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MULTIPLICITY RESULTS FOR SUBLINEAR AND SUPERLINEAR HAMMERSTEIN INTEGRAL EQUATIONS VIA MORSE THEORY

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ABSTRACT. In this paper we establish some results for nonlinear sublinear and superlinear Hammerstein integral equations. Using Morse theory and in particular critical groups we prove a number of multiplicity results.

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1. INTRODUCTION

In this paper we use critical point theory to establish some existence results for the Hammerstein integral equation

$$u(x) = \int_\Omega k(x,y)\, f(y,u(y))\, dy \quad \text{for} \quad x \in \Omega; \tag{1.1}$$

here Ω is a closed bounded subset of $\mathbf{R}^n$. We look for solutions to (1.1) in $C(\Omega)$. Throughout this paper we will also use the usual Lebesgue space $L^2(\Omega)$ with norm $|\cdot|_{L^2}$ and inner product $(\cdot,\cdot)$. A discussion of (1.1) using variational methods can be found in the books [3, 8]. For a more recent treatment using the ideas in [3, 8] we refer the reader to [4]. The results we present here are new and are based on critical groups and the multiplicity theory was motivated from ideas in [1, 7].

Throughout this paper we assume the kernel $k : \Omega\times\Omega \to \mathbf{R}$ satisfies the following:

$$k \in C(\Omega\times\Omega, \mathbf{R}) \tag{1.2}$$

$$k(x,y) = k(y,x) \quad \text{for} \quad x,y \in \Omega \tag{1.3}$$

and

$$\int_{\Omega\times\Omega} k(x,y)\, u(x)u(y)\, dx\, dy \geq 0 \;\forall u \in L^2(\Omega). \tag{1.4}$$

Let

$$K\, u(x) = \int_\Omega k(x,y)\, u(y)\, dy \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u \in C(\Omega).$$

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It is well known [3, 8] that $K : L^2(\Omega) \to L^2(\Omega)$ is a linear, completely continuous, self adjoint, nonnegative (i.e. $(Ky, y) \geq 0$ for all $y \in L^2(\Omega)$) operator. Also the square root operator of K, $K^{\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega)$ exists. From the spectral theory of such operators [9] we know that K has a countably infinite number of real eigenvalues (μ_i) (recall μ_i is an eigenvalue of K if there exists a $\psi_i \in L^2(\Omega)$ with $\mu_i K \psi_i = \psi_i$) with $\mu_i > 0$ for all i. ASSUME throughout this paper that if $K\psi = 0$ for some $\psi \in L^2(\Omega)$ then $\psi = 0$. Then we can relabel the eigenvalues (μ_i) so that

$$\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$$

and note $\mu_1 > 0$.

Also we assume

$$f : \Omega \times \mathbf{R} \to \mathbf{R} \text{ is continuous.} \tag{1.5}$$

Let

$$N_f\, u(x) = f(x, u(x)) \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u \in C(\Omega).$$

It is well known [3, 4, 8] that

$$u = K\, N_f\, u \tag{1.6}$$

has a solution in $C(\Omega)$ if and only if

$$v = K^{\frac{1}{2}}\, N_f\, K^{\frac{1}{2}}\, v \tag{1.7}$$

has a solution in $L^2(\Omega)$.

Let $\Phi : L^2(\Omega) \to \mathbf{R}$ be given by

$$\Phi(u) = \frac{1}{2}\, |u|_{L^2}^2 - \int_\Omega \int_0^{K^{\frac{1}{2}}\, u(x)} f(x, v)\, dv\, dx \quad \text{for} \quad u \in L^2(\Omega). \tag{1.8}$$

Again it is well known [3, 4, 8] that if there exists a $v \in L^2(\Omega)$ with

$$\Phi'(v) = 0 \tag{1.9}$$

then v is a solution of (1.7).

Remark 1.1. It is worth pointing out here that one could extend the results of this paper so that $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ above and indeed one could look at the more general problem

$$u(x) = \int_\Omega k(x, y)\, f(y, u(y))\, dy \quad \text{for a.e.} \quad x \in \Omega;$$

here we look for solutions in $L^p(\Omega)$. Also here the continuity of f is replaced by f is (p, q) Carathéodory of potential type (see [8, Chapter 6 and 7]) and (1.2) is replaced by a condition to guarantee that $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$; here p, q, p_0, q_0 are as in [8, Chapter 6 and 7].

We will obtain a variety of existence results for (1.7) in the next two sections using Morse theory. For convenience we recall here some results which we will need in Section 2 and Section 3. Let Φ be a real valued function on a real Banach space W and assume $\Phi \in C^1(W, \mathbf{R})$. For every $c \in \mathbf{R}$ let

$$\Phi^c = \{x \in W : \ \Phi(x) \leq c\} \ \ \text{(the sublevel sets at } c)$$

and

$$K = \{x \in W : \ \Phi'(x) = 0\} \ \ \text{(the set of critical points of } \Phi).$$

In Morse theory the local behavior of Φ near an isolated critical point u is described by the sequence of critical groups

$$C^q(\Phi, u) = H^q(\Phi^c \cap U, \Phi^c \cap U \backslash \{u\}), \ q \geq 0$$

where $c = \Phi(u)$ is the corresponding critical value and U is a neighborhood of u containing no other critical points of Φ. When the critical values are bounded from below and Φ satisfies (PS) the global behavior of Φ can be described by the critical groups at infinity

$$C^q(\Phi, \infty) = H^q(W, \Phi^a), \ q \geq 0$$

where a is less than all critical values. A critical point u of Φ with $C^1(\Phi, u) \neq 0$ is called a mountain pass point.

Next we discuss the variational eigenvalues from [7] in our situation here. Consider

$$u = \lambda \, K \, u$$

where $K : L^2(\Omega) \to L^2(\Omega)$. Let

$$M = \{u \in L^2(\Omega) : \ \frac{1}{2}\,(u, u) = 1\}, \ J(u) = \frac{1}{2}\,(K\,u, u)$$

with

$$\Psi(u) = \frac{1}{J(u)}, \ u \in L^2(\Omega)\backslash\{0\} \ \text{ and } \ \widetilde{\Psi} = \Psi|_M.$$

Let F denote the class of symmetric subsets of M and $i(M_0)$ the Fadell–Rabinowitz cohomological index of $M_0 \in F$. Then

$$\lambda_k = \inf_{M_0 \in F, \, i(M_0) \geq k} \ \sup_{u \in M_0} \widetilde{\Psi}(u), \quad 1 \leq k \leq \infty.$$

We know from Theorem 4.2.1 (ii) of [7] that $\lambda_1 = \mu_1$ where

$$\mu_1 = \min_{u \neq 0} \ \frac{(u, u)}{(Ku, u)}.$$

Remark 1.2. In fact in our situation (i.e. for the Hammerstein integral equation described above) $\lambda_k = \mu_k$ for all k as was noted by Kanishka Perera. Suppose for simplicity that all the μ_i's described above are simple with

$$\mu_1 < \mu_2 < \mu_3 < \cdots .$$

Fix $k \geq 1$ so if $\mu_k < \mu < \mu_{k+1}$ the set $\widetilde{\Psi}^\mu$ deformation retracts (see the introduction of [7]) to the intersection of M with the subspace spanned by the eigenvectors of

$\mu_1, \ldots, \mu_k$, which is a $(k-1)$-dimensional sphere, so (4.2.6) of [7] holds with all the $\lambda's$ replaced by μ's. This together with (4.2.6) of [7] itself gives $\lambda_k = \mu_k$.

Now from Proposition 9.4.1 (ii) of [7] (note $K^{\frac{1}{2}} N_f K^{\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega)$ is compact (completely continuous)) we have the following result in our situation.

Theorem 1.3. *Suppose*

$$\int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\, dw\, dx = \frac{\lambda}{2} (Ku, u) + o(|u|_{L^2}^2) \quad as \quad u \to 0$$

and zero is an isolated critical point.

(i). If $\lambda < \lambda_1$ *then* $C^q(\Phi, 0) = \delta_{q,0} \mathbf{Z}_2$.

(ii). If $\lambda_k < \lambda < \lambda_{k+1}$ *then* $C^k(\Phi, 0) \neq 0$.

In our main multiplicity result of Section 2 we will use Theorem 1.3 together with the following result in [2, 5].

Theorem 1.4. *Let* Φ *be a* C^1 *functional defined on a Banach space. If* Φ *is bounded from below, satisfies (PS) and* $C^k(\Phi, 0) \neq 0$ *for some* $k \geq 1$, *then* Φ *has two nontrivial critical points.*

2. MAIN RESULTS IN SUBLINEAR CASE

We begin with an easy result.

Theorem 2.1. *Suppose* (1.2), (1.3), (1.4) *and* (1.5) *hold. In addition assume there exists a* $\lambda < \mu_1$ *and a* $C > 0$ *with*

$$\int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\, dw\, dx \leq \frac{\lambda}{2} (Ku, u) + C \quad \forall\ u \in L^2(\Omega). \tag{2.1}$$

Then Φ *given in* (1.8) *is coercive, satisfies (PS) and is bounded from below. Also* Φ *has a global minimum on* $L^2(\Omega)$. *Moreover if* Φ *has a finite number of critical points then* $C^q(\Phi, \infty) = \delta_{q,0} \mathbf{Z}_2$ *and* Φ *has a global minimizer* u *with* $C^q(\Phi, u) = \delta_{q,0} \mathbf{Z}_2$.

Remark 2.2. Note $(Ku, u) = (K^{\frac{1}{2}} u, K^{\frac{1}{2}} u) = |K^{\frac{1}{2}} u|_{L^2}^2$ for $u \in L^2(\Omega)$.

Proof. Replacing λ with $\max\{\lambda, 0\}$ if necessary we may assume $\lambda \geq 0$. Then for $u \in L^2(\Omega)$ we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}|u|_{L^2}^2 - \int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\,dw\,dx \\
&\geq \frac{1}{2}|u|_{L^2}^2 - \frac{\lambda}{2}(Ku,u) - C \\
&\geq \frac{1}{2}|u|_{L^2}^2 - \frac{\lambda}{2}\frac{1}{\mu_1}|u|_{L^2}^2 - C \\
&= \frac{1}{2}\left(1 - \frac{\lambda}{\mu_1}\right)|u|_{L^2}^2 - C \\
&\to \infty \quad \text{as} \quad |u|_{L^2} \to \infty.
\end{aligned}$$

Every (PS) sequence is bounded by coercivity and hence has a convergent subsequence from Lemma 3.1.3 of [7]. Also since $\Phi(u) \geq -C$ from above then Φ is bounded from below. Now Theorem 4.4 of [6] (or Proposition 3.5.1 of [7]) guarantees that Φ has a global minimum on $L^2(\Omega)$ (note $\Phi : L^2(\Omega) \to \mathbf{R}$ is Fréchet differentiable; see [3, 4, 8]). If Φ has a finite number of critical points then Corollary 3.5.3 of [7] (note Φ is bounded from below) guarantees that $C^q(\Phi, \infty) = \delta_{q,0}\mathbf{Z}_2$ and Φ has a global minimizer u with $C^q(\Phi, u) = \delta_{q,0}\mathbf{Z}_2$. ☐

Remark 2.3. Suppose there exists $\lambda < \mu_1$, $0 \leq \gamma < 1$, $b > 0$ and $C > 0$ with

$$\int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\,dw\,dx \leq \frac{\lambda}{2}(Ku,u) + b\,[(Ku,u)]^\gamma + C \quad \forall\, u \in L^2(\Omega).$$

It follows immediately that we can find a $\mu < \mu_1$ and a constant $C_1 > 0$ with

$$\int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\,dw\,dx \leq \frac{\mu}{2}(Ku,u) + C_1 \quad \forall\, u \in L^2(\Omega).$$

We are now ready to prove our main result in this section.

Theorem 2.4. *Suppose* (1.2), (1.3), (1.4) *and* (1.5) *hold. In addition assume there exists a* $\lambda > \mu_1$ *with* $\lambda \neq \lambda_i (= \mu_i)$ *for* $i \in \{2, 3, \dots\}$ *and with*

$$\int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\,dw\,dx = \frac{\lambda}{2}(Ku,u) + o(|u|_{L^2}^2) \quad \textit{as} \quad u \to 0 \tag{2.2}$$

and also assume $f(x, 0) = 0$ *for* $x \in \Omega$. *Finally suppose there exists a constant* γ, $0 \leq \gamma < 1$, *a constant* $b < \mu_1$, *and a constant* $C \geq 0$ *with*

$$\int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\,dw\,dx \leq \frac{b}{2}(Ku,u) + C\left([(Ku,u)]^\gamma + 1\right) \quad \forall\, u \in L^2(\Omega). \tag{2.3}$$

Then Φ *has two nontrivial critical points.*

Proof. Without loss of generality assume $b \geq 0$. For any $u \in L^2(\Omega)$ we have

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2}\,|u|^2_{L^2} - \frac{b}{2}\,(Ku,u) - C\,[(Ku,u)]^\gamma - C \\ &\geq \frac{1}{2}\left(1-\frac{b}{\mu_1}\right)|u|^2_{L^2} - \frac{C}{\mu_1^\gamma}\,|u|^{2\,\gamma}_{L^2} - C.\end{aligned}$$

Since $0 \leq \gamma < 1$ we see that Φ is bounded from below and coercive, so Φ satisfies the (PS) condition. Note $\Phi'(0) = 0$ since $f(x,0) = 0$ for $x \in \Omega$. We may assume the origin is an isolated critical point (otherwise we have a sequence of nontrivial critical points of Φ and we are finished). Now Theorem 1.3 guarantees that $C^k(\Phi,0) \neq 0$ for some $k \geq 1$. Then Φ has two nontrivial critical points by Theorem 1.4. □

Our next two results follow immediately from Theorem 7.2.2 and Theorem 7.4.1 of [7] respectively (note $\lambda_k = \mu_k$ for all k).

Theorem 2.5. *Suppose* (1.2), (1.3), (1.4) *and* (1.5) *hold. In addition assume either*

$$\frac{\lambda_k}{2}\,(Ku,u) \leq \int_\Omega \int_0^{K^{\frac{1}{2}}u(x)} f(x,w)\,dw\,dx < \frac{\lambda_{k+1}}{2}\,(Ku,u) \quad \forall\; u \in B_\rho(0)\backslash\{0\}$$

for some k such that $\lambda_k < \lambda_{k+1}$ and $\rho > 0$, or

$$\frac{\lambda_\star}{2}\,(Ku,u) + o(|u|^2_{L^2}) \leq \int_\Omega \int_0^{K^{\frac{1}{2}}u(x)} f(x,w)\,dw\,dx \leq \frac{\lambda^\star}{2}\,(Ku,u) + o(|u|^2_{L^2}) \quad as \quad u \to 0$$

for some $\lambda_k < \lambda_\star \leq \lambda^\star < \lambda_{k+1}$. Also suppose (2.1) *holds and assume $f(x,0) = 0$ for $x \in \Omega$. Then* (1.7) *has a positive solution $v_1 \neq 0$. If $k \geq 2$ then there is a second solution $v_2 \neq 0$.*

Remark 2.6. The result in Theorem 2.5 can be improved if we use the ideas in Theorem 2.4. In fact in Theorem 2.5 there is a second solution $v_2 \neq 0$ if $k \geq 1$. To see this (again without loss of generality assume the origin is an isolated critical point) note [7, Lemma 7.2.1] guarantees that $C^k(\Phi,0) \neq 0$ and of course (2.1) guarantees that Φ is bounded from below and satisfies the (PS) condition. The result now follows from Theorem 1.4.

Theorem 2.7. *Suppose* (1.2), (1.3), (1.4) *and* (1.5) *hold. In addition assume*

$$f(x,-u) = -\,f(x,u) \quad for\ all \quad (x,u) \in \Omega \times \mathbf{R}$$

and that (2.1) *holds. Finally assume*

$$\int_\Omega \int_0^{K^{\frac{1}{2}}u(x)} f(x,w)\,dw\,dx \geq \frac{\lambda_\star}{2}\,(Ku,u) + o(|u|^2_{L^2}) \quad as \quad u \to 0$$

for some $\lambda_\star > \lambda_m$. Then (1.7) *has m distinct pairs (note Φ is even) of solutions (at positive levels).*

2. MAIN RESULTS IN SUPERLINEAR CASE

For notational purposes for $u \in L^2(\Omega)$ let

$$F(u) = \int_\Omega \int_0^{K^{\frac{1}{2}} u(x)} f(x,w)\, dw\, dx$$

and

$$H_\mu(u) = F(u) - \frac{1}{\mu} (K^{\frac{1}{2}} N_f K^{\frac{1}{2}} u, u), \ \ \mu > 0.$$

We begin with our main result in this section.

Theorem 3.1. *Suppose* (1.2), (1.3), (1.4) *and* (1.5) *hold. In addition assume*

$$f(x,0) = 0 \ \ \textit{for} \ \ x \in \Omega \tag{3.1}$$

$$F \ \ \textit{is bounded from below} \tag{3.2}$$

$$H_\mu \ \ \textit{is bounded from above for some} \ \ \mu > 2 \tag{3.3}$$

and

$$\lim_{t \to \infty} \frac{F(t\, u)}{t^2} = \infty \ \ \forall\ u \neq 0. \tag{3.4}$$

Finally suppose there exists λ *with* $\lambda \neq \lambda_i (= \mu_i)$ *for* $i \in \{1, 2, \dots\}$ *and with*

$$F(u) = \frac{\lambda}{2}\ (K\, u, u) + o(|u|^2_{L^2}) \ \ \textit{as} \ \ u \to 0.$$

Then Φ *has a nontrivial critical point.*

Proof. Note $\Phi'(0) = 0$ from (3.1). Suppose Φ has no nontrivial critical points. Then

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \backslash \{0\}), \ \ q \geq 0.$$

By the second deformation lemma (see Lemma 3.2.5 of [7] with $b = \infty$ and $b = 0$) Φ^0 is a deformation retract of $L^2(\Omega)$ and $\Phi^0 \backslash \{0\}$ deformation retracts to Φ^a for any $a < 0$ so

$$C^q(\Phi, 0) = H^q(L^2(\Omega), \Phi^a). \tag{3.5}$$

We know from Theorem 5.3.2 of [7] that Φ satisfies (PS). In Section 5.3 of [7] we proved for any u in the unit sphere S that

$$\frac{d}{d\, t}\, \Phi(t\, u) \leq \frac{2}{t}\ (\Phi(t\, u) - a_0)$$

where

$$a_0 = \inf\, \frac{1}{2}\, (\, [\mu - 2]\, F - \mu\, H_\mu)$$

so all the critical values of Φ are greater than or equal to a_0 (note $2\Phi(u) - \Phi'(u) = (\mu - 2) F(u) - \mu H_\mu(u)$). Note $a_0 \leq 0$ since $F(0) = H_\mu(0) = 0$. Also in Lemma 5.3.1 of [7] we showed for each $a < a_0$ there is a C^1 map $T_a : S \to (0, \infty)$ such that

$$\Phi^a = \{t\, u : \ u \in S : \ t \geq T_a(u)\} \cong S.$$

As a result if $|a|$ is sufficiently large (here $a < 0$), Φ^a is homotopic to S and hence contractible so from (3.5) we have

$$C^q(\Phi, 0) = 0 \ \forall \ q. \tag{3.6}$$

On the other hand since $\lambda \neq \lambda_i$ for $i \in \{1, 2, \dots\}$ and since zero is an isolated critical point (note $\Phi'(0) = 0$ and we assumed Φ has no nontrivial critical points) we know from Theorem 1.3 that $C^k(\Phi, 0) \neq 0$ for some $k \geq 0$. This is a contradiction. Thus Φ has a nontrivial critical point. □

Remark 3.2. To see how conditions (3.2), (3.3) and (3.4) relate to subcritical growth and the Ambrosetti–Rabinowitz condition we refer the reader to Example 5.3.3 in [7].

Our next two results follow immediately from Theorem 7.1.3 and Theorem 7.2.3 of [7] respectively (note $\lambda_k = \mu_k$ for all k).

Theorem 3.3. *Suppose* (1.2), (1.3), (1.4), (1.5), (3.1), (3.2), (3.3) *and* (3.4) *hold. In addition assume Φ has a finite number of critical points and either*

$$F(u) \leq \frac{\lambda_1}{2}\,(Ku, u) \ \ \forall \ u \in B_\rho(0)$$

for some $\rho > 0$, or

$$F(u) \leq \frac{\lambda^\star}{2}\,(Ku, u) + o(|u|^2_{L^2}) \quad as \quad u \to 0$$

for some $0 \leq \lambda^\star < \lambda_1$. Then (1.7) *has a mountain pass solution $v \neq 0$.*

Theorem 3.4. *Suppose* (1.2), (1.3), (1.4), (1.5), (3.1), (3.2), (3.3) *and* (3.4) *hold. In addition assume Φ has a finite number of critical points and either*

$$\frac{\lambda_k}{2}\,(Ku, u) \leq F(u) < \frac{\lambda_{k+1}}{2}\,(Ku, u) \ \ \forall \ u \in B_\rho(0)\backslash\{0\}$$

for some k such that $\lambda_k < \lambda_{k+1}$ and $\rho > 0$, or

$$\frac{\lambda_\star}{2}\,(Ku, u) + o(|u|^2_{L^2}) \leq F(u) \leq \frac{\lambda^\star}{2}\,(Ku, u) + o(|u|^2_{L^2}) \quad as \quad u \to 0$$

for some $\lambda_k < \lambda_\star \leq \lambda^\star < \lambda_{k+1}$. Then (1.7) *has a solution $v \neq 0$ with either $\Phi(v) < 0$ and $C^{k-1}(\Phi, v) \neq 0$ or $\Phi(v) > 0$ and $C^{k+1}(\Phi, v) \neq 0$.*

Remark 3.5. We can remove the condition that Φ has a finite number of critical points in Theorem 3.4 and then we deduce that (1.7) has a solution $v \neq 0$. Of course the result is immediate from Theorem 3.4. Alternately the proof is exactly the same as in Theorem 3.1 except here we deduce $C^k(\Phi, 0) \neq 0$ from [7, Lemma 7.2.1].

Our final result is a multiplicity result which follows immediately from Theorem 7.4.3 of [7].

Theorem 3.6. *Suppose* (1.2), (1.3), (1.4), (1.5), (3.2), (3.3) *and* (3.4) *hold. In addition assume*

$$f(x,-u) = -f(x,u) \quad \textit{for all} \quad (x,u) \in \Omega \times \mathbf{R}$$

and

$$F(u) \leq \frac{\lambda_\star}{2}(Ku,u) + o(|u|_{L^2}^2) \quad \textit{as} \quad u \to 0$$

for some $\lambda_\star < \lambda_k$. *Then* (1.7) *has infinite distinct pairs (note* Φ *is even) of solutions (at positive levels).*

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