbb{R}^d \\ \lim\limits_{|x| \longrightarrow +\infty} u(x) = 0 \end{cases} \tag{7.2}$$

where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^*$ is a continuous nondecreasing function, have at least one positive solution.

Proof. Consider the problem

$$\begin{cases} -\Delta_p u = V(x) \text{ in } \mathbb{R}^d \\ \lim\limits_{|x| \longrightarrow +\infty} u(x) = 0 \end{cases} \tag{7.3}$$

Like the proof of Theorem **2.1** we need 3 steps:

STEP 1: Boundedness.

Let us consider the approximation problem

$$\begin{cases} -\Delta_p u_k = V_k(x) \text{ in } B(0,k) \\ u_k(x) = 0 \text{ if } |x| = k, \end{cases}$$

where $V_k(x) = \min(V(x), k)\chi_{B(0,k)}$. It is clear that such solution u_k exists. Furthermore, using the weak comparison theorem and Proposition **7.1**, the sequence (u_k) is uniformly bounded in $L^\infty\left(\mathbb{R}^d\right)$ by $h\left(V\right)$. It follows that

$$\int_{\mathbb{R}^d} |\nabla u_k|^p \le \|V\|_{L^1} h(V),$$

and we deduce the boundedness of (u_k) in $W^{1,p}\left(\mathbb{R}^d\right)$. So there exist a subsequence which will be denoted (u_k) and $u \in W^{1,p}\left(\mathbb{R}^d\right)$ such that

$$\begin{cases} u_k \longrightarrow u \ a.e \text{ on } \mathbb{R}^d \\ u_k \longrightarrow u \text{ in } L^q_{loc}\left(\mathbb{R}^d\right),\ p \le q < p^* \\ u_k \rightharpoonup u \text{ in } W^{1,p}\left(\mathbb{R}^d\right) \end{cases}$$

$$\langle -\Delta_p u_k, u_k - u\rangle = \int_{\mathbb{R}^d} |\nabla u_k|^{p-2} \nabla u_k \left(\nabla u_k - \nabla u\right) = \int_{\mathbb{R}^d} V_k\left(u_k - u\right) \le \int_{\mathbb{R}^d} V\left(u_k - u\right).$$

Since $u_k \rightharpoonup u$ in $L^\infty(\Omega)$ weak-$*$, we have the convergence of $u_k \longrightarrow u$ in $W^{1,p}(\mathbb{R}^d)$. Thus we obtain the desired solution of (7.3) such that $\|u\|_{L^\infty} \le h(V)$.

Step 2: Super-solution.

Since $h(V) < \sup\limits_{\alpha>0} \frac{\alpha^{p-1}}{f(\alpha)}$, we have the existence of

$$\psi \in \mathcal{H} \text{ such that } V(x) \le \psi(|x|) \text{ and } N(\psi) < \sup_{\alpha>0} \frac{\alpha^{p-1}}{f(\alpha)}.$$

So, $\overline{u} = \int_{|x|}^{+\infty} \frac{1}{s^{d-1}} \left(\int_0^s t^{d-1}\psi(t)dt\right)^{\frac{1}{p-1}} ds$, is a super-solution of (7.2).

Step 3: Monotone iteration.

Now we consider the sequence $(u_k)_k$ defined by $u_0 = \overline{u}$ and

$$\begin{cases} -\Delta_p u_{k+1} = V(x) f(u_k) \\ \lim\limits_{|x| \longrightarrow +\infty} u_{k+1} = 0. \end{cases}$$

By the maximum principle, we have the fact that the sequence $(u_k)_k$ non increasing and $0 < u_k \le \overline{u}$. Using the same arguments of the first step, we obtain the convergence of u_k in $W^{1,p}(\mathbb{R}^d)$. □

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