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MULTIPLICITY RESULTS FOR p-BIHARMONIC PROBLEMS VIA MORSE THEORY

SHIBO LIU[1], EVERALDO MEDEIROS[2], AND KANISHKA PERERA[3]

[1]Department of Mathematics, Shantou University
Shantou 515063, Guangdong, P.R.China
E-mail: liusb@stu.edu.cn

[2]Departamento de Matemática, Universidade Federal da Paraíba
58051-900, João Pessoa - PB, Brazil
E-mail: everaldo@mat.ufpb.br

[3]Department of Mathematical Sciences, Florida Institute of Technology
Melbourne, FL 32901, USA
E-mail: kperera@fit.edu

ABSTRACT. We establish some multiplicity results for p-sublinear and p-superlinear p-biharmonic problems using Morse theory.

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1. INTRODUCTION

Consider the fourth order p-biharmonic problem with the Dirichlet boundary conditions

$$\begin{cases} \Delta_p^2 u = f(x,u) & \text{in } \Omega \\ \\ u = \nabla u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with C^2 boundary $\partial\Omega$,

$$\Delta_p^2 u = \Delta\big(|\Delta u|^{p-2}\,\Delta u\big)$$

is the p-biharmonic operator, $p \in (1,\infty)$, f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

$$|f(x,t)| \leq C\left(|t|^{q-1} + 1\right) \tag{1.2}$$

for some $q \in (1, p_2^*)$,

$$p_2^* = \begin{cases} \dfrac{np}{n-2p}, & p < n/2 \\ \\ \infty, & p \geq n/2 \end{cases}$$

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is the critical exponent for the Sobolev imbedding $W_0^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$, and C denotes a generic positive constant. Assume that $f(x,0) = 0$ a.e., so that (1.1) has the trivial solution $u(x) \equiv 0$. The purpose of this paper is to obtain others.

We assume that

$$F(x,t) := \int_0^t f(x,s)\, ds = \frac{\lambda}{p}\, |t|^p + \mathrm{o}(|t|^p) \;\text{ as }\; t \to 0, \;\text{uniformly a.e.} \tag{1.3}$$

for some $\lambda \in \mathbb{R}$. Note that (1.3) holds if

$$f(x,t) = \lambda\, |t|^{p-2}\, t + \mathrm{o}(|t|^{p-1}) \;\text{ as }\; t \to 0, \;\text{uniformly a.e.},$$

which leads us to the nonlinear eigenvalue problem

$$\begin{cases} \Delta_p^2\, u = \lambda\, |u|^{p-2}\, u & \text{in } \Omega \\[2ex] \quad u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

It turns out that this eigenvalue problem plays an important role in solving the problem (1.1).

Before giving some comments about (1.4), let us recall the eigenvalue problem for the p-biharmonic operator with the Navier boundary condition

$$\begin{cases} \Delta_p^2\, u = \mu\, |u|^{p-2}\, u & \text{in } \Omega \\[2ex] \quad u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Drábek and Ôtani [2] showed that the first eigenvalue of (1.5), given by

$$\mu_1 = \inf_{\substack{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ \int_\Omega |u|^p = 1}} \int_\Omega |\Delta u|^p,$$

is positive, simple, and isolated in the spectrum $\sigma(\Delta_p^2)$. So the second eigenvalue $\mu_2 = \inf \sigma(\Delta_p^2) \cap (\mu_1, \infty)$ is also well-defined. The fact that μ_1 is *isolated* enabled Liu and Squassina [6] to split the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ into two complementary subspaces, and using the idea of *local linking* (see Liu and Li [5]) to obtain existence and multiplicity results for p-biharmonic problems with the Navier boundary condition.

For the eigenvalue problem (1.4) with the Dirichlet boundary condition, El Khalil et al. [3] obtained a nondecreasing and unbounded sequence of eigenvalues using Ljusternik-Schnirelmann theory on C^1-manifolds (see Szulkin [8]). However, to the best of our knowledge, it is not known whether the properties of the first eigenvalue are as good as in the Navier case mentioned above. It particular, we do not know whether it is isolated. Therefore, we cannot split the underlying space $W_0^{2,p}(\Omega)$ as in Liu and Squassina [6] and solve the problem (1.1).

Here we will work with a new sequence of eigenvalues constructed in Perera et al. [7]. This sequence is based on the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [4] and gives nontrivial critical groups, a fact that will play a crucial role in our proofs.

We say that the nonlinearity f is p-sublinear if

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = 0 \text{ a.e.,} \tag{1.6}$$

and p-superlinear if

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = \infty \text{ a.e.} \tag{1.7}$$

In the p-sublinear case we will strengthen (1.6) to

$$|F(x,t)| \le C\,(|t|^r + 1) \tag{1.8}$$

for some $r \in (1,p)$. In the p-superlinear case we will assume the Ambrosetti-Rabinowitz condition

$$0 < \mu\, F(x,t) \le t f(x,t), \quad |t| \ge T \tag{1.9}$$

for some $\mu > p$ and $T > 0$. Note that integrating (1.9) gives

$$F(x,t) \ge c(x)\,|t|^\mu - C \tag{1.10}$$

where $c(x) = \min\, F(x, \pm T)/T^\mu > 0$, which implies (1.7). We will prove

Theorem 1.1. *Assume* (1.2) *with* $q \in (1, p_2^*)$, (1.3), *and* (1.8) *with* $r \in (1,p)$. *If* $\lambda > \lambda_1$ *is not an eigenvalue of* (1.4), *then problem* (1.1) *has two nontrivial solutions.*

Theorem 1.2. *Assume* (1.2) *with* $q \in (1, p_2^*)$, (1.3), *and* (1.9) *with* $\mu > p$ *and* $T > 0$. *If* λ *is not an eigenvalue of* (1.4), *then problem* (1.1) *has a nontrivial solution.*

Remark 1.3. We can also prove the corresponding theorems for the problem with the Navier boundary conditions

$$\begin{cases} \Delta_p^2\, u = f(x,u) & \text{in } \Omega \\[2ex] \quad u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.11}$$

by working in the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. This problem has been considered in Liu and Squassina [6]. However, there it is required that for $|t|$ small, $p\,F(x,t)/|t|^p$ is bounded between the first two eigenvalues of (1.5). Our approach here does not have such a restriction.

Our proofs will be based on an abstract framework for nonlinear eigenvalue problems introduced in Perera et al. [7], which we will recall in the next section.

2. PRELIMINARIES

In this section we recall an abstract framework for nonlinear eigenvalue problems introduced in Perera et al. [7].

Let $(W, \|\cdot\|)$ be a real reflexive Banach space with the dual $(W^*, \|\cdot\|^*)$ and the duality pairing $(\cdot,\cdot)$. We consider the nonlinear eigenvalue problem

$$A_p\, u = \lambda\, B_p\, u \tag{2.1}$$

in W^*, where $A_p \in C(W, W^*)$ is

(A_1) $(p-1)$-homogeneous and odd for some $p \in (1, \infty)$:

$$A_p\,(\alpha u) = |\alpha|^{p-2}\,\alpha\, A_p\, u \quad \forall u \in W,\ \alpha \in \mathbb{R},$$

(A_2) uniformly positive: $\exists\, c_0 > 0$ such that

$$(A_p\, u, u) \geq c_0\, \|u\|^p \quad \forall u \in W,$$

(A_3) a potential operator: there is a functional $I_p \in C^1(W, \mathbb{R})$, called a potential for A_p, such that

$$I_p'(u) = A_p\, u \quad \forall u \in W,$$

(A_4) of type (S): for any sequence $(u_j) \subset W$,

$$u_j \rightharpoonup u,\ \ (A_p\, u_j, u_j - u) \to 0 \implies u_j \to u,$$

and $B_p \in C(W, W^*)$ is

(B_1) $(p-1)$-homogeneous and odd,
(B_2) strictly positive:

$$(B_p\, u, u) > 0 \quad \forall u \neq 0,$$

(B_3) a compact potential operator.

The following proposition is often useful for verifying (A_4).

Proposition 2.1 ([7, Proposition 1.0.3])**.** *If W is uniformly convex and*

$$(A_p\, u, v) \leq r\, \|u\|^{p-1}\, \|v\|\,,\ \ (A_p\, u, u) = r\, \|u\|^p \quad \forall u, v \in W$$

for some $r > 0$, then (A_4) holds.

By Proposition 1.0.2 of [7], the potentials I_p and J_p of A_p and B_p satisfying $I_p(0) = 0 = J_p(0)$ are given by

$$I_p(u) = \frac{1}{p}\,(A_p\, u, u)\,, \qquad J_p(u) = \frac{1}{p}\,(B_p\, u, u)\,,$$

respectively, and are p-homogeneous and even. Let

$$\mathcal{M} = \big\{u \in W : I_p(u) = 1\big\}.$$

Then $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric C^1-Finsler manifold radially homeomorphic to the unit sphere in W, and the eigenvalues of (2.1) coincide with the critical values of the even C^1-functional

$$\Psi(u) = \frac{1}{J_p(u)}, \quad u \in \mathcal{M}$$

(see Section 4.1 of [7]).

Denote by $\mathcal{F}$ the class of symmetric subsets of $\mathcal{M}$ and by $i(M)$ the Fadell-Rabinowitz cohomological index of $M \in \mathcal{F}$ (see [4]). Then

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad 1 \leq k \leq \dim W \tag{2.2}$$

defines a nondecreasing sequence of eigenvalues of (2.1) that is unbounded when W is infinite dimensional (see Theorem 4.2.1 of [7]).

Now we consider the operator equation

$$A_p\, u = F'(u) \tag{2.3}$$

where $F \in C^1(W, \mathbb{R})$ with F' compact, whose solutions coincide with the critical points of the functional

$$\Phi(u) = I_p(u) - F(u), \quad u \in W.$$

The following proposition is useful for verifying the (PS) condition for Φ.

Proposition 2.2 ([7, Lemma 3.1.3])**.** *Every bounded* (PS) *sequence of* Φ *has a convergent subsequence.*

Suppose that $u = 0$ is a solution of (2.3) and the asymptotic behavior of F near zero is given by

$$F(u) = \lambda\, J_p(u) + \mathrm{o}(\|u\|^p) \ \text{ as } \ u \to 0. \tag{2.4}$$

Proposition 2.3 ([7, Proposition 9.4.1])**.** *Assume* (A_1) - (A_4), (B_1) - (B_3), *and* (2.4) *hold,* F' *is compact, and zero is an isolated critical point of* Φ.

(i) If $\lambda < \lambda_1$, *then* $C^q(\Phi, 0) \approx \delta_{q0}\, \mathbb{Z}_2$.
(ii) If $\lambda_k < \lambda < \lambda_{k+1}$, *then* $C^k(\Phi, 0) \neq 0$.

3. PROOFS OF THEOREMS 1.1 AND 1.2

First let us verify that our problem fits into the abstract framework of the previous section. Let $W = W_0^{2,p}(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p\right)^{\frac{1}{p}},$$

and define $A_p\,, B_p \in C(W, W^*)$ and $F \in C^1(W, \mathbb{R})$ by

$$(A_p\, u, v) = \int_\Omega |\Delta u|^{p-2}\, \Delta u\, \Delta v, \qquad (B_p\, u, v) = \int_\Omega |u|^{p-2}\, uv$$

and

$$F(u) = \int_\Omega F(x, u).$$

Then (A_1), (B_1), and (B_2) are clear, $(A_p\, u, u) = \|u\|^p$ in (A_2), and (A_3) and (B_3) hold with the potentials

$$I_p(u) = \frac{1}{p} \int_\Omega |\Delta u|^p, \qquad J_p(u) = \frac{1}{p} \int_\Omega |u|^p,$$

respectively. By the Hölder inequality,

$$(A_p\, u, v) \le \left(\int_\Omega |\Delta u|^p \right)^{1-\frac{1}{p}} \left(\int_\Omega |\Delta v|^p \right)^{\frac{1}{p}} = \|u\|^{p-1}\, \|v\|\,,$$

so (A_4) follows from Proposition 2.1. We have

$$\Phi(u) = \int_\Omega \frac{1}{p}\, |\Delta u|^p - F(x, u), \quad u \in W,$$

and (2.4) follows from (1.2) and (1.3).

Proof of Theorem 1.1. Since $\lambda > \lambda_1$ is not an eigenvalue of (1.4), it follows from Proposition 2.3 (ii) that $C^k(\Phi, 0) \neq 0$ for some $k \ge 1$. By (1.8) and the Sobolev imbedding,

$$\Phi(u) \ge \frac{1}{p}\, \|u\|^p - C\, (\|u\|^r + 1) \quad \forall u \in W.$$

Since $p > r$, it follows that Φ is bounded from below and coercive. Then every (PS) sequence of Φ is bounded and hence Φ satisfies the (PS) condition by Proposition 2.2. Thus, Φ has two nontrivial critical points by the following “three critical points theorem” of Chang [1] and Liu and Li [5].

Proposition 3.1. *Let Φ be a C^1-functional defined on a Banach space. If Φ is bounded from below, satisfies* (PS)*, and $C^k(\Phi, 0) \neq 0$ for some $k \ge 1$, then Φ has two nontrivial critical points.*

Remark 3.2. Note that it suffices to assume $\lambda > \lambda_1$ is not an eigenvalue from the particular sequence (λ_k) in (2.2).

Proof of Theorem 1.2. Note that $F(x, t)$ and $H(x, t) := tf(x, t) - \mu\, F(x, t)$ are bounded from below by (1.2) and (1.9).

Lemma 3.3. *If* (1.2) *with $q \in (1, p_2^*)$ and* (1.9) *with $\mu > p$ and $T > 0$ hold, then Φ satisfies the* (PS) *condition.*

Proof. Let (u_j) be a (PS) sequence, i.e., $\Phi(u_j) = \mathrm{O}(1)$ and $\Phi'(u_j) = \mathrm{o}(1)$. By Proposition 2.2, it suffices to show that (u_j) is bounded. We have

$$\mu\, \Phi(u_j) - (\Phi'(u_j), u_j) = \int_\Omega \left(\frac{\mu}{p} - 1\right) |\Delta u_j|^p + H(x, u_j),$$

and H is bounded from below, so

$$(\mu - p)\, \|u_j\|^p \le \mathrm{o}(1)\, \|u_j\| + \mathrm{O}(1).$$

Since $\mu > p > 1$, it follows that (u_j) is bounded. □

Next we show that the sublevel sets of Φ at infinity are homotopic to the unit sphere $S = \big\{u \in W : \|u\| = 1\big\}$. For $u \in S$ and $t > 0$,

$$\begin{aligned} \Phi(tu) &= \frac{t^p}{p} - \int_\Omega F(x, tu) \\ &\le \frac{t^p}{p} - t^\mu \int_\Omega c(x)\, |u|^\mu + C\, |\Omega| \to -\infty \text{ as } t \to \infty \end{aligned} \tag{3.1}$$

by (1.10), where $|\Omega|$ denotes the volume of Ω, and

$$\begin{aligned} t\, \frac{d}{dt}\big(\Phi(tu)\big) &= t^p - \int_\Omega tu f(x, tu) \\ &= p\, \Phi(tu) - \int_\Omega (\mu - p)\, F(x, tu) + H(x, tu) \\ &\le p\, \big(\Phi(tu) - a_0\big) \end{aligned} \tag{3.2}$$

where $a_0 := \inf\, ((\mu - p)\, F + H)\, |\Omega|/p > -\infty$. So all critical values of Φ are greater than or equal to a_0. Since $F(x, 0) \equiv 0 \equiv H(x, 0)$, $a_0 \le 0$.

Lemma 3.4. *For each $a < a_0$, there is a C^1-map $T_a : S \to (0, \infty)$ such that*

$$\Phi^a := \big\{u \in W : \Phi(u) \le a\big\} = \big\{tu : u \in S,\, t \ge T_a(u)\big\} \simeq S.$$

Proof. For fixed $u \in S$, we have $\Phi(tu) \le a$ for all sufficiently large $t > 0$ by (3.1), and

$$\Phi(tu) \le a \implies \frac{d}{dt}\big(\Phi(tu)\big) < 0$$

by (3.2), so there is a unique $T_a(u) > 0$ such that

$$t < \text{ (resp. } =, >) \ T_a(u) \implies \Phi(tu) > \text{ (resp. } =, <) \ a.$$

The map T_a is C^1 by the implicit function theorem. Then $W \setminus \{0\}$, which is homotopic to S, radially deformation retracts to $\Phi^a = \big\{tu : u \in S,\, t \geq T_a(u)\big\}$ via

$$(W \setminus \{0\}) \times [0,1] \to W \setminus \{0\}\,,$$

$$(u,t) \mapsto \begin{cases} (1-t)\, u + t\, T_a(\widehat{u})\, \widehat{u}, & u \in (W \setminus \{0\}) \setminus \Phi^a \\ u, & u \in \Phi^a \end{cases}$$

where $\widehat{u} = u / \|u\|$. □

We are now ready to prove Theorem 1.2. The critical groups of Φ at zero are given by

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}), \quad q \geq 0$$

where H^* denotes cohomology. Suppose that Φ has no nontrivial critical point. Then Φ^0 is a deformation retract of W and $\Phi^0 \setminus \{0\}$ deformation retracts to Φ^a for any $a < 0$ by the second deformation lemma, so

$$C^q(\Phi, 0) \approx H^q(W, \Phi^a).$$

If $|a|$ is sufficiently large, Φ^a is contractible by Lemma 3.4, so

$$C^q(\Phi, 0) = 0 \quad \forall q.$$

On the other hand, since λ is not an eigenvalue of (1.4), it follows from Proposition 2.3 that $C^k(\Phi, 0) \neq 0$ for some $k \geq 0$. This contradiction shows that Φ has a nontrivial critical point.

Remark 3.5. Again it suffices to assume that λ is not an eigenvalue from the sequence (λ_k).

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