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CRITICAL GROUPS OF FINITE TYPE FOR FUNCTIONALS DEFINED ON BANACH SPACES

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ABSTRACT. For a suitable class of functionals defined on Banach spaces, we prove that each isolated critical point has critical groups of finite type.

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1. INTRODUCTION AND MAIN RESULT

Let X be a Banach space and $f : X \longrightarrow \mathbb{R}$ a function of class C^1. A basic result of Morse theory, its global counterpart, says that, if $a, b \in \mathbb{R}$ with $a < b$, f satisfies $(PS)_c$ for any $c \in [a, b]$ and has a finite number of critical points in $f^{-1}([a, b])$, then there exists a formal series of powers Q with coefficients in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ such that

$$\sum_{\substack{a \le f(u) \le b \\ f'(u)=0}} P(f, u; \mathbb{K})(t) = P(f, a, b; \mathbb{K})(t) + (1 + t)Q(t)\,, \tag{1.1}$$

where $\mathbb{K}$ is an assigned field,

$$\begin{aligned} C_m(f, u; \mathbb{K}) &= H^m\left(\{f \le f(u)\}, \{f \le f(u)\} \setminus \{u\}; \mathbb{K}\right)\,, \\ P(f, u; \mathbb{K})(t) &= \sum_{m \ge 0} \dim\left(C_m(f, u; \mathbb{K})\right) t^m\,, \\ P(f, a, b; \mathbb{K})(t) &= \sum_{m \ge 0} \dim\left(H^m\left(\{f \le b\}, \{f < a\}; \mathbb{K}\right)\right) t^m\,, \end{aligned}$$

and H^* denotes for instance Alexander-Spanier cohomology (see e.g. [4, Theorem I.4.3] and [19, Theorem 8.2]). Then the task of the local counterpart of Morse theory is to provide a connection between $P(f, u; \mathbb{K})$ and other concepts of local nature describing the behavior of f near u. This is the part well established if X is a Hilbert space, but with only partial results in the Banach setting (see, in particular, [4, 22, 23] and, more recently, [5, 6]).

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Coming back to (1.1), already at this stage some conclusion can be derived. For instance, if $H^m(\{f \leq b\}, \{f < a\}; \mathbb{K})$ is nontrivial for some m, then there exists a critical point u in $f^{-1}([a,b])$ with $C_m(f,u;\mathbb{K})$ nontrivial.

However, if each $P(f,u;\mathbb{K})$ is a polynomial with coefficients in $\mathbb{N}$, rather than a formal series of powers with coefficients in $\overline{\mathbb{N}}$, then much more information can be deduced from (1.1). For instance, one is allowed to put $t = -1$ in (1.1), obtaining a typical relation involving the Euler-Poincaré characteristic of the pair $(\{f \leq b\}, \{f < a\})$. Again, if X is a Hilbert space, from the Shifting theorem of Gromoll and Meyer (see e.g. [4, Theorem I.5.4] or [19, Theorem 8.4]) one deduces a satisfactory information: if f is of class C^2 and the Hessian of f at u is a Fredholm operator, then $P(f,u;\mathbb{K})$ is a polynomial with coefficients in $\mathbb{N}$. For a generalization, with less regularity but still in the Hilbert setting, we refer the reader to [18, Theorem 2.2]. If X is a Banach space, partial results are contained in the papers we have already mentioned, but they do not cover, for instance, the functional

$$f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega G(x,u)\,dx\,,$$

with G of subcritical growth, defined on the Sobolev space $W_0^{1,p}(\Omega)$.

Our purpose is to provide a result which is applicable to such a functional. Consider a separable and reflexive Banach space X, whose dual space is denoted by X'. Let $f : X \longrightarrow \mathbb{R}$ be a function of the form $f = f_0 + f_1$, satisfying the following assumptions:

(A_1) $f_0 : X \longrightarrow \mathbb{R}$ is convex, of class C^1 and, for every sequence (u_k) weakly convergent to u in X with

$$\limsup_k f_0(u_k) \leq f_0(u)\,,$$

we have $\|u_k - u\| \to 0$;

(A_2) $f_1 : X \longrightarrow \mathbb{R}$ is of class C^1 and $f_1' : X \longrightarrow X'$ is completely continuous, namely, for every bounded sequence (u_k) in X, the sequence $(f_1'(u_k))$ admits a convergent subsequence in X'.

Our main result is the following

Theorem 1.1. *Let u be an isolated critical point of f. Then*

$$\bigoplus_{m\geq 0} C_m(f,u;\mathbb{K})$$

has finite dimension over $\mathbb{K}$, namely $P(f,u;\mathbb{K})$ is a polynomial with coefficients in $\mathbb{N}$. Moreover, there exists an open neighborhood U of u such that $\{f \leq f(u)\} \cap U$ and $(\{f \leq f(u)\} \setminus \{u\}) \cap U$ are both absolute neighborhood retracts.

In particular, the second information says that the choice of Alexander-Spanier cohomology, rather than singular cohomology, is not decisive, if critical points are isolated (see also [8, Remark 2]).

Let us point out that, since we have no generalized Morse lemma at our disposal, our proof is purely of topological rather than differential-topological type. The idea is to show that the cohomology of the pair $(\{f \leq f(u)\}, \{f \leq f(u)\} \setminus \{u\})$ is isomorphic to that of the same pair endowed with the weak topology. In this way we gain a kind of local compactness which allows us to show that this cohomology is in turn isomorphic to that of a compact ANR pair, which is of finite type by a well known result [24].

A feature of this argument is that, in spite of the fact that f is of class C^1, we are brought to consider also nonsmooth functionals, as this is the case of f with respect to the weak topology.

Actually, it would be interesting to complement Theorem 1.1 with a further information. It is easily seen that assumptions (A_1) and (A_2) imply that $f' : X \longrightarrow X'$ is of class $(S)_+$ in the sense of [2]. We conjecture that a Poincaré-Hopf theorem holds in the sense that

$$P(u, f; \mathbb{K})(-1) = \deg(f', V, 0)$$

for every sufficiently small neighborhood V of u, where deg denotes the degree for maps of class $(S)_+$, as defined in [2]. Let us recall that, in the Hilbert setting, a result of this type is well known (see [4, Theorem 3.2], [19, Theorem 8.5] and, with less regularity, [18, Theorem 3.2]).

Another possible conjecture is that, if f is even, then $P(0, f; \mathbb{K})(1)$ is an odd number. Again, this is true in finite dimension by approximation with Morse functions, hence whenever a Shifting theorem holds.

In Section 2 we recall some known properties concerning ANRs, while in Section 3 we recall some basic facts of nonsmooth critical point theory. In Section 4 we prove that $P(f, u; \mathbb{K})$ is a polynomial with coefficients in $\mathbb{N}$, when f is a continuous functional defined on a metric space satisfying suitable assumptions. Finally, in Section 5 we deduce Theorem 1.1.

2. ABSOLUTE NEIGHBORHOOD RETRACTS

All the results contained in this section are either well known or simple extensions of well known results. First of all, let us recall [14, Definition 11.1.1].

Definition 2.1. A topological space X is said to be an *absolute neighborhood retract* (ANR, for short) if it is metrizable and, for every metrizable space Y, every closed subset C of Y and every continuous map $\varphi : C \longrightarrow X$, there exist a neighborhood U of C in Y and a continuous map $\Phi : U \longrightarrow X$ such that $\Phi\big|_C = \varphi$.

Theorem 2.2. *Let X be an ANR and C a closed subset of X. Assume that C is also an ANR and that the quotient X/C is metrizable.*

Then X/C is an ANR.

Proof. Since X/C is metrizable, it is an ANR as a particular case of adjunction space (see e.g. [15, Theorem VI.1.2]). □

If X is a metric space, we denote by $\mathrm{B}_r(u)$ the open ball in X of center u and radius r. The proof of the next result is quite similar to that of [20, Theorem 16]. We sketch it for reader's convenience.

Theorem 2.3. *Let X, Y be two normed spaces, C a convex subset of Y and V an open subset of C. Let also $L : X \longrightarrow Y$ be a linear and continuous map such that $L(X) \cap C$ is dense in C and let $U = L^{-1}(V)$.*

Then $L\big|_U : U \longrightarrow V$ is a homotopy equivalence.

Proof. The sets C and $L^{-1}(C)$ are both ANRs, being convex subsets of normed spaces (see e.g. [14, Theorem 7.7.5]). In turn, V and U are also ANRs, being open subsets of ANRs (see e.g. [14, Proposition 11.1.2]). By [20, Theorem 14], V and U are dominated by simplicial complexes. By Whitehead's theorem [25, Theorem 1], it is therefore enough to prove that $L\big|_U : U \longrightarrow V$ is a weak homotopy equivalence.

Let $g : S^k \longrightarrow V$ be a continuous map. Let $\varepsilon > 0$ be such that $\|g(x) - v\| > \varepsilon$ whenever $x \in S^k$ and $v \in C \setminus V$. Let $v_1, \ldots, v_n \in L(X) \cap C$ be such that $g(S^k) \subseteq \bigcup_{j=1}^n \mathrm{B}_\varepsilon(v_j)$, say $v_j = Lu_j$ with $u_j \in X$, and let $\vartheta_1, \ldots, \vartheta_n : Y \longrightarrow [0,1]$ be continuous functions such that $\vartheta_j = 0$ outside $\mathrm{B}_\varepsilon(v_j)$ and $\sum_{j=1}^n \vartheta_j = 1$ on $g(S^k)$. Define a continuous map $f : S^k \longrightarrow X$ by

$$f(x) = \sum_{j=1}^n \vartheta_j(g(x))\, u_j \, .$$

Then we have

$$Lf(x) = \sum_{j=1}^n \vartheta_j(g(x))\, v_j \in C \, ,$$

$$\|Lf(x) - g(x)\| \leq \sum_{j=1}^n \vartheta_j(g(x))\, \|v_j - g(x)\| \leq \varepsilon \, ,$$

whence $Lf(x) \in V$, namely $f(x) \in U$. Define now $\mathcal{H} : S^k \times [0,1] \longrightarrow V$ by

$$\mathcal{H}(x,t) = \sum_{j=1}^n \vartheta_j(g(x))\, \left((1-t)g(x) + tv_j\right) \, .$$

Then $\mathcal{H}$ is a homotopy between g and Lf, so that $\left(L\big|_U\right)_\# : \pi_k(U) \longrightarrow \pi_k(V)$ is onto.

In a similar way it can be shown that $\left(L\big|_U\right)_\# : \pi_k(U) \longrightarrow \pi_k(V)$ is one-to-one and the assertion follows. □

3. METRIC CRITICAL POINT THEORY

Let X be a metric space endowed with the metric d and let $f : X \longrightarrow [-\infty, +\infty]$ be a function. We set

$$f^c = \{u \in X : \; f(u) \le c\} \qquad (c \in \mathbb{R}),$$

$$\mathrm{epi}\,(f) = \{(u, \lambda) \in X \times \mathbb{R} : \; f(u) \le \lambda\} \,.$$

In the following, $X \times \mathbb{R}$ will be endowed with the metric

$$d\left((u, \lambda), (v, \mu)\right) = \left(d(u, v)^2 + (\lambda - \mu)^2\right)^{\frac{1}{2}}$$

and $\mathrm{epi}\,(f)$ with the induced metric. Finally, as in [10], we define a continuous function $\mathcal{G}_f : \mathrm{epi}\,(f) \longrightarrow \mathbb{R}$ by $\mathcal{G}_f(u, \lambda) = \lambda$.

The next definition is taken from [3]. In an equivalent form, the notion was introduced in [11, 17], while a variant was considered in [16].

Definition 3.1. For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|\,(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist a neighborhood W of $(u, f(u))$ in $\mathrm{epi}\,(f)$, $\delta > 0$ and a continuous map $\mathcal{H} : W \times [0, \delta] \longrightarrow X$ satisfying

$$d(\mathcal{H}((v, \mu), t), v) \le t \,, \qquad f(\mathcal{H}((v, \mu), t)) \le \mu - \sigma t \,,$$

whenever $(v, \mu) \in W$ and $t \in [0, \delta]$.

The extended real number $|df|\,(u)$ is called the *weak slope* of f at u.

It is easily seen that, if (u_k) is a sequence convergent to u in X with $f(u) \in \mathbb{R}$ and $f(u_k) \to f(u)$, then

$$|df|\,(u) \le \liminf_k |df|\,(u_k) \,.$$

Moreover, according to [3, Proposition 2.3], for every $u \in X$ with $f(u) \in \mathbb{R}$, we have

$$|df|\,(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|\,(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|\,(u, f(u))^2}} & \text{if } |d\mathcal{G}_f|\,(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|\,(u, f(u)) = 1. \end{cases} \tag{3.1}$$

Finally, if X is an open subset of a normed space and $f : X \longrightarrow \mathbb{R}$ is of class C^1, then $|df|\,(u) = \|f'(u)\|$ for every $u \in X$ (see [11, Corollary 2.12]).

Proposition 3.2. *Let $\beta : X \longrightarrow \mathbb{R}$ be a locally Lipschitz function and set, for every $u \in X$,*

$$\mathrm{Lip}(\beta, u) = \inf\Big\{\inf\big\{\sigma \ge 0 : \; \beta \text{ is Lipschitz continuous of constant } \sigma \text{ on } \mathrm{B}_\delta\,(u)\big\} : \; \delta > 0\Big\}.$$

Then the following facts hold:

(*a*) *for every* $u \in X$ *with* $f(u) \in \mathbb{R}$, *we have*

$$|df|(u) - \mathrm{Lip}(\beta, u) \leq |d(f+\beta)|(u) \leq |df|(u) + \mathrm{Lip}(\beta, u)\,;$$

(*b*) *if we set* $Y = \{u \in X : \ f(u) \leq \beta(u)\}$, *for every* $u \in Y$ *with* $f(u) \in \mathbb{R}$ *and* $|df|(u) > \mathrm{Lip}(\beta, u)$, *we have*

$$\left|d\left(f\big|_Y\right)\right|(u) \geq |df|(u)\,.$$

Proof. Assertion (*a*) is proved in [13, Proposition 1.6]. To prove (*b*), consider $u \in Y$ with $f(u) \in \mathbb{R}$ and σ with $\mathrm{Lip}(\beta, u) < \sigma < |df|(u)$. Let $\delta > 0$ and

$$\mathcal{H} : (\mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f)) \times [0,\delta] \longrightarrow X$$

be a continuous map such that

$$d(\mathcal{H}((v,\mu),t), v) \leq t\,, \qquad f(\mathcal{H}((v,\mu),t)) \leq \mu - \sigma t\,,$$

for every $(v,\mu) \in (\mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f))$ and $t \in [0,\delta]$. Without loss of generality, we may also assume that β is Lipschitz continuous of constant σ on $\mathrm{B}_{2\delta}\,(u)$, hence on the range of $\mathcal{H}$. Then, if we define a continuous map

$$\mathcal{K} : (\mathrm{B}_\delta\,(u, f(u)) \cap \mathrm{epi}\,(f) \cap (Y \times \mathbb{R})) \times [0,\delta] \longrightarrow X$$

by

$$\mathcal{K}((v,\mu),t) = \begin{cases} \mathcal{H}((v,\mu),t) & \text{if } \mu \leq \beta(v)\,, \\ v & \text{if } \mu \geq \beta(v) \text{ and } \sigma t \leq \mu - \beta(v)\,, \\ \mathcal{H}\left((v,\beta(v)), t - \sigma^{-1}(\mu - \beta(v))\right) & \text{if } \mu \geq \beta(v) \text{ and } \sigma t \geq \mu - \beta(v)\,, \end{cases}$$

we have $d(\mathcal{K}((v,\mu),t), v) \leq t$. If $\mu \leq \beta(v)$, it holds

$$f(\mathcal{K}((v,\mu),t)) \leq \mu - \sigma t \leq \beta(v) - \sigma d(\mathcal{K}((v,\mu),t), v) \leq \beta(\mathcal{K}((v,\mu),t))\,.$$

If $\mu \geq \beta(v)$ and $\sigma t \leq \mu - \beta(v)$, we have

$$f(\mathcal{K}((v,\mu),t)) \leq \beta(\mathcal{K}((v,\mu),t)) = \beta(v) \leq \mu - \sigma t\,.$$

Finally, if $\mu \geq \beta(v)$ and $\sigma t \geq \mu - \beta(v)$, it holds

$$f(\mathcal{K}((v,\mu),t)) \leq \beta(v) - \sigma(t - \sigma^{-1}(\mu - \beta(v))) = \mu - \sigma t$$

and also

$$\begin{aligned} f(\mathcal{K}((v,\mu),t)) &\leq \beta(v) - \sigma(t - \sigma^{-1}(\mu - \beta(v))) \\ &\leq \beta(v) - \sigma d(\mathcal{K}((v,\mu),t), v) \leq \beta(\mathcal{K}((v,\mu),t))\,. \end{aligned}$$

Thus, in any case $f(\mathcal{K}((v,\mu),t)) \leq \mu - \sigma t$ and $\mathcal{K}$ has values in Y. Then we have $\left|d\left(f\big|_Y\right)\right|(u) \geq \sigma$ and the assertion follows by the arbitrariness of σ. □

Definition 3.3. An element $u \in X$ is said to be a (*lower*) *critical point* of f, if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. A real number c is said to be a (*lower*) *critical value* of f, if there exists a (lower) critical point u of f with $f(u) = c$. For every $c \in \mathbb{R}$, we set

$$K_c = \{u \in X : \ f(u) = c\,,\ |df|(u) = 0\}\ .$$

Definition 3.4. For every $u \in X$ with $f(u) \in \mathbb{R}$, every field $\mathbb{K}$ and every nonnegative integer m, we set

$$C_m(f, u; \mathbb{K}) = H^m\left(f^c, f^c \setminus \{u\}; \mathbb{K}\right)\,,$$

where $c = f(u)$ and H^* denotes Alexander-Spanier cohomology (see [21]). $C_m(f, u; \mathbb{K})$ is called the *m-th critical group* of f at u.

By excision, we have $H^m\left(f^c, f^c \setminus \{u\}; \mathbb{K}\right) \approx H^m\left(f^c \cap U, (f^c \setminus \{u\}) \cap U; \mathbb{K}\right)$ for every neighborhood U of u. Therefore, the concept has local nature. Moreover, if u is not a critical point of f, we have $C_m(f, u; \mathbb{K}) = \{0\}$ for any m (see [7, Proposition 3.4]). Finally, since the map

$$\begin{array}{ccc} (f^c, f^c \setminus \{u\}) & \longrightarrow & \left(\mathcal{G}_f^c, \mathcal{G}_f^c \setminus \{(u, f(u))\}\right) \\ u & \mapsto & (u, c) \end{array}$$

is a homotopy equivalence, we have

$$C_m(f, u; \mathbb{K}) \approx C_m(\mathcal{G}_f, (u, f(u)); \mathbb{K}) \tag{3.2}$$

(see [7, p. 1064]).

Definition 3.5. Let $c \in \mathbb{R}$. A sequence (u_k) in X is said to be a *Palais-Smale sequence at level c* (*$(PS)_c$-sequence*, for short) for f, if $f(u_k) \to c$ and $|df|(u_k) \to 0$.

The function f is said to satisfy the *Palais-Smale condition at level c* ($(PS)_c$, for short), if every $(PS)_c$-sequence for f admits a convergent subsequence in X.

Theorem 3.6. (Quantitative deformation theorem) *Let $f : X \longrightarrow \mathbb{R}$ be a continuous function and let $a, b \in \mathbb{R}$ with $a < b$. Assume that X is complete and that there exists $\sigma > 0$ such that $|df|(u) > \sigma$ for every $u \in f^{-1}(]a, b])$.*

Then $f^{-1}(a)$ is a strong deformation retract of $f^{-1}([a, b])$ by a map

$$\eta : f^{-1}([a, b]) \times [0, 1] \longrightarrow f^{-1}([a, b])$$

satisfying $f(\eta(u, t)) \leq f(u) - \sigma d(\eta(u, t), u)$ for any $u \in f^{-1}([a, b])$ and $t \in [0, 1]$.

Proof. It is a particular case of [9, Theorem 2.1]. □

Theorem 3.7. (Second deformation lemma) *Let $f : X \longrightarrow \mathbb{R}$ be a continuous function and let $a, b \in \mathbb{R}$ with $a < b$. Assume that X is complete, that f satisfies $(PS)_c$ for any $c \in [a, b]$, that there are no critical points of f in $f^{-1}(]a, b[)$ and only a finite number in $f^{-1}(a)$.*

Then $f^{-1}(a)$ is a strong deformation retract of $f^{-1}([a,b]) \setminus K_b$ by a map

$$\eta : \big(f^{-1}([a,b]) \setminus K_b\big) \times [0,1] \longrightarrow \big(f^{-1}([a,b]) \setminus K_b\big)$$

satisfying

$$\eta(u,t) \neq u \implies f(\eta(u,t)) < f(u)$$

for any $u \in f^{-1}([a,b]) \setminus K_b$ and $t \in [0,1]$.

Proof. See [7, Theorem 2.3] or [8, Theorem 4 and Remark 2]. The last assertion is not explicitly stated, but it is contained in the proof. □

4. CRITICAL GROUPS OF FINITE TYPE IN METRIC SPACES

Let X be a set endowed with two metrics d and d_0, let $f : X \longrightarrow \mathbb{R}$ be a function, let $u \in X$ and let $c = f(u)$. Assume that:

(M_1) for every $v, w \in X$, we have

$$d_0(v,w) \leq d(v,w)\,, \qquad |d_0 f|\,(v) \geq |df|\,(v)\,,$$

where $|d_0 f|$ and $|df|$ denote the weak slope with respect to the metrics d_0 and d, respectively;

(M_2) the space X is d-complete, is an ANR both in the d- and in the d_0-topology and, for every d_0-open subset U of X, the identity map of U is a homotopy equivalence from the d- to the d_0-topology;

(M_3) the function f is continuous with respect to the d_0-topology and there exists $\varepsilon > 0$ such that $\{v \in X :\ c - \varepsilon \leq f(v) \leq c + \varepsilon\}$ is d_0-compact and f satisfies $(PS)_a$ with respect to the metric d for every $a \in [c-\varepsilon, c+\varepsilon]$;

(M_4) u is an isolated critical point of f with respect to the metric d.

Our main result is the following

Theorem 4.1. *There exists a d_0-open neighborhood U of u such that:*

- (a) *$(f^c \cap U, (f^c \setminus \{u\}) \cap U)$ is a pair of ANRs both in the d- and in the d_0-topology;*
- (b) *the identities of $f^c \cap U$ and of $(f^c \setminus \{u\}) \cap U$, as maps from the d- to the d_0-topology, are homotopy equivalences;*
- (c) *the identity of $(f^c \cap U, (f^c \setminus \{u\}) \cap U)$, as a map from the d- to the d_0-topology, induces an isomorphism in cohomology;*
- (d) *for every field $\mathbb{K}$, the linear space*

$$\bigoplus_{m \geq 0} C_m(f, u; \mathbb{K})$$

has finite dimension over $\mathbb{K}$.

The section will be devoted to the proof of this result.

Lemma 4.2. *There exist two d_0-open neighborhoods U, V of u and a d_0-closed neighborhood W of u such that:*

(i) *the d_0-closure of V is contained in U;*
(ii) *$f^c \cap U$ is a strong deformation retract of U both in the d- and in the d_0-topology;*
(iii) *$f^c \cap (U \setminus V)$ is a strong deformation retract of $(f^c \setminus \{u\}) \cap U$ both in the d- and in the d_0-topology;*
(iv) *$f^c \cap (U \setminus V)$ is a strong deformation retract of $U \setminus W$ both in the d- and in the d_0-topology;*
(v) *if $f^c \cap U$ is endowed with the d_0-topology, then the quotient space*

$$(f^c \cap U)/\,(f^c \cap (U \setminus V))$$

is a compact ANR.

Proof. By (M_3) and (M_4), there exist $r, \sigma > 0$ such that $\sigma r \le \varepsilon$ and

$$|df|\,(v) > 0 \qquad \text{for every } v \in X \text{ with } 0 < d_0(v,u) \le 2r\,,$$
$$|df|\,(v) > \sigma \qquad \text{for every } v \in X \text{ with } r \le d_0(v,u) \le 2r\,.$$

Define $\beta : X \longrightarrow \mathbb{R}$ by $\beta(v) = \left[r - (d_0(v,u) - r)^+\right]^+$, so that β is globally Lipschitz continuous of constant 1 and satisfies

$$\mathrm{Lip}(\beta, v) = 0 \qquad \text{if } d_0(v,u) < r \text{ or } d_0(v,u) > 2r$$

with respect to d_0, hence also to d. Then set

$$U = \{v \in X : \ |f(v) - c| < \sigma\beta(v)\}\ ,$$

which is clearly a d_0-open neighborhood of u. Again by (M_3), there exists $\sigma' \in]0, \sigma/2]$ such that

$$|df|\,(v) > \sigma' \qquad \text{for every } v \in X \text{ with } c + \frac{\sigma}{2}\,\beta(v) < f(v) \le c + \sigma\beta(v)\,.$$

If we set $g = f - (\sigma - \sigma')\beta$ and

$$\begin{aligned} Y = \{v \in X :\ & c + (\sigma - \sigma')\beta(v) \le f(v) \le c + \sigma\beta(v)\} \\ &= \{v \in X :\ c \le g(v) \le c + \sigma'\beta(v)\}\ , \end{aligned}$$

then Y is d-complete and d_0-compact. Moreover, by Proposition 3.2 and (M_1) we have

$$|dg|\,(v)\,,\ |d_0 g|(v) > \sigma' \qquad \text{for every } v \in X \text{ with } r \le d_0(v,u) \le 2r\,,$$
$$|dg|\,(v) = |df|\,(v)\,,\ |d_0 g|(v) = |d_0 f|(v) \qquad \text{otherwise}\,,$$

hence

$$|dg|\,(v)\,,\ |d_0 g|(v) > \sigma' \qquad \text{for every } v \in X \text{ with } c < g(v) \le c + \sigma'\beta(v)\,.$$

Again by Proposition 3.2 and (M_1), it follows

$$\left|d\left(g\big|_Y\right)\right|(v) \geq |dg|(v) > \sigma'\,, \qquad \left|d_0\left(g\big|_Y\right)\right|(v) \geq |d_0 g|(v) > \sigma'\,,$$

for every $v \in Y$ with $g(v) > c$. By Theorem 3.6 we infer that

$$\{v \in X :\ f(v) = c + (\sigma - \sigma')\beta(v)\} = \{v \in Y :\ g(v) = c\}$$

is a strong deformation retract, both in the d- and in the d_0-topology, of Y by maps η satisfying

$$f(\eta(v,t)) \leq f(v) - \sigma' d(\eta(v,t), v) \quad (\text{resp. } f(\eta(v,t)) \leq f(v) - \sigma' d_0(\eta(v,t), v))\,.$$

It easily follows that in both cases

$$f(v) < c + \sigma\beta(v) \implies f(\eta(v,t)) < c + \sigma\beta(\eta(v,t))\,.$$

Therefore,

$$\{v \in X :\ d_0(v,u) < 2r\,,\ f(v) = c + (\sigma - \sigma')\beta(v)\}$$

is a strong deformation retract, both in the d- and in the d_0-topology, of

$$\{v \in X :\ c + (\sigma - \sigma')\beta(v) \leq f(v) < c + \sigma\beta(v)\}\ .$$

Consider now

$$Z = \{v \in X :\ c - \varepsilon \leq f(v) \leq c + (\sigma - \sigma')\beta(v)\}\ ,$$

which is again d-complete and d_0-compact. Arguing as before, it turns out that

$$\left|d\left(f\big|_Z\right)\right|(v) \geq |df|(v)\,, \qquad \left|d_0\left(f\big|_Z\right)\right|(v) \geq |d_0 f|(v) \geq |df|(v)\,,$$

for every $v \in Z$ with $f(v) > c - \varepsilon$. In particular, $f\big|_Z$ satisfies $(PS)_a$ for every $a \geq c$ and there is no critical point of $f\big|_Z$ in $\{v \in Z \setminus \{u\} :\ f(v) \geq c\}$, with respect to both metrics. By Theorem 3.7 we infer that $\{v \in X :\ f(v) = c\}$ is a strong deformation retract of

$$\{v \in X :\ c \leq f(v) \leq c + (\sigma - \sigma')\beta(v)\}\ ,$$

with respect to both topologies, by maps η satisfying

$$\eta(v,t) \neq v \implies f(\eta(v,t)) < f(v)\,.$$

Therefore

$$\{v \in X :\ d_0(v,u) < 2r\,,\ f(v) = c\}$$

is a strong deformation retract of

$$\{v \in X :\ d_0(v,u) < 2r\,,\ c \leq f(v) \leq c + (\sigma - \sigma')\beta(v)\}$$

with respect to both topologies. Combining this fact with the previous step, we infer that

$$\{v \in X :\ d_0(v,u) < 2r\,,\ f(v) = c\}$$

is a strong deformation retract of

$$\{v \in X : \ d_0(v,u) < 2r\,, \ c \le f(v) < c + \sigma\beta(v)\}$$

with respect to both topologies. It easily follows that $f^c \cap U$ is a strong deformation retract of U with respect to both topologies and assertion (ii) is proved.

Define now

$$V = \{v \in X : \ |f(v) - c| < \sigma\beta(v) - \sigma r/2\}\ ,$$

which is clearly a d_0-open neighborhood of u whose d_0-closure is contained in U.

Consider $h = f + \sigma\beta$ and

$$E = \{v \in X : \ c - \sigma\beta(v) \le f(v) \le c\} = \{v \in X : \ c \le h(v) \le c + \sigma\beta(v)\}\ ,$$

which is d-complete and d_0-compact. As before, we see that $h\big|_E$ satisfies $(PS)_a$ for every $a > c$, that $h(u) = c + \sigma r$ and there is no critical point of $h\big|_E$ in

$$\{v \in E \setminus \{u\} : \ c < h(v) \le c + \sigma r\}$$

with respect to both metrics. By Theorem 3.7 we infer that

$$\{v \in X : \ f(v) \le c\,, \ f(v) = c - \sigma\beta(v) + \sigma r/2\} = \{v \in E : \ h(v) = c + \sigma r/2\}$$

is a strong deformation retract of

$$\begin{aligned}\{v \in X \setminus \{u\} : \ c - \sigma\beta(v) + \sigma r/2 \le f(v) \le c\}& \\ = \{v \in E \setminus \{u\} : \ c + \sigma r/2 \le h(v) \le c + \sigma r\}&\end{aligned}$$

with respect to both topologies. It easily follows that $f^c \cap (U \setminus V)$ is a strong deformation retract of $(f^c \setminus \{u\}) \cap U$ with respect to both topologies and assertion (iii) is proved.

Finally, let

$$W = \{v \in X : \ f(v) \ge c - \sigma\beta(v) + 3\sigma r/4\}\ ,$$

which is clearly a d_0-closed neighborhood of u. Arguing as in the proof of (b), it turns out that

$$\{v \in X : \ d_0(v,u) < 2r\,, \ f(v) = \min\{c\,, \, c - \sigma\beta(v) + \sigma r/2\}\}$$

is a strong deformation retract of

$$\{v \in U : \ \min\{c\,, \, c - \sigma\beta(v) + \sigma r/2\} \le f(v) < c - \sigma\beta(v) + 3\sigma r/4\}$$

with respect to both topologies. It easily follows that $f^c \cap (U \setminus V)$ is a strong deformation retract of $U \setminus W$ with respect to both topologies and assertion (iv) is proved.

Consider now only the d_0-metric. The closure $\overline{f^c \cap U}$ is d_0-compact by assumption (M_3). Since $\left(\overline{f^c \cap U}, \overline{f^c \cap (U \setminus V)}\right)$ is a compact metrizable pair, then the quotient $\overline{f^c \cap U}/\overline{f^c \cap (U \setminus V)}$ is Hausdorff and compact, hence metrizable by [1, Proposition IX.2.17]. In turn, $f^c \cap U/f^c \cap (U \setminus V)$, which is clearly homeomorphic to

$\overline{f^c \cap U}/\overline{f^c \cap (U \setminus V)}$, is also compact and metrizable. Being retracts of the open subsets U and $U \setminus W$ of X, $f^c \cap U$ and $f^c \cap (U \setminus V)$ are ANRs by (M_2) and $f^c \cap (U \setminus V)$ is closed in $f^c \cap U$. Since the quotient $f^c \cap U / f^c \cap (U \setminus V)$ is metrizable, it is an ANR by Theorem 2.2. □

Proof of Theorem 4.1. We have already observed that $f^c \cap U$ is an ANR in the d_0-topology and the same argument applies to the d-topology. Then $(f^c \setminus \{u\}) \cap U$ is also an ANR, with respect to both topologies, being open in $f^c \cap U$ and assertion (a) follows.

By (ii) and (iv) of Lemma 4.2, the inclusions $f^c \cap U \longrightarrow U$ and $f^c \cap (U \setminus V) \longrightarrow U \setminus W$ are homotopy equivalences with respect to both topologies. From (M_2) we deduce that the identities of $f^c \cap U$ and of $f^c \cap (U \setminus V)$, as maps from the d- to the d_0-topology, are homotopy equivalences. By (iii) of Lemma 4.2 assertion (b) follows.

Assertion (c) is an obvious consequence of (b) and the Five lemma (see [21]). From (iii) of Lemma 4.2 we also see that

$$H^m\left(f^c \cap U, (f^c \setminus \{u\}) \cap U; \mathbb{K}\right) \approx H^m\left(f^c \cap U, f^c \cap (U \setminus V); \mathbb{K}\right).$$

On the other hand, from [21, Theorem 6.6.5], we infer that

$$H^m\left(f^c \cap U, f^c \cap (U \setminus V); \mathbb{K}\right) \approx H^m\left(f^c \cap U / f^c \cap (U \setminus V), \{f^c \cap (U \setminus V)\}; \mathbb{K}\right).$$

From (v) of Lemma 4.2 and [24, Corollary 5.3], we conclude that

$$\bigoplus_{m \geq 0} H^m\left(f^c \cap U / f^c \cap (U \setminus V), \{f^c \cap (U \setminus V)\}; \mathbb{K}\right)$$

has finite dimension and assertion (d) follows. □

5. CRITICAL GROUPS OF FINITE TYPE IN BANACH SPACES

Let X be a separable and reflexive Banach space, whose dual space is denoted by X', and let $f : X \longrightarrow \mathbb{R}$ be a function of the form $f = f_0 + f_1$ satisfying (A_1) and (A_2).

From (A_2) it easily follows that

$$f_1 \text{ is sequentially continuous with respect to the weak topology}. \tag{5.1}$$

Proposition 5.1. *Let C be a bounded, closed and convex subset of X and let $\varphi = f\big|_C$. Then φ satisfies $(PS)_c$ for every $c \in \mathbb{R}$.*

Proof. Let (u_k) be a sequence in C with $|d\varphi|(u_k) \to 0$. By [11, Proposition 2.10 and Theorem 2.11], there exists a sequence (w_k) in X' with

$$\|w_k\| = |d\varphi|(u_k),$$

$$f_0(v) \geq f_0(u_k) - \langle f_1'(u_k), v - u_k \rangle + \langle w_k, v - u_k \rangle \qquad \forall v \in C.$$

Up to a subsequence, (u_k) is weakly convergent to some $u \in C$ and $(f_1'(u_k))$ is strongly convergent to some z in X'. Since

$$f_0(u) \geq f_0(u_k) - \langle f_1'(u_k), u - u_k \rangle + \langle w_k, u - u_k \rangle \,,$$

we have

$$\limsup_k f_0(u_k) \leq f_0(u)$$

and the assertion follows. □

Let now (e_k) be a dense sequence in the unit ball of X' and let, for every $u, v \in X$,

$$(u|v)_0 = \sum_{k=1}^{\infty} 2^{-k} \langle e_k, u \rangle \langle e_k, v \rangle \,.$$

Then $(\ |\ )_0$ is a scalar product on X whose associated norm $\|\ \|_0$ satisfies

$$\|u\|_0 \leq \|u\| \qquad \forall u \in X$$

and induces the weak topology of X on every $\|\ \|$-bounded subset of X. We also consider the product $X \times \mathbb{R}$ endowed with the norms

$$\|(u,\lambda)\| = \left(\|u\|^2 + |\lambda|^2\right)^{\frac{1}{2}} \,,$$

$$\|(u,\lambda)\|_0 = \left(\|u\|_0^2 + |\lambda|^2\right)^{\frac{1}{2}} \,.$$

Proposition 5.2. *Let C be a $\|\ \|$-bounded and convex subset of X and let $\varphi = f\big|_C$. Then, for every $u \in C$ and $\lambda > \varphi(u)$, we have*

$$|d_0\varphi|(u) \geq |d\varphi|(u) \,,$$

$$|d_0\mathcal{G}_\varphi|(u,\varphi(u)) \geq |d\mathcal{G}_\varphi|(u,\varphi(u)) \,,$$

$$|d_0\mathcal{G}_\varphi|(u,\lambda) = |d\mathcal{G}_\varphi|(u,\lambda) = 1 \,,$$

where d_0 denotes the weak slope with respect to the norm $\|\ \|_0$.

Proof. Let $u \in C$. By substituting f_0 with $f_0(v) + f_1(u) + \langle f_1'(u), v - u \rangle$ and f_1 with $f_1(v) - f_1(u) - \langle f_1'(u), v - u \rangle$, we may assume that $f_1(u) = 0$ and $f_1'(u) = 0$. By [11, Proposition 2.10 and Theorem 2.11], it follows that $|d\varphi|(u) = \left|d\left(f_0\big|_C\right)\right|(u)$.

If $|d\varphi|(u) = 0$, the first inequality is obvious. Otherwise, let $0 < \sigma < |d\varphi|(u)$ and let $v \in C$ with $\|v - u\| < 1$ and $f_0(v) < f_0(u) - \sigma\|v - u\|$. Then the map

$$\mathcal{H} : \mathrm{epi}\,(\varphi) \times [0,1] \longrightarrow C$$

defined as $\mathcal{H}((w,\mu),t) = w + t(v - w)$ is $\|\ \|_0$-continuous and satisfies

$$\|\mathcal{H}((w,\mu),t) - w\|_0 \leq t(\|v - u\| + \|w - u\|) \,.$$

Observe also that, if (w_k, μ_k) is a sequence in $\mathrm{epi}\,(\varphi)$ with $\|(w_k,\mu_k) - (u,\varphi(u))\|_0 \to 0$, then we have

$$\limsup_k f(u_k) \leq \lim_k \mu_k = f(u) \,,$$

which yields, by (5.1),

$$\limsup_k f_0(u_k) \le f_0(u)\,.$$

From (A_1) it follows that $\|w_k - u\| \to 0$.

Now it is easy to see that

$$\|\mathcal{H}((w,\mu),t) - w\|_0 \le t\,, \qquad \varphi(w + t(v-w)) \le \varphi(w) - \sigma t \le \mu - \sigma t\,,$$

provided that $\|(w,\mu) - (u,\varphi(u))\|_0$ and t are small enough. Therefore, $|d_0\varphi|\,(u) \ge \sigma$ and the first inequality follows from the arbitrariness of σ. Taking into account (3.1), the second inequality also follows.

Now let $\lambda > \varphi(u)$. From [11, Proposition 2.3] it follows that $|d\mathcal{G}_\varphi|\,(u,\lambda) = 1$. On the other hand, by (A_2) and (5.1), for every $\varrho > 0$ there exists $\delta \in]0,1]$ such that $\delta \le \varrho$ and

$$|f_1(w) - f_1(u)| + |\langle f_1'(z), w-u\rangle| < \varrho$$

whenever $w, z \in C$ with $\|w-u\|_0 < \delta$. If

$$\mathcal{H} : \{w \in C : \ \|w-u\|_0 < \delta\,,\ f(w) < \lambda + \delta\} \times [0,\delta] \longrightarrow C$$

is defined as $\mathcal{H}(w,t) = w + t(u-w)$, we have

$$\|\mathcal{H}(w,t) - w\|_0 \le \varrho t\,,$$

and, by the convexity of f_0,

$$f(w + t(u-w)) \le f_0(w) + t(f_0(u) - f_0(w)) + f_1(w) + t\langle f_1'(z), u-w\rangle$$

for some $z \in C$. It follows

$$f(w + t(u-w)) \le f(w) + t(f(u) - f(w) + \varrho)\,.$$

By [12, Corollary 2.11] we conclude that $|d_0\mathcal{G}_\varphi|\,(u,\lambda) = 1$. □

Proposition 5.3. *Let C be a $\|\ \|$-bounded and convex subset of X and let $\varphi = f\big|_C$. Then the following facts hold:*

(*a*) *the map*

$$\begin{array}{ccc} \mathrm{epi}\,(f_0) \cap (C \times \mathbb{R}) & \longrightarrow & \mathrm{epi}\,(\varphi) \\ (u,\lambda) & \mapsto & (u, \lambda + f_1(u)) \end{array}$$

is a homeomorphism when both spaces are endowed with the $\|\ \|$-topology or the $\|\ \|_0$-topology;

(*b*) *for every $\|\ \|_0$-open subset V of* epi$\,(\varphi)$*, the identity of V is a homotopy equivalence from the $\|\ \|$-topology to the $\|\ \|_0$-topology.*

Proof. Assertion (*a*) easily follows from (5.1). By Theorem 2.3, for every $\|\ \|_0$-open subset V of epi$\,(f_0) \cap (C \times \mathbb{R})$, the identity of V is a homotopy equivalence from the $\|\ \|$-topology to the $\|\ \|_0$-topology. Then assertion (*b*) follows from (*a*). □

Proof of Theorem 1.1. Let u be an isolated critical point of f, let

$$C = \{v \in X : \|v - u\| \leq 1\}$$

and let $\varphi = f\big|_C$

By the excision property and (3.2), we have

$$C_m(f, u; \mathbb{K}) \approx C_m(\varphi, u; \mathbb{K}) \approx C_m(\mathcal{G}_\varphi, (u, \varphi(u)); \mathbb{K})$$

with respect to the $\| \ \|$-topology. Consider now the set $\mathrm{epi}\,(\varphi)$ endowed with the metrics d and d_0 induced by $\| \ \|$ and $\| \ \|_0$, respectively. Consider also $\mathcal{G}_\varphi : \mathrm{epi}\,(\varphi) \longrightarrow \mathbb{R}$ and $(u, \varphi(u)) \in \mathrm{epi}\,(\varphi)$. We aim to apply Theorem 4.1. By (3.1) and Proposition 5.2, assumption (M_4) holds.

From the definition of $\| \ \|_0$ and Proposition 5.2, we see that (M_1) holds. Since C is closed in the Banach space $(X, \| \ \|)$ and f is $\| \ \|$-continuous, it is clear that $\mathrm{epi}\,(\varphi)$ is d-complete. On the other hand, $\mathrm{epi}\,(f_0) \cap (C \times \mathbb{R})$ is a convex subset of $X \times \mathbb{R}$, hence an ANR both in the d and in the d_0-topology. Then the same fact is true for $\mathrm{epi}\,(\varphi)$, by (a) of Proposition 5.3. Taking into account (b) of Proposition 5.3, we conclude that (M_2) is satisfied.

By (3.1) and Propositions 5.1, 5.2, the function $\mathcal{G}_\varphi$ satisfies $(PS)_a$ with respect to the metric d, for any $a \in \mathbb{R}$. Of course, $\mathcal{G}_\varphi$ is d_0-continuous and C is d_0-compact, as X is reflexive. From (A_1) and (5.1) we see that φ is lower semicontinuous with respect to the weak topology, namely the d_0-topology. In particular, the set

$$\{(v, \mu) \in \mathrm{epi}\,(\varphi) : \ a \leq \mu \leq b\}$$

is d_0-compact for any $a, b \in \mathbb{R}$. Therefore (M_3) also holds. From Theorem 4.1 the assertion follows. □

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REFERENCES

[1] N. Bourbaki, *Éléments de Mathématique. Topologie Générale. Chapitres 5 à 10*, Hermann, Paris, 1974.

[2] F. E. Browder, Fixed point theory and nonlinear problems, *Bull. Amer. Math. Soc. (N.S.)* **9** (1983), 1–39.

[3] I. Campa and M. Degiovanni, Subdifferential calculus and nonsmooth critical point theory, *SIAM J. Optim.* **10** (2000), 1020–1048.

[4] K.-C. Chang, *Infinite-dimensional Morse Theory and Multiple Solution Problems*, Progress in Nonlinear Differential Equations and their Applications, **6**, Birkhäuser Boston Inc., Boston, MA, 1993.

[5] S. Cingolani and G. Vannella, Critical groups computations on a class of Sobolev Banach spaces via Morse index, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003), 271–292.

[6] S. Cingolani and G. Vannella, Marino-Prodi perturbation type results and Morse indices of minimax critical points for a class of functionals in Banach spaces, *Ann. Mat. Pura Appl. (4)* **186** (2007), 157–185.

[7] J.-N. Corvellec, Morse theory for continuous functionals, *J. Math. Anal. Appl.* **196** (1995), 1050–1072.

[8] J.-N. Corvellec, On the second deformation lemma, *Topol. Methods Nonlinear Anal.* **17** (2001), 55–66.

[9] J.-N. Corvellec and A. Hantoute, Homotopical stability of isolated critical points of continuous functionals, in: *Calculus of Variations, Nonsmooth Analysis and Related Topics*, (Ed.: Degiovanni and Ioffe), *Set-Valued Anal.* **10** (2002), 143–164.

[10] E. De Giorgi, A. Marino, and M. Tosques, Problemi di evoluzione in spazi metrici e curve di massima pendenza, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **68** (1980), 180–187.

[11] M. Degiovanni and M. Marzocchi, A critical point theory for nonsmooth functionals, *Ann. Mat. Pura Appl. (4)* **167** (1994), 73–100.

[12] M. Degiovanni, M. Marzocchi, and V. D. Rădulescu, Multiple solutions of hemivariational inequalities with area-type term, *Calc. Var. Partial Differential Equations* **10** (2000), 355–387.

[13] M. Degiovanni and F. Schuricht, Buckling of nonlinearly elastic rods in the presence of obstacles treated by nonsmooth critical point theory, *Math. Ann.* **311** (1998), 675–728.

[14] J. Dugundji and A. Granas, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[15] S.-T. Hu, *Theory of Retracts*, Wayne State University Press, Detroit, 1965.

[16] A. Ioffe and E. Schwartzman, Metric critical point theory. I. Morse regularity and homotopic stability of a minimum, *J. Math. Pures Appl. (9)* **75** (1996), 125–153.

[17] G. Katriel, Mountain pass theorems and global homeomorphism theorems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11** (1994), 189–209.

[18] C. Li, S. Li, and J. Liu, Splitting theorem, Poincaré-Hopf theorem and jumping nonlinear problems, *J. Funct. Anal.* **221** (2005), 439–455.

[19] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, **74**, Springer-Verlag, New York, 1989.

[20] R. S. Palais, Homotopy theory of infinite dimensional manifolds, *Topology* **5** (1966), 1–16.

[21] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York, 1966.

[22] A. J. Tromba, A general approach to Morse theory, *J. Differential Geometry* **12** (1977), 47–85.

[23] K. Uhlenbeck, Morse theory on Banach manifolds, *J. Functional Analysis* **10** (1972), 430–445.

[24] J. E. West, Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk, *Ann. Math. (2)* **106** (1977), 1–18.

[25] J. H. C. Whitehead, Combinatorial homotopy. I, *Bull. Amer. Math. Soc.* **55** (1949), 213–245.