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"ON REVIENT TOUJOURS À SON PREMIER AMOUR": ON THE MATHEMATICAL WORK OF ESPEDITO DE PASCALE

JÜRGEN APPELL

University of Würzburg, Department of Mathematics,
Am Hubland 9, D-97074 Würzburg, Germany
E-mail: jurgen@dmuw.de

All'amico Espedito, con tanti auguri,
affinché la freschezza dell'adolescenza non lo abbandoni mai.

ABSTRACT. In the first part of this note, we briefly describe Espedito De Pascale's mathematical work during the last 30 years. In the second part we discuss some aspects of numerical ranges for nonlinear operators which have been a field of particular interest of De Pascale.

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INTRODUCTION

In this note we give a brief review, to the best of our knowledge, of the mathematical *œvre* of Espedito De Pascale whose retirement was celebrated during an International Conference in Amantea, Calabria, in June 2008.

In the paper [35], one of Espedito's first papers written 30 years ago, he studies numerical ranges for both linear and nonlinear operators. Since he has always been interested in this topic during the last 30 years, in the final part of this note we mention some directions in which current research on numerical ranges is moving. Incidentally, one of the last chapters of our monograph [15] (jointly with Alfonso Vignoli) on nonlinear spectral theory is entirely dedicated to the state-of-the-art of numerical ranges. So Espedito returned, after so many years, to the topic he was interested in right at the beginning of his research activity; this explains the somewhat cryptic French saying in the title.

In the following sections we try to summarize even superficially some of De Pascale's contributions to the following fields: Functional analysis and geometry of Banach spaces; numerical functional analysis; differential, integral, and integro-differential equations; measures of noncompactness and condensing maps; nonlinear

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superposition operators; fixed points and topological degree; nonlinear spectral theory. Of course, this classification is highly random, since most contributions could be attributed to more than one of these fields. Moreover, it goes without saying that we do not aim for a complete coverage, but just choose some typical features which show that Espedito's influential research contributions cover a wide range of topics and illustrate his broad view of mathematical analysis as a whole.

1. FUNCTIONAL ANALYSIS AND GEOMETRY OF BANACH SPACES

As far as we know, the note [36] on so-called quasi-bounded P-compact maps is De Pascale's first paper in classical functional analysis, and probably his very first paper. There are three similar articles whose contents is concerned with geometrical properties of Banach spaces. The first is the paper [49], where the authors introduce some "constant of quasi-normality" for cones and subspaces and apply this notion to operator equations in Banach spaces. The second paper [20] is concerned with a new class of Banach limits as multiplicative functionals on ℓ_∞ which are not shift-invariant (as classical Banach limits are) but have stronger geometrical properties. Finally, in the more recent important paper [28] the authors study Cohen's definition of injective envelope and give an explicit description of the algebraic structure hidden in the metric injective envelope of a Banach space, thus solving also an old conjecture raised by Isbell [68] in 1969.

Another interesting contribution is the joint paper [29] with Cianciaruso. It is well-known that, given a metric space X and a closed subset $A \subset X$, one may extend every nonexpansive map $f : A \to \mathbb{R}$ to a nonexpansive map $f : X \to \mathbb{R}$. Any target space Y which may replace here the real line $\mathbb{R}$ and still has the mentioned extension property is called hyperconvex. In [29] the authors study some "measure of non-hyperconvexity" and use it to characterize hyperconvex metric spaces.

2. NUMERICAL FUNCTIONAL ANALYSIS

In the last 15 years, De Pascale became more and more interested in application-oriented questions, including (but not restricted to) numerical analysis, iteration schemes, and error estimates for approximate solutions, the most prominent topic being the Newton-Kantorovich iteration

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \qquad (n = 1, 2, 3, \ldots)$$

for approximate solutions of the operator equation $F(x) = 0$ in a (usually, infinite dimensional) Banach space X. The crucial condition is here a Hölder condition (also called Vertgejm condition in the Russian literature) on the Fréchet derivative of F on a closed ball $B_r(x_0) = \{x \in X : \|x - x_0\| \leq r\}$, i.e.,

$$\|F'(x_1) - F'(x_2)\| \leq k\|x_1 - x_2\|^\alpha \quad (x_1, x_2 \in B_r(x_0)) \tag{1}$$

which may also be imposed in the weaker form

$$\|F'(x) - F'(x_0)\| \leq k_0\|x - x_0\|^\alpha \quad (x \in B_r(x_0)). \tag{2}$$

In a long series of papers with several coauthors [8-10,18,19,21,24,30-34,47, 60,61] De Pascale has analyzed the rather sophisticated "interaction" between the constants k in (1) and k_0 in (2), on the one hand, and the constants

$$a := \|F'(x_0)^{-1}F(x_0)\|, \quad b := \|F'(x_0)^{-1}\|, \tag{3}$$

on the other. It is clear that always $k_0 \leq k$, and examples show that the ratio k/k_0 may be arbitrarily large. Building on the constants a and b in (3), one may construct very subtle lower and upper bounds for solutions, reducing the "terra incognita" of non-existence step by step by sharpening these bounds. This applies, in particular, to approximate solutions of nonlinear integral equations of Uryson or Hammerstein type, see [9,10,18,60]. Results of this type, both of theoretical and practical interest (but too technical to be presented here in detail) have been further developed subsequently by De Pascale's PhD student Filomena Cianciaruso, see e.g. [27].

3. DIFFERENTIAL, INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

There is a vast literature on existence, uniqueness, regularity, boundedness, and stability results for solutions of boundary value problems for differential equations, both linear and nonlinear. In the papers [1] and [59] the authors prove exponential stability of Poincaré-Ljapunov type of solutions by extending Bohl's theorem on bounded solutions for perturbed systems. Lienard equations with delay are considered in the conference proceedings [44], while an essential enlargement of the existence interval in abstract Cauchy-Kovalevskaja type theorems is obtained in [46]. In this connection, we also mention the paper [45], where the authors study the structure of compact subsets in spaces of bounded continuous functions and subsequently apply their results to boundary value problems on unbounded intervals.

Existence theorems for nonlinear integral equations are often based on topological, variational, or monotonicity methods. When applying topological methods, such as fixed point theorems or degree theory (see Sections 4 and 6 below) to problems with lack of compactness, the celebrated fixed point principle for condensing maps by Darbo [38] and Sadovskij [75] are a very useful tool, see also [79]. A typical example is the paper [16], where this fixed point principle is applied to strongly singular nonlinear integral equations of the type

$$u(x) = \lambda \int_a^b \frac{k(x,y)}{x-y} f(y, u(y))\, dy \qquad (a \le x \le b) \tag{4}$$

in generalized Hölder spaces. Hammerstein integral equations in L_p spaces are considered in [22] in the case when the kernel function of the linear part is not symmetric, but "splits" into two parts which makes it possible to reduce everything to the Hilbert space case $p = 2$. Integral equations with Ljapunov-Schmidt kernel are studied in [50].

Another field of study of De Pascale is the theory of integro-differential equations of Barbashin type which in the simplest setting have the form

$$\frac{\partial x(t,s)}{\partial t} = c(s)x(t,s) + \int_a^b k(s,\sigma)x(t,\sigma)\, d\sigma + f(t,s). \tag{5}$$

Looking at the multiplication and integral part in (5) as operator functions with values in a Banach space, and identifying the scalar solution $(t,s) \mapsto x(t,s)$ with the abstract function $t \mapsto \mathbf{x}(t) := x(t,\cdot)$, one may rewrite (5) equivalently as (ordinary) differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{C}\mathbf{x} + \mathbf{K}\mathbf{x} + \mathbf{f}(t) \tag{6}$$

in this Banach space, where $\mathbf{C}$ and $\mathbf{K}$ are suitable linear operators generated by the multiplication and integral terms in the right-hand side of (5), and $\mathbf{f}(t) := f(t,\cdot)$. In the paper [7] the authors give some conditions for the existence of a Green function for (6) and its continuous dependence on a parameter, obtaining some analogue to the classical Bogoljubov-Krylov averaging principle.

4. MEASURES OF NONCOMPACTNESS AND CONDENSING MAPS

Loosely speaking, a measure of noncompactness shows how far a bounded set in a Banach space is from being precompact. Classical examples are the Hausdorff measure of noncompactness

$$\chi(M) = \inf\{\varepsilon > 0 : M \text{ admits a finite } \varepsilon\text{-net in } X\} \tag{7}$$

and the Kuratowski measure of noncompactness

$$\alpha(M) = \inf\{\delta > 0 : M \text{ admits a finite covering by sets of diameter } \le \delta\}. \tag{8}$$

Darbo's fixed point principle [38] refers to α-condensing operators $A : X \to X$, i.e., those which strictly decrease the measure of noncompactness (8) in the sense that

$$\alpha(A(M)) \le q\alpha(M) \tag{9}$$

for some $q < 1$ and each bounded subset M of X. This condition is some kind of topological counterpart to the usual metric condition

$$\|A(u) - A(v)\| \leq q\|u - v\| \tag{10}$$

which is crucial in the Banach-Caccioppoli contraction mapping principle. Subsequently, many other measures of noncompactness have been constructed, often in special Banach spaces, in order to apply a Darbo type result to a large variety of nonlinear problems with lack of compactness.

Sometimes such new measures of noncompactness are also of theoretical interest. Thus, in the papers [51,52] the authors study relations between the function (7) in the metric space $M(\Omega)$ of (classes of equivalent) measurable functions on Ω, on the one hand, and other measure-theoretic characteristics, on the other. Similar relations, but in the Lebesgue space $L_p(\Omega)$, are given in [4], with a particular emphasis on the cases $p = 1$ and $p = \infty$.

5. NONLINEAR SUPERPOSITION OPERATORS

Given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, the nonlinear superposition operator (or Nemytskij operator) generated by f is defined by

$$F(u)(x) = f(x, u(x)). \tag{11}$$

In spite of its simple form, this operator exhibits a quite surprising behaviour. For example, if the operator F acts between so-called ideal spaces of measurable functions (or Banach lattices, in different terminology) X and Y, its boundedness behaviour essentially depends on the space X, and its continuity behaviour on the space Y, while the analytical form of the generating function f is, surprisingly enough, less important. The role of the source space X in the boundedness properties of Nemytskij operators is studied in [5]. On the other hand, in [6] it is shown that the condition (9) is very often satisfied for the Nemytskij operator F in spaces of continuous or differentiable functions, while imposing a Lipschitz condition like (10) leads to a strong degeneracy for the generating function f.

There is also a vast literature on superposition operators generated by multivalued functions f. Some contributions in this direction may be found in [17], applications to nonlinear integral inclusions of Hammerstein type in [11]. The survey [12] contains a rather complete presentation of the theory and applications of multivalued Nemytskij operators, including many illuminating examples and surprising counterexamples.

6. FIXED POINTS AND TOPOLOGICAL DEGREE

The fixed point principles by Brouwer, Schauder, and Darbo-Sadovskij belong to the most powerful topological tools in proving existence results for nonlinear problems. They may be obtained as simple consequences of the Brouwer degree for continuous maps in $\mathbb{R}^N$, the Leray-Schauder degree for compact perturbations of the identity in Banach spaces, and the Nussbaum-Sadovskij degree for condensing perturbations of the identity in Banach spaces, respectively. For a detailed account of degree and fixed point theory we refer the reader to [79].

In one of his first papers [41] (see also [42]), De Pascale showed that the degree for certain multivalued noncompact maps depends, as the Leray-Schauder degree in the singlevalued case or Ma's degree in the multivalued case, only on the boundary values of the vector field. In the paper [53] the authors introduce and study a topological degree for weakly continuous maps in reflexive Banach spaces, while in [54] the same authors provide a comparison between the Browder-Petryshyn degree and the Canfora-Pacella degree for certain classes of monotone maps.

Concerning fixed point theory, in [3] and [16] it is shown that the strongly singular integral equation (4) may serve as a natural example for a problem to which the Darbo-Sadovskij fixed point principle applies, but neither Schauder's fixed point principle nor Banach's contraction mapping principle do. Positive fixed points are studied in [43], particular classes of fixed point theorems in [40] and [62]. More precisely, in the paper [40] Espedito and his son Luigi (who, by the way, meanwhile started a brillant mathematical career on his own) consider, for given $\alpha, \beta, \gamma > 0$, the nonstandard contraction-type condition

$$(12) \quad \|A(u)(t)-A(v)(t)\|_E \leq \beta\|u(t)-v(t)\|_E+\frac{\gamma}{t^\alpha}\int_0^t \|u(s)-v(s)\|_E\, ds \quad (0 < t \leq T)$$

in the space $C([0,T],E)$ of continuous functions with values in a Banach space E. They show that a previous theorem by Lou [72] may be obtained, even in a stronger version, by applying the theory of K-normed spaces developed mainly by Petr P. Zabrejko, and also that the term $\beta\|u(t) - v(t)\|_E$ in (12) cannot be replaced by the term $\beta\|u-v\|_{C([0,T],E)}$. In [62] De Pascale and Zabrejko present some new facts in the theory, methods, and applications of fixed point theorems in such spaces. Typically, the classical contraction condition (10) for some number $q < 1$ is replaced in this type of results by the far more general condition

$$]|A(u) - A(v)|[\leq Q\,]|u - v|[,$$

where $]|\cdot|[$ is some "generalized norm" taking its values in the nonnegative cone K of some Banach space, and Q is a bounded linear operator in this Banach space with spectral radius less than 1. In particular, in the paper [62] the authors also consider the non-standard contraction condition

$$\|A(u)(t) - A(v)(t)\|_E \leq Q(\|u - v\|_E)(t) \qquad (0 \leq t \leq t)$$

in the space $C([0,T], E)$, where E is a Banach space and Q is a "sufficiently well-behaved" nonnegative linear operator in $C([0,T], \mathbb{R})$.

De Pascale also contributed to abstract fixed point theory in metric spaces. For example, contraction type operators in so-called (o)-metric spaces are discussed in the joint work [48] with Paolamaria Pietramala and Giuseppe Marino (who together with Gennaro Infante organized the nice celebration event in Amantea), as well as in [26]. The papers [39] and [55-58] are related to the so-called Schauder conjecture which claims that the Schauder-Tychonov fixed point principle holds not only in Banach spaces or, more generally, locally convex spaces, but also in complete topological vector spaces which are not necessarily locally convex. Although the papers [39] and [58] give a deeper insight into this conjecture, as far as we know the full conjecture has not been proven yet.

7. NONLINEAR SPECTRAL THEORY

In view of the enormous importance of spectral theory for linear operators, in both mathematics and quantum mechanics, it is not surprising that various attempts have been made to define and study spectra also for nonlinear operators. A brief state-of-the-art of nonlinear spectral theory at the beginning of the new century is contained in the survey paper [13], while a very detailed, self-contained, and virtually complete overview of what has been done in nonlinear spectral theory between 1965 and 2005, say, may be found in the monograph [15] mentioned at the beginning. A new contribution is the paper [14], where the authors introduce a spectrum $\sigma(L, F)$ for pairs of operators, where L is a linear densely defined (but not necessarily bounded) Fredholm operator of index zero between two Banach spaces, and F is a continuous nonlinear operator between the same spaces. Such pairs occur frequently in the theory of differential equations with non-invertible linear part, e.g., in the presence of ω-periodic boundary conditions of the form

$$\begin{cases} \ddot{x}(t) = f(t, x(t), \dot{x}(t)), \\ x(0) = x(\omega), \ \dot{x}(0) = \dot{x}(\omega). \end{cases}$$

In this case F is a nonlinear Nemytskij operator of type (11), and L is a linear differential operator containing lower order terms and acting between suitable spaces of ω-periodic functions. In case $L = I$ (the identity operator), the spectrum $\sigma(L, F)$ defined in [14] reduces to the familiar Furi-Martelli-Vignoli spectrum from [65]. In fact, the definition of the spectrum $\sigma(F, L)$ in [14] imitates the construction of another spectrum defined by Feng and Webb in [64] which in the case $L = I$ reduces to Feng's spectrum from [63]. In the joint paper [67] with Webb, De Pascale's friend and former postgraduate student Gennaro Infante has written an interesting paper on nonlinear spectra which may be defined through finite dimensional approximations, see also [37].

An important part of any spectrum, linear or nonlinear, is the point spectrum which in the classical setting is defined by

$$\sigma_p(F) = \{\lambda \in \mathbb{K} : \ F(u) = \lambda u \text{ for some } u \neq 0\}; \tag{13}$$

here it is tacitly assumed that $F(0) = 0$, and the elements $\lambda \in \sigma_p(F)$ are usually called eigenvalues of F. However, in nonlinear spectral theory it is not reasonable to compare the nonlinear operator F in (13) with the identity; instead, it is useful to replace the identity with some "well-behaved" (but also nonlinear) operator J. This amounts to replacing (13) by the point spectrum

$$\sigma_p(F, J) = \{\lambda \in \mathbb{K} : \ F(u) = \lambda J(u) \text{ for some } u \neq 0\}, \tag{14}$$

i.e., to considering eigenvalues of *pairs of operators*. This has two advantages: Firstly, one may consider the operator F between *different* spaces X and Y, while (13) makes sense only in case $X = Y$; secondly, spectra of pairs of operators apply to much larger classes of problems than spectra of single operators, see Section 8 below.

In some problems it is reasonable to replace the "global" point spectrum (14) by the "asymptotic" point spectrum

$$\sigma_\psi^\infty(F, J) := \{\lambda \in \mathbb{K} : \ \liminf_{\|u\|\to\infty} \frac{\|F(u) - \lambda J(u)\|}{\psi(\|u\|)} = 0\}, \tag{15}$$

which in the special case $X = Y$, $J = I$ and $\psi(t) = t$ goes back to Furi, Martelli and Vignoli (see [65,66] or Chapter 6 in the book [15]), and was further developed for other choices of ψ by Weber [76]. This concept of eigenvalue is different from the classical one, but has interesting applications to nonlinear problems, where one is interested only in the asymptotic behaviour of operators, i.e., their values "on large spheres".

8. A CLOSER LOOK AT NUMERICAL RANGES

As announced at the beginning, we will now discuss some recent contributions (some of them due to or inspired by Espedito De Pascale) to the theory of numerical ranges for nonlinear operators. More precisely, we will concentrate on the application of so-called numerical ranges with gauge functions to nonlinear eigenvalue problems involving the p-Laplace operator.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz continuous boundary, and $1 < p < \infty$. It is well known that the *p-Laplace operator* on the domain Ω defined by

$$\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u)$$

acts from the Sobolev space $X = W_0^{1,p}(\Omega)$ to its dual, $X^* = W^{-1,p'}(\Omega)$, where $p' = p/(p-1)$. The *nonlinear eigenvalue problem* with Dirichlet boundary condition

$$\begin{cases} -\Delta_p u(x) = \mu |u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

for this operator consists in finding $\mu \in \mathbb{R}$ for which (16) has a nontrivial solution u and arises in many fields of applied mathematics and mechanics, see e.g. [70] and the references therein. Of course, in case $p = 2$ this problem just reduces to the *linear* eigenvalue problem $-\Delta u = \mu u$ for the Laplace operator $\Delta = \Delta_2$ which has been studied over and over in the last 150 years.

If we denote by J the differential operator defined by $-\Delta_p$ in the weak form, i.e.,

$$\langle J(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx \qquad (u, v \in W_0^{1,p}(\Omega)), \tag{17}$$

and by F the Nemytskij operator (11) generated by the nonlinearity $f(x, u) = |u|^{p-2}u$ on the right hand side of (16), also in weak form, i.e.,

$$\langle F(u), v\rangle = \int_\Omega |u(x)|^{p-2}u(x)v(x)\,dx \qquad (u, v \in W_0^{1,p}(\Omega)), \tag{18}$$

we obtain two operators acting simultaneously from X to its dual X^*. Here the norm we consider on X is

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p\,dx\right)^{1/p}$$

which is, by the classical Poincaré inequality, equivalent to the usual norm on X which also involves the L_p-norm of u.

In this way, the eigenvalue problem (16) may be rewritten, for $\mu \neq 0$ and $\lambda = 1/\mu$, equivalently as the operator equation

$$F(u) = \lambda J(u) \tag{19}$$

which is nothing else but the nonlinear eigenvalue problem for the operator pair (F, J) in the sense of (14). A survey of methods and results for such eigenvalue problems may be found in Chapter 10 of the monograph [15].

Now, since spectra are intimately related to numerical ranges, it seems reasonable to connect the study of equation (19) (or, equivalently, of the set (14)) to some numerical range for operator pairs (F, J) from a Banach space X to its dual X^*, like those given in (17) and (18). To this end, we make the following general hypotheses.

In what follows, X is a reflexive real Banach space, and X^* denotes its dual. Let $J : X \to X^*$ and $F : Y \to Y^*$ be two hemicontinuous operators such that $F(0) = J(0) = 0$ and J is *strictly monotone*, i.e.,

$$\langle J(u) - J(v), u - v \rangle > 0 \qquad (u, v \in X,\ u \neq v), \tag{20}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing between X^* and X.

Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such a function will be called a *gauge function* in what follows; typical examples are $\varphi(t) = t^{p-1}$ for $1 < p < \infty$ (polynomial growth), $\varphi(t) = e^t - 1$ (superpolynomial growth), or $\varphi(t) = \log(1+t)$ (subpolynomial growth). We say that an operator $A : X \to X^*$ is *φ-coercive* if

$$\lim_{\|u\| \to \infty} \frac{\langle A(u), u \rangle}{\varphi(\|u\|)} = \infty,$$

and *φ-monotone* if there is some $C_1 > 0$ such that

$$\langle A(u) - A(v), u - v \rangle \geq C_1 \varphi(\|u - v\|)\|u - v\| \quad (u, v \in X).$$

Moreover, we say that an operator $A : X \to X^*$ satisfies a *φ-Hölder condition* if there is some $C_2 > 0$ such that

$$\|A(u) - A(v)\| \leq C_2 \varphi(\|u - v\|) \quad (u, v \in X).$$

For example (see, e.g., [80]), the operator (17) satisfies for $p \geq 2$ in the space $X = W_0^{1,p}(\Omega)$ the estimate

$$\langle J(u) - J(v), u - v \rangle \geq C\|u - v\|^p \quad (u, v \in X) \tag{21}$$

which means that J is φ-monotone for $\varphi(t) = t^{p-1}$. Putting $v = 0$ in (21) yields

$$\frac{\langle J(u), u \rangle}{\|u\|^{p-1}} \geq C\|u\| \to \infty \qquad (\|u\| \to \infty) \tag{22}$$

which shows that J is, again for $\varphi(t) = t^{p-1}$, also φ-coercive. Clearly, any φ-monotone operator is also strictly monotone in the sense of (20), and hence injective. In particular, the operator (17) is invertible on its range, and (21) immediately implies that its inverse satisfies the *global Hölder condition*

$$\|J^{-1}(f) - J^{-1}(g)\| \leq \frac{1}{C^{1/(p-1)}}\|f - g\|^{1/(p-1)} \tag{23}$$

in case $p \geq 2$. For $1 < p < 2$, however, the situation is far more delicate. In fact, it was shown in [25] by means of a highly nontrivial computation that in this case (23) has to be replaced by

$$\|J^{-1}(f) - J^{-1}(g)\| \leq \frac{2^{2-p}}{C}(\|f\| + \|g\|)^{(2-p)(p-1)}\|f - g\|,$$

i.e., a *local Lipschitz condition* for J^{-1}, where the Lipschitz constant depends on the size of $\|f\|$ and $\|g\|$. This gives some kind of homeomorphism result like that of G.

Minty ([74], see also [80]) for classical monotone hemicontinuous coercive operators from a reflexive Banach space into its dual.

From standard estimates of scalar functions it follows that the operator (18) satisfies, again for $\varphi(t) = t^{p-1}$, a φ-Hölder condition of type

$$\|F(u) - F(v)\| \leq \begin{cases} \varphi(\|u-v\|) & \text{if } 1 < p \leq 2, \\ 2^{p-2}\varphi(\|u-v\|) & \text{if } 2 \leq p < \infty. \end{cases} \tag{24}$$

We consider now the numerical range [2]

$$W_0(F, J) = \left\{ \frac{\langle F(u), u\rangle}{\langle J(u), u\rangle} : u \in X, \ \langle J(u), u\rangle \neq 0 \right\}. \tag{25}$$

In the special case when X is a Hilbert space, $Y = X$, $J = I$, and F is continuous, the numerical range (25) reduces to the numerical range in the sense of Wenying Feng [63]. Moreover, Věra Burýšková [23] considered the numerical range (25) for continuous positively homogeneous operators (of the same degree) F and J, which is of course motivated by the eigenvalue problem (16).

The importance of the conditions (21), (22) and (24) is illustrated by the following proposition whose proof with additional remarks may be found in [2].

Proposition. *Suppose that $J : X \to X^*$ is φ-monotone, and $F : X \to X^*$ satisfies a φ-Hölder condition. Then the following holds.*

(a) *The numerical range* (25) *is bounded.*

(b) *The inclusion*

$$\sigma_p(F, J) \subseteq W_0(F, J) \tag{26}$$

holds, where $\sigma_p(F, J)$ denotes the classical point spectrum (14).

(c) *If, in addition, F is compact, and both J and F are odd, then the operator $\lambda J - F$ is surjective for any $\lambda \in \mathbb{R} \setminus (\overline{W_0(F, J)} \cup \{0\})$.*

(d) *If the map ψ defined by $\psi(t) = \varphi(t)/t$ is also a gauge function and*

$$\liminf_{\|u\|\to\infty} \frac{\|J(u)\|}{\varphi(\|u\|)} > 0, \tag{27}$$

then

$$\sigma_\psi^\infty(F, J) \subseteq \overline{W_0(F, J)}, \tag{28}$$

where $\sigma_\psi^\infty(F, J)$ denotes the asymptotic point spectrum (15).

We point out that the inclusions (26) and (28) are certain analogues to Zarantonello's localization theorems [77,78] for the spectrum of Lipschitz continuous nonlinear operators which has been studied in detail in [73]. We also remark that the assertion (c) contains a nonlinear Fredholm alternative: If λ does not belong to the numerical range (25) (and so, by (26), it does not belong to the point spectrum (14) either), then $\lambda J - F$ is not only injective, but also surjective. Thus, *uniqueness implies existence.*

To apply the mentioned Proposition, we put $X = W_0^{1,p}(\Omega)$, $X^* = W^{-1,p'}(\Omega)$, and $\varphi(t) = t^{p-1}$. As we have seen, the operator J defined by (17) is then φ-monotone and φ-coercive, and the Nemytskij operator F defined by (18) satisfies the φ-Hölder condition (24). Clearly, both J and F are odd and satisfy $J(0) = F(0) = 0$. Moreover, from Krasnosel'skij's classical theorem on Nemytskij operators between Lebesgue spaces [69] it follows immediately that F is both bounded and continuous from $L_p(\Omega)$ into $L_{p'}(\Omega)$, and so even compact from X to X^*, by standard embedding theorems between Sobolev spaces. Finally, from (22) and the Cauchy-Schwarz inequality it follows that

$$\liminf_{\|u\|\to\infty} \frac{\|J(u)\|}{\varphi(\|u\|)} = \liminf_{\|u\|\to\infty} \frac{\|J(u)\|}{\|u\|^{p-1}} \geq \liminf_{\|u\|\to\infty} \frac{\langle J(u), u\rangle}{\|u\|^p} \geq C,$$

which shows that (27) is satisfied as well. So the localization results (26) and (28) apply to the eigenvalue problem (16) and the numerical range (25).

There is another interesting point about nonstandard numerical ranges for the operator pair given by (17) and (18). The extension of numerical ranges from the Hilbert space case to the Banach space case is usually done by means of either semi-inner products (in the sense of Lumer, see again Chapter 11 of [15]) or, equivalently, duality maps. Recall that the *duality map* $\mathcal{D} : X \to X^*$ of a Banach space is defined by

$$\mathcal{D}(u) = \{\ell_u \in X^* : \langle u, \ell_u\rangle = \|u\|^2,\ \|\ell_u\| = \|u\|\}. \tag{29}$$

More generally, given a gauge function φ as before, one may define a *duality map with gauge function* $\mathcal{D}_\varphi : X \to X^*$ by

$$\mathcal{D}_\varphi(u) = \{\ell_u^\varphi \in X^* : \langle u, \ell_u^\varphi\rangle = \varphi(\|u\|)\|u\|,\ \|\ell_u^\varphi\| = \varphi(\|u\|)\}. \tag{30}$$

As far as we know, the generalized duality map (30) was considered first by J. L. Lions [71] and has useful applications in the theory of partial differential equations. In general, the duality maps (29) and (30) may be multivalued; in special spaces, however, they are singlevalued. For example, the map (29) is singlevalued if and only if the underlying space X is smooth (i.e., its norm is Gâteaux differentiable on $X \setminus \{0\}$).

Obviously, (29) is a special case of (30) for $\varphi(t) = t$, so there is some interest in asking for the explicit form of (30) for other choices of φ. In particular, if we take $\varphi(t) = t^{p-1}$ as in our preceding discussion, the answer comes as a very pleasant surprise: In this case the duality map (30) in $X = W_0^{1,p}(\Omega)$ is precisely the operator (17), while in $X = L_p(\Omega)$ it is precisely the operator (18). This gives yet another approach to the p-Laplace operator through numerical ranges and nonlinear spectra; for details we refer to [2].

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