

A GENERALIZED FOUR DIMENSIONAL EMDEN-FOWLER EQUATION WITH EXPONENTIAL NONLINEARITY

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ABSTRACT. In this paper we prove the existence of singular limits for solutions of four-dimensional Emden-Fowler equation in Bi-laplace form with exponential nonlinearity by using some nonlinear domain decomposition method. The proofs combine variational methods with potential theory techniques.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\Omega \subset \mathbb{R}^4$ be a regular bounded domain. We are interested in positive solutions of the problem

$$\begin{cases} \Delta(a\Delta u) = \rho^4 a e^u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

when the parameter ρ tends to 0 and $a(x)$ is a smooth function over Ω satisfying

$$(H) \quad 0 < a_1 \leq a(x) \leq a_2 < \infty.$$

We suppose that $a(x) = a(|x|)$ is a radial symmetric function defined in some ball $\Omega \subset \mathbb{R}^4$.

With the dilation, we can assume that $\Omega = B_R$, the ball of radius R centered on the origin with $R > 1$ large enough. $a(x)$ replaced by $a(x)/a(0)$, is now a function defined on B_R satisfying $a(0) = 1$ and also we can suppose that

$$\|\nabla^2 a\|_{L^\infty(B_R)} = c/R^2 < \eta, \quad (1.2)$$

for some $\eta > 0$ small enough, which will be determined later.

We will construct a family of solutions for (1.1) which blow up at the origin. To describe our result, let us denote by

$$\Delta_a^2 u = \frac{1}{a} \Delta(a\Delta u) = \Delta^2 u + \Sigma_a u, \quad (1.3)$$

where

$$\Sigma_b u = \frac{\Delta b}{b} \Delta u + 2 \frac{\nabla b}{b} \nabla(\Delta u) \quad (1.4)$$

and we denote by G_a the Green's function, solution of the problem

$$\begin{cases} \Delta_a^2 G_a = 64\pi^2 \delta_0 & \text{in } B_R \\ G_a = \Delta G_a = 0 & \text{on } \partial B_R. \end{cases} \tag{1.5}$$

It is easy to check that the function

$$H_a(x) := G_a(x) + 8 \log |x|$$

is a smooth function, H_a is the regular part of the Green's function.

Our main result is the following

Theorem 1.1. *Let $\Omega = B_R \subset \mathbb{R}^4$ and $a(x)$ be a radial function over Ω verifying (H). Then there exist a one parameter family of radially solutions $(u_\rho)_{0 < \rho < \rho_0}$ of (1.1) such that*

$$\lim_{\rho \rightarrow 0} u_\rho = G_a$$

in $C_{loc}^\infty(B_R - \{0\})$.

Problem (1.1) with $a = \text{const.}$ is given by

$$\begin{cases} \Delta^2 u = \rho^4 e^u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

In [2], Baraket *et al.* construct blow-up solutions for the problem (1.6) when ρ tends to 0 for general domains Ω and give the singular limit of such a family of solutions by using some nonlinear domain decomposition method.

Semilinear equations involving fourth order elliptic operator and exponential non-linearity appear naturally in conformal geometry and in particular in the prescription of the so called Q-curvature on 4-dimensional Riemannian manifolds, [5] and [6],

$$Q_g = \frac{1}{12}(-\Delta_g S_g + S_g^2 - 3|Ric_g|^2),$$

where Ric_g denotes the Ricci tensor and S_g is the scalar curvature of the metric g . The Q-curvature changes under a conformal change of metric

$$g_w = e^{2w} g$$

according to

$$P_g w + 2Q_g = 2\tilde{Q}_{g_w} e^{4w} \tag{1.7}$$

where

$$P_g := \Delta_g^2 + \delta\left(\frac{2}{3}S_g I - 2Ric_g\right)d \tag{1.8}$$

is the Paneitz operator, which is an elliptic fourth order partial differential operator [6] and which transforms according to

$$e^{4w} P_{e^{2w}g} = P_g, \tag{1.9}$$

under a conformal change of metric $g_w = e^{2w}g$. In the special case where the manifold is the Eucliden space, the Paneitz operator is given by $P_{g_{eucl}} = \Delta^2$ in which case (1.7) reduces to

$$\Delta^2 w = \tilde{Q}e^{4w}$$

the solution of which give rise to conformal metric $g_w = e^{2w}g_{eucl}$ whose Q-curvature is given by \tilde{Q} .

When $n = 2$, the analogue of the Q-curvature is the Gauss curvature and the corresponding problem is

$$\begin{cases} -\Delta u = \rho^2 e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

which has been studied for a long time by many authors. Let us mention that (1.10) corresponds to the case $a = \text{const.}$ in the following generalized Emden-Fowler equation

$$\begin{cases} -\text{div}(a\nabla u) = \rho^2 a e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

when ρ is a constant who tends to 0 and $a(x)$ is a smooth function over Ω satisfying (H).

The study of the equation (1.10) goes back to 1853 when Liouville derived a representation formula for all solutions of (1.10) which are defined in \mathbb{R}^2 , for more details see [9]. When ρ tends to 0, the asymptotic behavior of nontrivial branches of solutions of (1.10) is well understood thanks to the pioneer work of Suzuki [14] and Nagasaki-Suzuki [12] that characterizes the possible limits of nontrivial branches of solutions of (1.10). Many authors tried to construct blow-up solutions for problem (1.10), see [16] and [3]. We also refer to [8] for the Emden-Fowler equation with singular potential and convection term. The problem (1.11) has been studied by many authors, see [7], [17] and [15].

In order to construct solutions of problem (1.1) and prove our result, rotationally symmetric solutions are given and some properties of linearized operators are studied. We also use a matching argument inspired of [2]. Such ideas have been used also in [1]. Throughout the paper, the symbol c denotes always a positive constant independent of ρ , it could be changed from one line to another.

2. KNOWN RESULTS IN [2] AND REFINEMENTS

Following [2], we denote by ε the smallest positive number satisfying

$$\rho^4 = \frac{384 \varepsilon^4}{(1 + \varepsilon^2)^4}. \tag{2.1}$$

We define, for $\tau > 0$,

$$u_{\varepsilon,\tau}(x) = 4 \log(1 + \varepsilon^2) - 4 \log(\varepsilon^2 + \tau^2|x|^2) + 4 \log \tau. \tag{2.2}$$

Then $u_{\varepsilon,\tau}$ are solutions of

$$\Delta^2 u = \rho^4 e^u \text{ on } \mathbb{R}^4. \tag{2.3}$$

We consider the following associated linear operator

$$\mathbb{L}w = \Delta^2 w - \frac{384}{(1 + |x|^2)^4} w, \tag{2.4}$$

which corresponds to the linearization of (2.3) about the solution $u_{1,1}$ and which play an important role in the study of the inverse problem. To this purpose, we introduce some weighted Hölder spaces, which were firstly introduced by Caffarelli, Hardt and Simon [4] (see also [10], [11] and [13]).

Definition 2.1. Given $k \in \mathbb{N}$, $0 < \alpha < 1$ and $\mu \in \mathbb{R}$. The Hölder weighed space $C_{\mu}^{k,\alpha}(\mathbb{R}^4)$ is defined to be the collection of functions $w \in C_{loc}^{k,\alpha}(\mathbb{R}^4)$ for which the following norm

$$\|w\|_{C_{\mu}^{k,\alpha}(\mathbb{R}^4)} := \|w\|_{C^{k,\alpha}(\bar{B}_1)} + \sup_{r \geq 1} \left(r^{-\mu} \|w(r \cdot)\|_{C^{k,\alpha}(\bar{B}_1 - B_{1/2})} \right)$$

is finite.

We define the subspace of functions in $C_{\mu}^{k,\alpha}(\mathbb{R}^4)$ which are radially,

$$C_{rad,\mu}^{k,\alpha}(\mathbb{R}^4) = \{f \in C_{\mu}^{k,\alpha}(\mathbb{R}^4); \text{ such that } f(x) = f(|x|), \forall x \in \mathbb{R}^4\}.$$

The following result holds

Proposition 2.2. [2] *Suppose that $\mu > 0$ and $\mu \notin \mathbb{Z}$, then $\mathbb{L} : C_{rad,\mu}^{4,\alpha}(\mathbb{R}^4) \longrightarrow C_{rad,\mu-4}^{0,\alpha}(\mathbb{R}^4)$ is surjective.*

In the following, we will denote J_{μ} a right inverse for \mathbb{L} . Let $B_R^* = B_R - \{0\}$.

Definition 2.3. Given $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighed space $C_{\nu}^{k,\alpha}(\bar{B}_R^*)$ as the space of functions $w \in C_{loc}^{k,\alpha}(\bar{B}_R^*)$ which is endowed with the norm

$$\|w\|_{C_{\nu}^{k,\alpha}(\bar{B}_R^*)} := \sup_{0 < r \leq R} \left(r^{-\nu} \|w(r \cdot)\|_{C^{k,\alpha}(\bar{B}_1 - B_{1/2})} \right).$$

When $k \geq 2$, we denote by $[C_{\nu}^{k,\alpha}(\bar{B}_R^*)]_0$ the subspace of functions $w \in C_{\nu}^{k,\alpha}(\bar{B}_R^*)$ satisfying $w = \Delta w = 0$ on ∂B_R .

The mapping properties of Δ^2 we will need are included in the

Proposition 2.4. [2] *Assume that $\nu < 0$ and $\nu \notin \mathbb{Z}$, then*

$$\Delta^2 : [C_{\nu}^{4,\alpha}(\bar{B}_R^*)]_0 \rightarrow C_{\nu-4}^{0,\alpha}(\bar{B}_R^*)$$

is surjective.

Remark 2.5. Given γ and $\tilde{\gamma}$ two reals, we define $H_\gamma^i(x) = \gamma|x|^2$ the Bi-harmonic function on B_1 satisfying

$$H^i = \gamma, \Delta H^i = 8\gamma \text{ on } \partial B_1,$$

and the Bi-harmonic function on $\mathbb{R}^4 - B_1$, $H_\gamma^e(x) = \frac{\tilde{\gamma}}{|x|^2}$ satisfying

$$H^e = \tilde{\gamma}, \Delta H^e = 0 \text{ on } \partial B_1.$$

3. SOLUTIONS WITHOUT BOUNDARY CONDITIONS

Let

$$r_\varepsilon := \sqrt{\varepsilon},$$

we suppose ε is small such that $B_{r_\varepsilon} \subset B_1$. We would like to found a solution u of

$$\Delta_a^2 u = \rho^4 e^u \text{ in } B_{r_\varepsilon} \tag{3.1}$$

which is a perturbation of $u_{\varepsilon,\tau}$.

Let $\tau > 0$, we define

$$R_\varepsilon := \tau/\sqrt{\varepsilon}.$$

Using the following transformation

$$w(x) = u\left(\frac{\varepsilon x}{\tau}\right) + 8 \log \varepsilon - 4 \log \frac{\tau(1 + \varepsilon^2)}{2}, \tag{3.2}$$

found a solution u of (3.1) in B_{r_ε} is equivalent to found a solution w of

$$\Delta_a^2 w = 24 e^w \text{ in } B_{R_\varepsilon}, \tag{3.3}$$

where $\tilde{a}(x) = a(\frac{\varepsilon x}{\tau})$. Next, we will find a solution of (3.3) which is a perturbation of $u_{1,1}$.

Writing $w = u_{1,1} + f$ where $f \in C_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$, this amounts to solve the equation

$$\mathbb{L}f = -\Sigma_{\tilde{a}} u_{1,1} - \Sigma_{\tilde{a}} f + \frac{384}{(1 + |\cdot|^2)^4} (e^f - f - 1) \text{ in } B_{R_\varepsilon} \tag{3.4}$$

where $\Sigma_{\tilde{a}}$ was defined by (1.4). We recall the following

Definition 3.1. Given $\bar{r} \geq 1$, $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$, the weighted space $C_\mu^{k,\alpha}(\bar{B}_{\bar{r}})$ is defined to be the space of functions $w \in C_{loc}^{k,\alpha}(\bar{B}_{\bar{r}})$ endowed with the norm

$$\|w\|_{C_\mu^{k,\alpha}(\bar{B}_{\bar{r}})} = \|w\|_{C^{k,\alpha}(\bar{B}_1)} + \sup_{1 \leq r \leq \bar{r}} \left(r^{-\mu} \|w(r \cdot)\|_{C^{k,\alpha}(\bar{B}_1 - B_{1/2})} \right).$$

For all $\sigma \geq 1$, we denote by $\xi_\sigma : C_\mu^{0,\alpha}(\bar{B}_\sigma) \rightarrow C_\mu^{0,\alpha}(\mathbb{R}^4)$ the extension operator defined by

$$\xi_\sigma(f) = f \text{ in } \bar{B}_\sigma \text{ and } \xi_\sigma(f)(x) = \chi\left(\frac{|x|}{\sigma}\right) f\left(\sigma \frac{x}{|x|}\right) \text{ on } \mathbb{R}^4 - \bar{B}_\sigma,$$

where $t \mapsto \chi(t)$ is a smooth nonnegative cutoff function identically equal to 1 for $t \leq 1$ and identically equal to 0 for $t \geq 2$.

It is easy to check that there exists a constant $c = c(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\xi_\sigma(w)\|_{C_\mu^{0,\alpha}(\mathbb{R}^4)} \leq c \|w\|_{C_\mu^{0,\alpha}(\bar{B}_\sigma)}. \tag{3.5}$$

We fix $\mu \in (0, 1)$. To find a solution of (3.4), it is enough to find $f \in C_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$ solution of

$$f = \mathcal{M}(f), \tag{3.6}$$

where we define

$$\mathcal{M}(f) = \mathcal{M}(\varepsilon, \tau, f) = J_\mu \circ \xi_{R_\varepsilon} \left(-\Sigma_{\tilde{a}}u_{1,1} - \Sigma_{\tilde{a}}f + \frac{384}{(1+|\cdot|^2)^4}(e^f - f - 1) \right). \tag{3.7}$$

Given $\kappa > 0$ (whose value will be fixed later), we assume that the constant $\tau > 0$ satisfy

$$|\ln(\tau/\tau_*)| \leq 2\kappa \varepsilon \ln(1/\varepsilon) \tag{3.8}$$

where $\tau_* > 0$ is fixed. We have the following technical lemma.

Lemma 3.2. *Given $\kappa > 0$ and $\mu \in (0, 1)$. There exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$*

$$\|\mathcal{M}(0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon \tag{3.9}$$

and

$$\|\mathcal{M}(f_1) - \mathcal{M}(f_2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon \|f_1 - f_2\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \tag{3.10}$$

provided $f_i \in C_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$ and $\|f_i\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon$; $i = 1, 2$.

Proof: By (3.7), we have $\mathcal{M}(f) = J_\mu \circ \xi_{R_\varepsilon} \circ T(f)$ where

$$\begin{aligned} T(f) &= -\Sigma_{\tilde{a}}u_{1,1} - \Sigma_{\tilde{a}}f + \frac{384}{(1+|\cdot|^2)^4}(e^f - f - 1) \\ &= -\Sigma_{\tilde{a}}u_{1,1} - \frac{\Delta(\tilde{a})}{\tilde{a}}\Delta f - 2\frac{\nabla(\tilde{a})}{\tilde{a}}\nabla(\Delta f) + \frac{384}{(1+|\cdot|^2)^4}(e^f - f - 1) \\ &= -\Sigma_{\tilde{a}}u_{1,1} + L_1(f) + L_2(f) + L_3(f). \end{aligned}$$

We remark that L_1 and L_2 are linear mappings. Let c_κ denote a constant that only depends on κ (provided ε is chosen small enough). It follows from (1.4) that

$$\Sigma_{\tilde{a}}u_{1,1} = \frac{\Delta(a(\frac{\varepsilon}{\tau}))}{a(\frac{\varepsilon}{\tau})}\Delta u_{1,1} + 2\frac{\nabla(a(\frac{\varepsilon}{\tau}))}{a(\frac{\varepsilon}{\tau})}\nabla(\Delta u_{1,1}) = I + II.$$

Since a satisfy the condition (H),

$$\begin{aligned} \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \|I(r)\|_{C^{0,\alpha}(\bar{B}_1 - B_{1/2})} &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \varepsilon^2 \|\nabla^2 a\|_\infty \frac{2+r^2}{(1+r^2)^2} \\ &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} \varepsilon^2 r^{2-\mu} \leq c_\kappa \varepsilon^{(2+\mu)/2}. \end{aligned}$$

So, $\|I\|_{C_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon^{(2+\mu)/2}$. Also,

$$\begin{aligned} \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \|II(r)\|_{C^{0,\alpha}(\bar{B}_1 - B_{1/2})} &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \varepsilon^2 r^2 \|\nabla^2 a\|_\infty \frac{(3+r^2)}{(1+r^2)^3} \\ &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} \varepsilon^2 r^{2-\mu}. \end{aligned}$$

Then, we have $\|II\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon^{(2+\mu)/2}$ and thus we prove estimate (3.9).

Recall that $L_1(f) = -\frac{\Delta(a(\frac{\varepsilon}{\tau} \cdot))}{a(\frac{\varepsilon}{\tau} \cdot)} \Delta f$ and $L_2(f) = -2\frac{\nabla(a(\frac{\varepsilon}{\tau} \cdot))}{a(\frac{\varepsilon}{\tau} \cdot)} \nabla(\Delta f)$ for $f \in \mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$. We have

$$\begin{aligned} \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \|L_1(f)\|_{\mathcal{C}^{0,\alpha}(\bar{B}_1 - B_{1/2})} &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \varepsilon^2 \|\nabla^2 a\|_\infty r^{\mu-2} \|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \\ &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} \varepsilon^2 r^2 \|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \end{aligned}$$

So

$$\|L_1(f)\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon \|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \quad (3.11)$$

Also,

$$\begin{aligned} \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \|L_2(f)(r \cdot)\|_{\mathcal{C}^{0,\alpha}(\bar{B}_1 - B_{1/2})} &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \varepsilon^2 r \|\nabla^2 a\|_\infty r^{\mu-3} \|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \\ &\leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} \varepsilon^2 r^2 \|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \end{aligned}$$

Thus,

$$\|L_2(f)\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon \|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \quad (3.12)$$

Finally, consider $L_3(f) = \frac{384}{(1 + |\cdot|^2)^4} (e^f - f - 1)$, and let $f_1, f_2 \in B(0, 2c_\kappa \varepsilon)$ in $\mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$. We have

$$\|L_3(f_1) - L_3(f_2)\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon \|f_1 - f_2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.$$

Then, for any $f_1, f_2 \in \mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$ verifying $\|f_i\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon$, we have

$$\|(T(f_1) - T(f_2))\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon \|f_1 - f_2\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}.$$

Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that $c_\kappa \varepsilon \leq 1/2$ for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then (3.9) and (3.10) in lemma 3.1 are enough to show that $v \rightarrow \mathcal{M}(\varepsilon, \tau, v)$ is a contraction from $\{v \in \mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4); \|v\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon\}$ into itself and hence has a unique fixed point f in this set and which is a solution of (3.4) on $\mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$. We summarize this in the

Proposition 3.3. *Given $\kappa > 1$ and $\mu \in (0, 1)$, there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ (only depending on κ) such that for $\tau > 0$ satisfying (3.8) the function $w(x) = u_{1,1} + f$ solves (3.3) in B_{R_ε} . In addition*

$$\|f\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon.$$

4. THE NONLINEAR INTERIOR PROBLEM

Given γ a real, we define

$$\mathbf{u} := u_{1,1} + f + H_\gamma^i(\cdot/R_\varepsilon).$$

We would like to find a function u solution of

$$\Delta^2 u + \Sigma_{\bar{a}} u - 24e^u = 0 \quad (4.1)$$

that is defined in B_{R_ε} and that is a perturbation of \mathbf{u} . Writing $u = \mathbf{u} + v$ and using the fact that H_γ^i is Bi-harmonic, we see that this amounts to solve the equation

$$\mathbb{L}v = -\Sigma_{\bar{a}}H_\gamma^i(\cdot/R_\varepsilon) - \Sigma_{\bar{a}}v + \frac{384}{(1 + |\cdot|^2)^4}(e^{f+H_\gamma^i(\cdot/R_\varepsilon)+v} - e^f - v). \tag{4.2}$$

We denote by $S(v) = S_{\varepsilon,\tau,\gamma}(v)$ the right hand of (4.2). We fix $\mu \in (0, 1)$ and recall that J_μ is the right inverse provided by proposition 2.1.

To find a solution v of (4.2), it is enough to find $v \in C_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$ solution of

$$v = J_\mu \circ \xi_{R_\varepsilon} \circ S(v) \text{ in } B_{R_\varepsilon}. \tag{4.3}$$

We denote by $\mathcal{N}(v) = \mathcal{N}_{\varepsilon,\tau,\gamma}(v) = J_\mu \circ \xi_{R_\varepsilon} \circ S(v)$, the nonlinear operator.

Given $\kappa > 0$ (whose value will be fixed later), we suppose that the constant η that is introduced by (1.2) and the constant γ satisfy

$$|\gamma| \leq \kappa\varepsilon \tag{4.4}$$

and

$$\eta \leq \kappa\varepsilon, \tag{4.5}$$

we have the following

Lemma 4.1. *Let $\mu \in (0, 1)$. Under the assumptions (4.4) and (4.5) there exist a constant $c_\kappa > 0$ such that*

$$\|\mathcal{N}(0)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon$$

and

$$\|\mathcal{N}(v_2) - \mathcal{N}(v_1)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon^{1-\mu/2} \|v_2 - v_1\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)}$$

provided that $v_1, v_2 \in C_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$ and $\|v_i\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa \varepsilon$.

Proof. The proof of these estimates follows from the result of Remark 2.1 together with assumptions (4.4) and (4.5). Let c_κ denote constant that only depends on κ (provided ε is chosen small enough). We have

$$S(0) = -\Sigma_{\bar{a}}H_\gamma^i(\cdot/R_\varepsilon) + \frac{384}{(1 + |\cdot|^2)^4}e^f(e^{H_\gamma^i(\cdot/R_\varepsilon)} - 1) = I + II.$$

It follows from Remark 2.1 that

$$\|H_\gamma^i(r \cdot /R_\varepsilon)\|_{C^{4,\alpha}(\bar{B}_1 - B_{1/2})} \leq c_\kappa r^2 R_\varepsilon^{-2} |\gamma|,$$

for all $r \leq R_\varepsilon$.

According to (1.4), we obtain

$$\sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \|I(r \cdot)\|_{C^{0,\alpha}(\bar{B}_1 - B_{1/2})} \leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \varepsilon^2 R_\varepsilon^{-2} \|\nabla^2 a\|_\infty |\gamma|.$$

By using (1.2), (4.4) and (4.5), we have

$$\|I(r \cdot)\|_{C_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon^{(2+\mu)/2}.$$

Since $\mu \in (0, 1)$, by proposition 3.1, we have $\|f\|_{\mathcal{C}^{4,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa r_\varepsilon^2$, then for all $1 \leq r \leq R_\varepsilon$

$$|f(r)| \leq c_\kappa r_\varepsilon^2 r^\mu \leq c_\kappa \varepsilon^{1-\mu/2} \tag{4.6}$$

and $\varepsilon^{1-\mu/2}$ tends to 0 as ε tends to 0. Then

$$\sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \|II(r)\|_{\mathcal{C}^{0,\alpha}(\bar{B}_1 - B_{1/2})} \leq c_\kappa \sup_{1 \leq r \leq R_\varepsilon} r^{4-\mu} \frac{1}{(1+r^2)^4} r^2 R_\varepsilon^{-2} |\gamma| \leq c_\kappa \varepsilon^{(2+\mu)/2} .$$

Taking into account the fact that the extension operator does not modify the estimate, we have

$$\|\mathcal{N}(0)\|_{\mathcal{C}^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon^{1+\mu/2} .$$

For the second estimation, let $v_1, v_2 \in B(0, 2c_\kappa \varepsilon)$ in $\mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$, then

$$S(v_2) - S(v_1) = -\Sigma_{\bar{a}}(v_2 - v_1) + \frac{384}{(1+|\cdot|^2)^4} \left(e^{f+H_\gamma^i(\cdot/R_\varepsilon)}(e^{v_2} - e^{v_1}) + v_1 - v_2 \right).$$

According to (3.11) and (3.12), we have

$$\|\Sigma_{\bar{a}}(v_2 - v_1)\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon \|v_2 - v_1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$

and

$$\left\| \frac{1}{(1+|\cdot|^2)^4} \left(e^{f+H_\gamma^i(\cdot/R_\varepsilon)}(e^{v_2} - e^{v_1}) + v_1 - v_2 \right) \right\|_{\mathcal{C}_{\mu-4}^{0,\alpha}(\bar{B}_{R_\varepsilon})} \leq c_\kappa \varepsilon^{1-\mu/2} \|v_2 - v_1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} .$$

So

$$\|\mathcal{N}(v_2) - \mathcal{N}(v_1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon^{1-\mu/2} \|v_2 - v_1\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)},$$

provided that $v_1, v_2 \in \mathcal{C}_{rad,\mu}^{4,\alpha}(\mathbb{R}^4)$ and $\|v_i\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_k \varepsilon$.

Observe that these estimates are uniform in τ provided that τ remains in a fixed compact subset of $(0, \infty)$. Applying a fixed point Theorem for contraction mappings, we have the following result

Proposition 4.2. *Given $\kappa > 0$ and $\tau^-, \tau^+ > 0$, then there exists $\varepsilon_\kappa, c_\kappa > 0$ (depending on κ and τ^\pm) such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, τ in $[\tau^-, \tau^+]$, any given γ , satisfying (4.4), there exists a unique $v (= v_{\varepsilon,\tau,\gamma})$, solution of (4.3) such that $\|v\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa \varepsilon$.*

In particular, the function

$$u_{\varepsilon,\tau,\gamma} = u_{1,1} + f + H_\gamma^i(\cdot/R_\varepsilon) + v_{\varepsilon,\tau,\gamma}$$

solves (4.1) in B_{R_ε} . Observe that the function $v_{\varepsilon,\tau,\gamma}$ being obtained as a fixed point for contraction mapping, it depends smoothly on the parameter τ and on the boundary data γ . Moreover, we claim that the mapping $(\tau, \gamma) \mapsto v_{\varepsilon,\tau,\gamma}|_{B_{R_\varepsilon}} \in \mathcal{C}^{4,\alpha}(B_{R_\varepsilon})$ is compact. This follows from the fact that the equation which we solve is semilinear and in (4.3) the right hand side belongs to $\mathcal{C}^{8,\alpha}(B_{R_\varepsilon})$.

5. THE NONLINEAR EXTERIOR PROBLEM

Let $\varepsilon \in (0, 1/4)$,

$$r_\varepsilon = \sqrt{\varepsilon},$$

and $\lambda, \tilde{\gamma}$ two reals close to 0. We define

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_{\varepsilon, \lambda, \tilde{\gamma}} := (1 + \lambda)G_a + \chi H_{\tilde{\gamma}}^e(\cdot/r_\varepsilon),$$

where χ is a cut-off function identically equal to 1 in $B_{1/2}$ and equal to 0 outside B_1 . We recall that ρ is defined as

$$\rho^4 = \frac{384\varepsilon^4}{(1 + \varepsilon^2)^4}.$$

Recall that $\Omega = B_R$, we would like to find a solution u of the equation

$$\Delta_a^2 u - \rho^4 e^u = 0 \tag{5.1}$$

that is defined in $\bar{\Omega} - B_{r_\varepsilon}$ and that is a perturbation of $\tilde{\mathbf{u}}$. Writing $u = \tilde{\mathbf{u}} + \tilde{v}$, this amounts to solve

$$\Delta^2 \tilde{v} = \rho^4 e^{\tilde{\mathbf{u}} + \tilde{v}} - \Sigma_a(\tilde{\mathbf{u}} + \tilde{v}) - \Delta^2 \tilde{\mathbf{u}} = \tilde{S}(\tilde{v}), \tag{5.2}$$

where we have defined Σ_a by (1.4). We need to define an auxiliary weighted spaces.

Definition 5.1. Given $\bar{r} \in (0, 1/2)$, $\alpha \in (0, 1)$, $k \in \mathbb{R}$ and $\nu \in \mathbb{R}$. We define the Hölder weighted space $C_{\nu}^{k, \alpha}(\bar{\Omega} - B_{\bar{r}})$ as the space of functions $w \in C^{k, \alpha}(\bar{\Omega} - B_{\bar{r}})$ that endowed with the norm

$$\|w\|_{C_{\nu}^{k, \alpha}(\bar{\Omega} - B_{\bar{r}})} := \|w\|_{C^{k, \alpha}(\bar{\Omega} - B_{1/2})} + \sup_{\bar{r} \leq r < 1/2} \left(r^{-\nu} \|w(r \cdot)\|_{C^{k, \alpha}(\bar{B}_2 - B_1)} \right).$$

For $\sigma \in (0, 1/2)$, we denote by $\tilde{\xi}_\sigma : C_{\nu}^{0, \alpha}(\bar{\Omega} - B_\sigma) \rightarrow C_{\nu}^{0, \alpha}(\bar{\Omega}^*)$, the extension operator defined by

$$\tilde{\xi}_\sigma(f) = f \text{ in } \bar{\Omega} - B_\sigma, \quad \tilde{\xi}_\sigma(f)(x) = \tilde{\chi}\left(\frac{|x|}{\sigma}\right) f\left(\sigma \frac{x}{|x|}\right) \text{ in } \bar{B}_\sigma - B_{\sigma/2} \text{ and } \tilde{\xi}_\sigma(f) = 0 \text{ in } B_{\sigma/2},$$

where $t \rightarrow \tilde{\chi}(t)$ is a cutoff function identically equal to 1 for $t \geq 1$ and to 0 for $t \leq 1/2$. Then, there exists a constant $c = c(\nu) > 0$, only depending on ν , such that

$$\|\tilde{\xi}_\sigma(w)\|_{C_{\nu}^{0, \alpha}(\bar{\Omega}^*)} \leq c \|w\|_{C_{\nu}^{0, \alpha}(\bar{\Omega} - B_\sigma)}.$$

We fix $\nu \in (-1, 0)$, and denote by \tilde{G}_ν a right inverse of Δ^2 provided by proposition 2.2. To solve the equation (5.2), it is enough to find $\tilde{v} \in C_{\nu}^{4, \alpha}(\bar{\Omega}^*)$ solution of $\tilde{v} = \tilde{G}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{S}(\tilde{v})$. We denote by

$$\tilde{\mathcal{N}}(\tilde{v}) = \tilde{\mathcal{N}}_{\varepsilon, \lambda, \tilde{\gamma}}(\tilde{v}) = \tilde{G}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{S}(\tilde{v}).$$

Given $\kappa > 0$ (whose value will be fixed later on), we assume that the constant $\tilde{\gamma}$ satisfy

$$|\tilde{\gamma}| \leq \kappa \varepsilon \tag{5.3}$$

and the parameter λ is chosen to satisfy

$$|\lambda| \leq \kappa \varepsilon. \tag{5.4}$$

Then we have the following result

Lemma 5.2. *Given $\kappa > 1$ and $\nu \in (-1, 0)$. There exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$*

$$\|\tilde{\mathcal{N}}(0)\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq c_\kappa \varepsilon \tag{5.5}$$

and

$$\|\tilde{\mathcal{N}}(\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1)\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq c_\kappa \varepsilon \|\tilde{v}_2 - \tilde{v}_1\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \tag{5.6}$$

provided that $\tilde{v}_1, \tilde{v}_2 \in C_\nu^{4,\alpha}(\bar{\Omega}^*)$ and $\|\tilde{v}_i\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq 2c_\kappa \varepsilon$.

Proof. Reducing ε is necessary, we can assume that $\kappa \varepsilon < 1/4$. Let c_κ denotes a constant that only depends on κ (provided ε is chosen small enough). Let

$$\tilde{S}(0) = \rho^4 e^{\tilde{\mathbf{u}}} - \Sigma_a \tilde{\mathbf{u}} - \Delta^2 \tilde{\mathbf{u}} = I + II + III.$$

By the definition 5.1 and the fact that the extension operator does not modify the estimate, then for the proof of (5.5), we have to estimate

$$\sup_{r_\varepsilon \leq r < 1/2} r^{4-\nu} \|\tilde{S}(0)(r.)\|_{C^{0,\alpha}(\bar{B}_2 - B_1)}.$$

For $x \in \bar{B}_2 - B_1$ and $r \in [r_\varepsilon, 1/2)$, we have $rx \in B_1 - B_{r_\varepsilon}$ and

$$|I(rx)| \leq c_\kappa \varepsilon^4 r^{-8(1+\lambda)} e^{(1+\lambda)H_a(rx)} e^{\chi(rx)H_\gamma^\varepsilon(r \ x/r_\varepsilon)}.$$

From Remark 2.1 and condition (5.3), $\|I(r.)\|_{C^{0,\alpha}(\bar{B}_2 - B_1)} \leq c_\kappa \varepsilon^4 r^{-8(1+\lambda)}$. Since the parameter λ satisfy (5.4), we have

$$\sup_{r_\varepsilon \leq r < 1/2} r^{4-\nu} \|I(r.)\|_{C^{0,\alpha}(\bar{B}_2 - B_1)} \leq c_\kappa r_\varepsilon^{1-\nu/2}.$$

It is easy to check that the function G_a solution of (1.5) satisfy, for $x \in B_1$

$$\Delta G_a(x) = \frac{16}{a(x)} \frac{|x|^2 - 1}{|x|^2}. \tag{5.7}$$

Then $\|II(r.)\|_{C^{0,\alpha}(\bar{B}_2 - B_1)} \leq c_\kappa r^{-3} \|\nabla^2 a\|_\infty$ and $\sup_{r_\varepsilon \leq r < 1/2} r^{4-\nu} \|II(r.)\|_{C^{0,\alpha}(\bar{B}_2 - B_1)} \leq c_\kappa \varepsilon$.

Using the Remark 2.2, we have $\Delta^2 \tilde{\mathbf{u}} = 0$, in $B_1 - B_{r_\varepsilon}$. We conclude that $\|\tilde{S}(0)\|_{C_{\nu-4}^{0,\alpha}(\bar{\Omega} - B_{r_\varepsilon})} \leq c_\kappa \varepsilon$ and so

$$\|\tilde{\mathcal{N}}(0)\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq c_\kappa \varepsilon.$$

For the second estimation, let \tilde{v}_1, \tilde{v}_2 be in $C_\nu^{4,\alpha}(\bar{\Omega}^*)$ such that $\|\tilde{v}_i\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq 2c_\kappa r_\varepsilon^2$

$$|\tilde{S}(\tilde{v}_2) - \tilde{S}(\tilde{v}_1)| = \rho^4 (e^{\tilde{\mathbf{u}}+\tilde{v}_2} - e^{\tilde{\mathbf{u}}+\tilde{v}_1}) - \Sigma_a(\tilde{v}_2 - \tilde{v}_1) = I + II.$$

We have $\|I(r.)\|_{C^{0,\alpha}(\bar{B}_2-B_1)} \leq c_\kappa \varepsilon^4 r^{-8(1+\lambda)} r^{-\nu} \|\tilde{v}_2 - \tilde{v}_1\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)}$. Then

$$\sup_{r_\varepsilon \leq r < 1/2} r^{4-\nu} \|I(r.)\|_{C^{0,\alpha}(\bar{B}_2-B_1)} \leq c_\kappa r_\varepsilon^2 \|\tilde{v}_2 - \tilde{v}_1\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)}.$$

Also $\|II(r.)\|_{C^{0,\alpha}(\bar{B}_2-B_1)} \leq c_\kappa r^{\nu-2} \|\nabla^2 a\|_\infty \|\tilde{v}_2 - \tilde{v}_1\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)}$, so

$$\sup_{r_\varepsilon \leq r < 1} r^{4-\nu} \|II(r.)\|_{C^{0,\alpha}(\bar{B}_2-B_1)} \leq c_\kappa \|\nabla^2 a\|_\infty \|\tilde{v}_2 - \tilde{v}_1\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq c_\kappa \varepsilon \|\tilde{v}_2 - \tilde{v}_1\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)}.$$

Reducing ε_κ if necessary, we can assume that, $c_\kappa \varepsilon^2 < 1/2$, for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then (5.5) and (5.6) are enough to show that $\tilde{v} \rightarrow \tilde{\mathcal{N}}(\tilde{v})$ is a contraction from $\{\tilde{v} \in C_\nu^{4,\alpha}(\bar{\Omega}^*); \|\tilde{v}\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq 2c_\kappa \varepsilon\}$ into itself and hence has a unique fixed point which is a solution of (5.2).

We summarize this in the

Proposition 5.3. *Given $\kappa > 1$, there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ (only depending on κ) such that for $\varepsilon \in (0, \varepsilon_\kappa)$ and for λ, η and a boundary constant $\tilde{\gamma}$ satisfying (5.3), (4.5) and (5.4), the function*

$$\tilde{u} = (1 + \lambda)G_a + \chi H_{\tilde{\gamma}}^\varepsilon(\cdot/r_\varepsilon) + \tilde{v}$$

solves (5.1) in $\bar{\Omega} - B_{r_\varepsilon}$. In addition

$$\|\tilde{v}\|_{C_\nu^{4,\alpha}(\bar{\Omega}^*)} \leq c_\kappa \varepsilon.$$

6. THE NONLINEAR CAUCHY-DATA MATCHING

Keeping the notations of the previous sections, we gather the results of Propositions 4.1 and 5.1. From now on $\kappa > 1$ is fixed large enough and $\varepsilon \in (0, \varepsilon_\kappa)$.

We define $\tau_* > 0$ by

$$-4 \log \tau_* = H_a(0). \quad (6.1)$$

We assume that $\lambda \in \mathbb{R}$ satisfying (5.4), $\tau \in (0, +\infty)$ satisfying (3.8) and $\eta \in (0, +\infty)$ satisfying (4.5).

First, we consider some constant boundary data $\gamma \in \mathbb{R}$ satisfying (4.4). According to the result of proposition 4.1 and provided $\varepsilon \in (0, \varepsilon_\kappa)$, we can find u_{int} a solution of

$$\Delta_a^2 u - \rho^4 e^u = 0 \quad \text{in } B_{r_\varepsilon}.$$

By (3.2), these solutions can be decomposed as

$$u_{int}(x) = u_{\varepsilon,\tau}(x) + f(R_\varepsilon x/r_\varepsilon) + H_\gamma^i(x/r_\varepsilon) + v(R_\varepsilon x/r_\varepsilon),$$

with the function $v = v_{\varepsilon,\tau,\gamma}$ satisfying

$$\|v\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq c_\kappa r_\varepsilon^2$$

and $R_\varepsilon = \tau/\sqrt{\varepsilon}$.

Similarly, given some constant boundary data $\tilde{\gamma}$ satisfying (5.3), we can use the result of Proposition 5.1 to find a solution u_{ext} of

$$\Delta_a^2 u - \rho^4 e^u = 0$$

in $\bar{\Omega} - B_{r_\varepsilon}$ (provided $\varepsilon \in (0, \varepsilon_k)$), which can be decomposed as

$$u_{ext}(x) = (1 + \lambda)G_a(x) + \chi(x)H_{\tilde{\gamma}}^\varepsilon(x/r_\varepsilon) + \tilde{v}(x),$$

with the function $\tilde{v} = \tilde{v}_{\varepsilon, \lambda, \tilde{\gamma}} \in C_\nu^{4, \alpha}(\bar{\Omega}^*)$ verifying

$$\|\tilde{v}\|_{C_\nu^{4, \alpha}(\bar{\Omega}^*)} \leq c_\kappa \varepsilon. \tag{6.2}$$

It remains to determine the parameters and the boundary constants in such a way that the function that is equal to u_{int} in B_{r_ε} and that is equal to u_{ext} in $\bar{\Omega} - B_{r_\varepsilon}$ is a smooth function. This amounts to find the parameters so that

$$u_{int} = u_{ext}, \quad \partial_r u_{int} = \partial_r u_{ext}, \quad \Delta u_{int} = \Delta u_{ext} \quad \text{and} \quad \partial_r \Delta u_{int} = \partial_r \Delta u_{ext} \tag{6.3}$$

on $\partial B_{r_\varepsilon}$.

Assuming we have already done so, this provides for each ε small enough a function $u_\varepsilon \in C^3(\bar{\Omega})$ (which is obtained by patching together the function u_{int} and the function u_{ext}) weak solution of $\Delta_a^2 u - \rho^4 e^u = 0$ and elliptic regularity theory implies that this solution is in fact smooth and when ε tend to 0 and away from the point 0, the sequence u_ε converges to G_a .

Before we proceed, the following remarks are due. The function $u_{\varepsilon, \tau}$ can be expanded as

$$u_{\varepsilon, \tau}(x) = -8 \log |x| - 4 \log \tau + \mathcal{O}(\varepsilon) \quad \text{near } \partial B_{r_\varepsilon}. \tag{6.4}$$

Moreover, the function $(1 + \lambda)G_a$ that appears in the expression of u_{ext} can be expanded as

$$(1 + \lambda)G_a(x) = -8(1 + \lambda) \log |x| + H_a(0) + \frac{1}{2} \nabla^2 H_a(0) x^2 + \mathcal{O}(\varepsilon) \tag{6.5}$$

near $\partial B_{r_\varepsilon}$ (since H_a is a radial function).

It will be convenient to solve, instead of (6.3), the following set of equations

$$\begin{aligned} (u_{int} - u_{ext})(r_\varepsilon y) &= 0, & \partial_r (u_{int} - u_{ext})(r_\varepsilon y) &= 0, \\ \Delta (u_{int} - u_{ext})(r_\varepsilon y) &= 0, & \partial_r \Delta (u_{int} - u_{ext})(r_\varepsilon y) &= 0, \end{aligned} \tag{6.6}$$

on S^3 . Since H_a is a radial function and by a simple calculation, we have

$$\nabla^2 H_a(0) = 4I, \tag{6.7}$$

where I is the identity matrix. Then, equations (6.6) yield to the system

$$\begin{cases} -4 \log \tau + \gamma + 8 \lambda \log r_\varepsilon - H_a(0) - \tilde{\gamma} + \mathcal{O}(r_\varepsilon^2) = 0 \\ 2\gamma + 8 \lambda + 2\tilde{\gamma} + \mathcal{O}(r_\varepsilon^2) = 0 \\ 8\gamma + 16 \lambda + \mathcal{O}(r_\varepsilon^2) = 0 \\ -32\lambda + \mathcal{O}(r_\varepsilon^2) = 0. \end{cases} \tag{6.8}$$

Here and from now on, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on the variables $\tau, \gamma, \tilde{\gamma}$ and λ , but it is bounded (in the appropriate norm) by a constant (independent of ε and κ) time ε .

Note that these equations come from (6.6) when expansions (6.4) and (6.5) are used, together with the expression of H^i and H^e . The system (6.8) can be readily simplified into

$$\lambda = \mathcal{O}(r_\varepsilon^2), \quad \gamma = \mathcal{O}(r_\varepsilon^2), \quad \tilde{\gamma} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\log r_\varepsilon} [4 \log \tau + H_a(0)] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ^- and τ^+ since, according to the above, as ε tends to 0 we expect that τ will converge to τ_* satisfying (6.1) and hence it is enough to choose τ^- and τ^+ in such a way that

$$4 \log \tau^- < -H_a(0) < 4 \log \tau^+.$$

If we define

$$t = \frac{1}{\log r_\varepsilon} [4 \log \tau + H_a(0)],$$

then the system we have to solve reads

$$(t, \lambda, \gamma, \tilde{\gamma}) = \mathcal{O}(r_\varepsilon^2). \tag{6.9}$$

The nonlinear mapping which appears on the right hand side of (6.9) is continuous, compact and by Schauder’s fixed point Theorem in the ball of radius $\kappa\varepsilon$ in the product space, the nonlinear system (6.9) can be solved and this completes the proof of Theorem 1.1.

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