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PAIRS OF SOLUTIONS OF ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We establish the existence of two nontrivial solutions for the semilinear elliptic problem

$$\begin{aligned} -\Delta u &= g(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $g \in C^1(\Omega \times \mathbb{R} \setminus \{0\}, \mathbb{R})$ is such that $g(x,0) = 0$ and asymptotically linear. Our proofs are based on minimax methods and critical groups.

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1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. We study the existence of two nontrivial solutions of the elliptic problem

$$\begin{aligned} -\Delta u &= g(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $g \in C^1(\Omega \times \mathbb{R} \setminus \{0\}, \mathbb{R})$ is such that $g(x,0) = 0$. Furthermore, it is required that there exist

$$\alpha_\pm = \lim_{t\to\pm\infty} \frac{g(x,t)}{t}, \quad \alpha_\pm \in \mathbb{R}, \quad \text{uniformly in } \Omega; \tag{1.2}$$

and

$$\beta_\pm = \lim_{t\to 0^\pm} \frac{g(x,t)}{t}, \quad \beta_\pm \in \mathbb{R}, \quad \text{uniformly in } \Omega. \tag{1.3}$$

Without loss of generality, we can assume that $\alpha_- \leq \alpha_+$. It is well known that existence and multiplicity of solutions for problem (1.1) rely strongly on the position of the pairs (α_-, α_+) and $(\beta_\pm, \beta_\mp)$, with respect to eigenvalues of $(-\Delta, H_0^1(\Omega))$.

Similar multiplicity results for problem (1.1) were investigated by many authors. See for instance Ahmad [1], Ambrosetti and Mancine [2], Bartsch *et al.* [3], Castro and Lazer [5], Hirano [9, 10], Li *et al.* [12], Li and Su [14], Li and Willem [15], Mizoguchi [17], de Paiva [18, 19], Zou [21] and references therein. There are several techniques

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used to study this problem. For example, Morse theory (critical groups and Morse inequalities) was used in [3, 12, 14, 15, 18, 19, 21]. In [1, 2, 9], the authors used degree theory based on Leray-Schauder degree. The Lyapunov-Schmmidt method combined with critical point theory was applied in [5, 15]. In [17], the author used minimax theorems combined with Conley index theory. Our approach is based on minimax theorems and critical groups (Morse theory).

The paper is organized as follows: in Section 2 we collect some preliminaries needed to prove the theorems. Section 3 is devoted to present and prove the main results.

2. PRELIMINARIES

For the convenience of the reader, we recall some notation and results of Morse Theory. Let H be a Hilbert space and $f : H \to \mathbb{R}$ be a functional of class C^1. We assume that f satisfies the Palais-Smale condition (we write it (PS) for short). We will also assume that the set of critical points of f, denoted by K, is finite. Let $y \in H$ be a critical point of f with $c = f(y)$. The group

$$C_p(f, y) = H_p(f^c, f^c \setminus \{y\}), \quad p = 0, 1, 2, ...,$$

is called the p^{th} critical group of f at y, where $f^c = \{x \in H : f(x) \leq c\}$ and $H_p(\cdot, \cdot)$ is the singular relative homology group with integer coefficients. The p^{th} critical groups of f at infinity is defined as

$$C_p(f, \infty) = H_p(H, f^\alpha),$$

where $\alpha < \inf f(K)$. The next result is extremely useful in Morse theory (see for instance [4]).

Proposition 2.1. *If $a < b$ are regular values of f and $H_k(f^b, f^a) \neq 0$, for some $k \in \mathbb{N}$, then f has a critical point y with $C_k(f, y) \neq 0$. Moreover, if there is a critical point y with $f(y) \in (a, b)$ and $C_k(f, y) \neq H_k(f^b, f^a)$, then there are other critical points than y.*

Now, we present an application of the previous proposition (see the proof of Theorem 3.8 in [3]).

Proposition 2.2. *Assume that 0 is a critical point of f with $f(0) = 0$ and $C_k(f, 0) \neq 0$. If $H_k(H, f^\alpha) = 0$, then there is a critical point y such that either $C_{k-1}(f, y) \neq 0$ or $C_{k+1}(f, y) \neq 0$.*

Proof. By $C_k(f,0) \neq 0$, for $\epsilon > 0$ small enough we have $H_k(f^\epsilon, f^{-\epsilon}) \neq 0$. Consider the following diagram

$$\begin{array}{ccccc} H_{k+1}(H, f^\epsilon) & \rightarrow & H_k(f^\epsilon, f^\alpha) & \rightarrow & H_k(H, f^\alpha) \\ & & \downarrow & & \\ & & H_k(f^\epsilon, f^{-\epsilon}) & & \\ & & \downarrow & & \\ & & H_{k-1}(f^{-\epsilon}, f^\alpha). & & \end{array}$$

The property of exactness of the Homology implies that either

$$H_{k-1}(f^{-\epsilon}, f^\alpha) \neq 0 \quad \text{or} \quad H_{k+1}(H, f^\epsilon) \neq 0,$$

since $H_k(f^\epsilon, f^{-\epsilon}) \neq 0$ and $H_k(H, f^\alpha) = 0$. By the previous proposition, we conclude that there is a nontrivial critical point y that satisfies either

$$C_{k-1}(f, y) \neq 0 \quad \text{or} \quad C_{k+1}(f, y) \neq 0,$$

which is the desired conclusion. □

The classical solutions of problem (1.1) correspond to critical points of the C^{2-0}-functional, denoted by F, defined on $H_0^1 = H_0^1(\Omega)$ by

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)dx, \quad u \in H_0^1, \tag{2.1}$$

where $G(x,t) = \int_0^t g(x,s)ds$. The following nonresonance condition will be assumed throughout the paper: the problem

$$-\Delta u = \alpha_+ u^+ - \alpha_- u^-, \quad u \in H_0^1,$$

has only the trivial solution. Under this assumption the functional F satisfies the Palais-Smale compactness condition. In order to apply Morse theory to obtain multiplicity of critical points of F, we need to compute the critical groups of known critical points and the critical groups at infinity. In this direction we have the following result (see for instance [6, 13]).

Proposition 2.3. (i) *If $\lambda_m < \alpha_\pm < \lambda_{m+1}$, then $C_p(F,\infty) = \delta_{pm}\mathbb{Z}$.*

(ii) *If there is $\delta > 0$ such that $\lambda_k \leq \frac{g(x,t)}{t} \leq \lambda_{k+1}$, $\forall\, x \in \Omega$ and $0 \leq |t| < \delta$, then $C_p(F,0) = \delta_{pk}\mathbb{Z}$.*

Remark 2.4. Assume the hypotheses of the previous proposition with $m \neq k$. Proposition 2.1 implies that there is a nontrivial critical point u_1 with $C_m(F,u_1) \neq 0$. Moreover, Proposition 2.2 implies that there is a nontrivial critical point u_2 such that either $C_{k-1}(F,u_2) \neq 0$ or $C_{k+1}(F,u_2) \neq 0$. In order to prove that $u_1 \neq u_2$ we will assume some additional conditions, see Theorems 3.1 and 3.2 in the next section. Our results were motivated by [3, 15], where similar results have been obtained.

Now, we present a version of the Shifting Theorem for C^{2-0}-functionals. Take $X = C_0^1(\Omega)$ and u a nontrivial critical point of F. We have that $F' \in C^1(D, H_0^1)$ and $F''(u_0)$ is a bounded linear operator from X to H_0^1, where D is a neighborhood of u_0 in the X-topology. The Morse index $\mu(u_0)$ of u_0 measures the dimension of the maximal subspace of X on which $F''(u_0)$ is negative definite. The nullity of u_0 is the dimension of the kernel of $F''(u_0)$, we denote it by $\nu(u_0)$. The authors in [12] were able to give a version of the Shifting Theorem to this case. We summarize it in the next proposition.

Proposition 2.5. *Assume that u is a nontrivial critical point of F with finite Morse index μ and nullity ν, then either*

(i) $C_p(F, u) = 0$ *for* $p \leq \mu$ *and* $p \geq \mu + \nu$, *or*
(ii) $C_p(F, u) = \delta_{p\mu}\mathbb{Z}$, *or*
(iii) $C_p(F, u) = \delta_{p(\mu+\nu)}\mathbb{Z}$.

Another useful tool that we will make use of are the spectral properties of weighted eigenvalue problems. Let $p(x)$ be a bounded function in Ω with nontrivial positive part. Consider the eigenvalue problem

$$\begin{aligned} -\Delta v &= \lambda p(x) v \quad \text{in} \quad \Omega \\ v &= 0 \qquad\quad\ \ \text{on} \quad \partial\Omega. \end{aligned} \tag{2.2}$$

This problem have a sequence of eigenvalues $0 < \lambda_1(p) < \lambda_2(p) \leq \cdots \leq \lambda_j(p) \to \infty$, and the associated eigenfunctions satisfies the Unique Continuation Property. Moreover, if $p(x) \leq q(x)$, with strict inequality holding on a set of positive measure, then $\lambda_j(p) > \lambda_j(q)$. For all this properties and more we refer to [8]. We remark that in the case $p \equiv 1$ we denote $\lambda_j(1)$ by λ_j.

3. MAIN RESULTS AND PROOFS

We will denote by φ_j the normalized eigenfunction associated to λ_j and $H_j := span\{\varphi_1, \cdots, \varphi_j\}$.

Theorem 3.1. *Suppose that there exist k and $m \geq k+1$ such that $\lambda_{k-1} \leq \beta_\pm < \lambda_k$ and $\lambda_m < \alpha_\pm < \lambda_{m+1}$. Moreover, assume that one of the following conditions holds:*

(a) $g'(x,t) \geq g(x,t)/t$, *for all* $x \in \Omega$ *and all* $t \in \mathbb{R}$;
(b) $k = 2$ *and* $g'(x,t) \leq \lambda_{m+1}$, *for all* $x \in \Omega$ *and* $t \in \mathbb{R}$.

Then problem (1.1) *has at least two nontrivial solutions.*

Proof. From $\lambda_m < \alpha_\pm < \lambda_{m+1}$ the functional F satisfies the (PS) condition and has the geometry of Saddle Point Theorem. More precisely, we have

(i) $F(u) \to -\infty$, as $||u|| \to \infty$, for $u \in H_m$; and

(ii) $F(u) \to \infty$, as $||u|| \to \infty$, for $u \in H_m^\perp$.

It is follows that there exists a critical point u_1 of F such that, see [6, 16],

$$C_m(F, u_1) \neq 0. \tag{3.1}$$

Using $\beta_\pm < \lambda_k$, we can prove that $\langle F''(0)\varphi_j, \varphi_j\rangle > 0$ for all $j \geq k$. Then $\mu(0)+\nu(0) \leq k-1$ and, by the Proposition 2.5, we have $C_p(F, 0) = 0$ for all $p \geq k$. Therefore u_1 is a nontrivial critical point of F provided $m \geq k+1$.

Proof of (a): By $\lambda_{k-1} \leq \beta_\pm < \lambda_k$ and $g'(x,t) \geq g(x,t)/t$, we can show that

(i) there is $R > 0$ such that $F(u) \leq 0$ for all $u \in \{t\varphi_k + u;\ t \geq 0, u \in H_{k-1}\}$ with $||u|| \leq R$; and

(ii) there are $r > 0$ and $a > 0$ such that $F(u) \geq a$ for all $u \in H_{k-1}^\perp$ with $||u|| = r$.

So, the functional F satisfies the hypotheses of the Linking Theorem. We can conclude that there is a critical point u_2 that satisfies

$$C_k(F, u_2) \neq 0, \tag{3.2}$$

and so u_2 is nontrivial. The proof of (a) is complete by the assumption $m > k$, (3.1), and showing that:

Claim: $C_m(F, u_2) = 0$.

Indeed, consider the eigenvalue problem

$$-\Delta v = \lambda \frac{g(x, u_2)}{u_2} v, \quad v \in H_0^1.$$

Using that u_2 solves (1), we can conclude that 1 is an eigenvalue of the above problem and u_2 is the associated eigenfunction. Moreover, we can certainly assume that $g(x,u_2)/u_2 > \lambda_{k-1}$ in a set of positive measure. Indeed, if not, then $g(x,u_2)/u_2 = \lambda_{k-1}$, so $u_2 = c\varphi_{k-1}$ and we can also show that $g'(x,u_2) = \lambda_{k-1}$. But, in this case, we have that $s\varphi_{k-1}$ is a solution of (1.1) for all $0 < s < c$, and it is easy to see that the claim is true in that case. Now, assuming $g(x,u_2)/u_2 > \lambda_{k-1}$ in a set of positive measure to hold, we have $\lambda_i(g(x,u_2)/u_2) < \lambda_i(\lambda_{k-1}) \leq 1$ for all $i \leq k-1$. Since 1 is an eigenvalue, we can conclude that $\lambda_k(g(x,u_2)/u_2) \leq 1$. If we assume that $g'(x,u_2) > g(x,u_2)/u_2$ in a set of positive measure, then we have $\lambda_k(g'(x,u_2)) < 1$. This implies that $\mu(u_2) \geq k$, but $\mu(u_2) \leq k$ since (3.2) holds, so $\mu(u_2) = k$. Then, by (3.2) and the item (ii) in Proposition 2.5, we have the desired conclusion in this case. On the other hand, if we have $g'(x,u_2) = g(x,u_2)/u_2 < \lambda_{m+1}$ for all $x \in \Omega$, then $\lambda_{m+1}(g'(x,u_2)) > 1$, which implies $\mu(u_2) + \nu(u_2) \leq m$. Now, if $\mu(u_2) = k$ the conclusion follows by item (ii) in Proposition 2.5, and if $\mu(u_2) < k$ the conclusion follows by item (i) in Proposition 2.5. ♢

Proof of (b): Let u_1 be a nontrivial solution such that $C_m(F, u_1) \neq 0$. Let us first prove that:

Claim 1: $C_p(F,u_1)=\delta_{pm}G$.

In fact, by the Proposition 2.5, we have that $\mu(u_1)+\nu(u_1)\geq m$. Let $\varphi\in H_m^{\perp}$, by $g'(x,t)\leq\lambda_{m+1}$ and the strict inequality holding in a set of positive measure, we have

$$\begin{aligned}\langle F''(u_1)\varphi,\varphi\rangle &= \int_\Omega |\nabla\varphi|^2-g'(x,u_1)\varphi^2\\ &> \int_\Omega |\nabla\varphi|^2-\lambda_{m+1}\varphi^2\geq 0,\end{aligned}$$

where we use the variational characterization of λ_{m+1}. Follows that $\mu(u_1)+\nu(u_1)\leq m$, and so $\mu(u_1)+\nu(u_1)=m$. The claim follows from (3.1) and the item (iii) of the Proposition 2.5. ♢

The proof (b) follows from the next claim and by the assumption $m>2$.

Claim 2: There exists a critical point u_2 of F such that

$$C_2(F,u_2)\neq 0.$$

First note that the flux of $-\nabla F$ is well defined in $X=C_0^1$ and $D=P\cup(-P)$ is an invariant set, where $P=\{u\in X;\ u\geq 0\}$ (see [7]). Moreover, we have that

(i) there is $R>0$ such that $F(u)<0$ for any $u\in H_2$ with $||u||\geq R$; and
(ii) there are $a,r>0$ such that $F(u)>a$ for any $u\in H_1^{\perp}$ with $||u||=r$.

The rest of the proof follows as in [3, Theorem 3.6]. We only show the main ideas of the proof. Set

$$B:=\{u=s\varphi_1+t\varphi_2\ ;\ |s|\leq R,\ 0\leq t\leq R\}$$

and

$$\partial B=\{s\varphi_1+t\varphi_2\,;\,|s|=R \text{ or } t\in\{0,R\}\}.$$

Denoting $\widetilde{F}=F|_X$ and using (i), we have $\partial B\subset\widetilde{F}^0\cup D$. Let $\gamma=\max\widetilde{F}(B)$ so that $(B,\partial B)\overset{i}{\hookrightarrow}(\widetilde{F}^\gamma\cup D,\widetilde{F}^0\cup D)$. Now, by (ii), we have

$$(B,\partial B)\overset{i}{\hookrightarrow}(\widetilde{F}^\gamma\cup D,\widetilde{F}^0\cup D)\overset{j}{\hookrightarrow}(X,X\setminus\{u\in H_1^{\perp};\ ||u||=r\}).$$

Using that $H_2(B,\partial B)\overset{j_*}{\to}H_2(X,X\setminus\{u\in H_1^{\perp};\ ||u||=r\})$ is nontrivial, follows that $H_2(B,\partial B)\overset{i_*}{\to}H_2(\widetilde{F}^\gamma\cup D,\widetilde{F}^0\cup D)$ is nontrivial. Let $\xi\in H_2(\widetilde{F}^\gamma\cup D,\widetilde{F}^0\cup D)$ defined by $\xi=i_*(1)$, where $1\in\mathbb{Z}\cong H_2(B,\partial B)$. Define

$$\Gamma=\{\delta\in\mathbb{R};\ \xi\in\text{image}(i_\delta)\}\quad\text{and}\quad c=\inf\Gamma,$$

where $i_\delta:H_2(\widetilde{F}^\delta\cup D,\widetilde{F}^0\cup D)\to H_2(\widetilde{F}^\gamma\cup D,\widetilde{F}^0\cup D)\}$ is induced by the inclusion. It was proved in [3] that c is a critical value of F and $0<c\leq\gamma$. Furthermore, there is a critical point of F at level c that satisfies the conditions required in the claim. ♢ □

Theorem 3.2. *Suppose that there exist m and $k\geq m+2$ such that $\lambda_m<\alpha_\pm<\lambda_{m+1}$ and $\lambda_k<\beta_\pm\leq\lambda_{k+1}$. Moreover, assume that one of the following conditions holds:*

(a) $g'(x,t) \leq g(x,t)/t$, *for all* $x \in \Omega$ *and all* $t \in \mathbb{R}$;
(b) $m = 1$ *and* $g(x,t)/t \leq \lambda_{k+1}$, *for all* $x \in \Omega$ *and all* $t \in \mathbb{R}$.

Then problem (1.1) *has at least two nontrivial solutions.*

Proof. *Proof of (a):* As in the proof of previous theorem, we have that there exists u_1 a critical point of F such that

$$C_m(F, u_1) \neq 0. \tag{3.3}$$

Moreover, we can show that

(i) $\exists\, r > 0$ such that $\sup_{u \in S} F(u) < 0$, where $S := \{u \in H_k;\ ||u|| = r\}$;
(ii) $F(u) \geq 0$ for all $u \in H_k^{\perp}$; and
(ii) F is bounded below on $\{s\varphi_k + u;\ s \geq 0, u \in H_k^{\perp}\}$.

Then, by [20, Theorem 3.2], there is a critical point u_2 of F such that

$$C_{k-1}(F, u_2) \neq 0.$$

By $\lambda_k < \beta_{\pm} \leq \lambda_{k+1}$ and $g(x,t)/t \leq \lambda_{k+1}$, follows that $C_p(F, 0) = \delta_{pk}\mathbb{Z}$. Thus u_1 and u_2 are nontrivial critical points of F. The proof follows from the next claim and by the assumption $k - 1 > m$.

Claim: $C_m(F, u_2) = 0$.

In fact, by the item (i) in Proposition 2.5, we have that $\mu(u_2) \geq k - 1$. We can assume that $g(x, u_2)/u_2 < \lambda_{k+1}$ in a set of positive measure. Thus $\lambda_i(g(x, u_2)/u_2) > \lambda_i(\lambda_{k+1}) \geq 1$, for all $i \geq k+1$. Now, using that u_2 solves (1.1), we have

$$-\Delta u_2 = \frac{g(x, u_2)}{u_2} u_2.$$

This implies that $\lambda_k(g(x, u_2)/u_2) \geq 1$. Then, assuming $g'(x, u_2) < g(x, u_2)/u_2$ in a set of positive measure, we have $\mu_k(g'(x, u_2)) > 1$. This implies that $\mu(u_2) \leq k - 1$, and so $\mu(u_2) = k - 1$. The item (i) of the Proposition 2.5 and (3.2) imply the *Claim*. If $g'(x, u_2) = g(x, u_2)/u_2$ for all $x \in \Omega$, then $g'(x, u_2) > \lambda_m$. Hence $\lambda_m(g'(x, u_2)) < 1$ and follows that $\mu(u_2) \geq m$. Now, if $\mu(u_2) = k - 1$ the conclusion follows by item (ii) in Proposition 2.5, and if $\mu(u_2) > k - 1$ the conclusion follows by item (i) in Proposition 2.5. ♢

Proof of (b): As in the proof of (a), we have a nontrivial critical point u_2 such that

$$C_{k-1}(F, u_2) \neq 0.$$

The proof follows from the next claim and the assumption $k > 2$.

Claim 2: There exists a critical point u_1 of F such that

$$C_p(F, u_1) = \delta_{p,1}\mathbb{Z}.$$

By the characterization of mountain pass, it is sufficient to prove the existence of a critical point u_1 of F such that $C_1(F, u_1) \neq 0$ (see [6]). But it follows as in the case (a) since $m = 1$. □

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