

## CORES FOR SECOND-ORDER DIFFERENTIAL OPERATORS ON REAL INTERVALS

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**ABSTRACT.** We investigate several general conditions in order to determine some cores for generators of strongly continuous positive semigroups of the form  $Au := \alpha u''$  on weighted spaces of continuous functions on an arbitrary noncompact real interval. As an application we consider a degenerate differential operator of the above mentioned form on the interval  $[0, +\infty[$  and we establish an approximation formula for the corresponding positive semigroup in terms of iterates of an integral modification of Szász-Mirakjan operators.

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### 1. INTRODUCTION

For a given generator  $(A, D(A))$  of a strongly continuous semigroup on a Banach space  $E$  several subspaces of  $D(A)$  can be considered which, to various extent, are useful to describe properties of  $A$  or of the semigroup itself. Among them the cores have an important rôle because, on one hand, they determine the operator  $A$  and, on the other, very often they allow more easily to compute it, while generally the action of  $A$  on  $D(A)$  is more difficult to obtain (see, e.g., [7, Theorem 6.2.6], [25, Chapter II, Section 2.8, and Chapter III, Corollary 5.8]).

Moreover, because of Trotter approximation theorem ([34, Theorem 5.2]; see also [30, Chapter 3, Theorem 6.7]), cores are also strongly involved when investigating the possibility to approximate a semigroup in terms of simpler semigroups or by iterates of bounded linear operators.

This last aspect has been investigated in many respects in the last two decades and the relevant results have successfully led to several applications concerning the approximation and the qualitative analysis (especially, spatial regularity) of the solutions of a large class of evolution equations in spaces of real valued continuous functions defined on compact intervals (see, e.g., [3], [10]-[12], [17], [19], [20], [31]-[33]), on the interval  $[0, +\infty[$  (see, e.g., [6], [9], [13], [14], [18], [23], [26], [27]), on the whole real line ([15], [16], [29]) and on convex compact subsets of  $\mathbb{R}^n$ ,  $n \geq 1$  (see, e.g., [1], [2], [7, Chapter 6], [8]).

In all these papers the determination of cores has a crucial rôle. However, developing these researches it appeared to be more and more pressing the need to have general results to help to find cores when dealing with general differential operators.

In this paper we try to give some contributions to the above mentioned problem by starting with a simple situation involving a second-order differential operator on a real interval. More precisely, we provide some (rather) general and simple conditions under which suitable linear spaces of smooth functions are cores for a large class of (possibly degenerate) elliptic differential operators of the form  $Au := \alpha u''$ , in the setting of weighted spaces of continuous functions on a noncompact real interval.

We shall assume that the coefficient  $\alpha \in C(J)$  is strictly positive on  $\overset{\circ}{J}$ , that  $0 < \alpha(x) \leq \frac{(x-r_1)(r_2-x)}{2}$  ( $x \in \overset{\circ}{J}$ ) if  $J$  is bounded (here  $r_1$  and  $r_2$  are the endpoints of  $J$ ) and that  $\alpha$  has at most a quadratic growth at infinity whenever  $J$  is unbounded.

The domain of the operator  $A$  is a linear subspace of the weighted space  $C_0^w(J) := \{f \in C(J) \mid wf \in C_0(J)\}$ , where  $w$  is a bounded continuous weight on  $J$ , and it incorporates a kind of weighted Wentzell conditions at the endpoints. However, the construction of cores will be possible just checking  $\alpha$  at the endpoints  $r_i$ ,  $i = 1, 2$ , without no further requirements on the weight  $w$ .

Our results would be compared with the ones of [12, Theorem 3.4 and final note added in proof], [16, Section 2] and [22, Proposition 3.1] where similar problems are treated in other different settings.

As an application we consider a degenerate differential operator of the above mentioned form on the interval  $[0, +\infty[$  and we establish an approximation formula for the corresponding positive semigroup in terms of iterates of an integral modification of Szász-Mirakjan operators introduced in [28].

## 2. NOTATIONS AND PRELIMINARIES

Let  $J$  be an arbitrary noncompact real interval and set  $r_1 := \inf J \in \mathbb{R} \cup \{-\infty\}$  and  $r_2 := \sup J \in \mathbb{R} \cup \{+\infty\}$ . Throughout this paper the symbol  $C(J)$  (resp.,  $C_b(J)$ ) will stand for the space of all real valued continuous (resp., continuous and bounded)

functions on  $J$ . The space  $C_b(J)$  endowed with the natural (pointwise) order and the sup-norm  $\|\cdot\|_\infty$  is a Banach lattice.

We shall also consider the spaces

$$C_0(J) := \{f \in C(J) \mid \lim_{x \rightarrow r_i} f(x) = 0 \text{ whenever } r_i \notin J, i = 1, 2\}$$

and

$$C_*(J) := \{f \in C(J) \mid \lim_{x \rightarrow r_i} f(x) \in \mathbb{R} \text{ whenever } r_i \notin J, i = 1, 2\}$$

which are closed subspaces of  $C_b(J)$ .

Note that a function  $f \in C(J)$  belongs to  $C_0(J)$  if and only if for every  $\varepsilon > 0$  there exists a compact subset  $K$  of  $J$  such that  $|f(x)| \leq \varepsilon$  for every  $x \in J \setminus K$ .

If  $w$  is a weight on  $J$ , i.e.,  $w \in C_b(J)$  and  $w(x) > 0$  for all  $x \in J$ , we shall denote by  $C_b^w(J)$  (resp.,  $C_0^w(J)$ ) the Banach lattice of all functions  $f \in C(J)$  such that  $wf \in C_b(J)$  (resp.,  $wf \in C_0(J)$ ). The space  $C_b^w(J)$  will be endowed with the natural order and the weighted norm  $\|\cdot\|_w$  defined by  $\|f\|_w := \|wf\|_\infty$  ( $f \in C_b^w(J)$ ).

Observe that  $C_b(J) \subset C_b^w(J)$  and  $\|\cdot\|_w \leq \|w\|_\infty \|\cdot\|_\infty$  on  $C_b(J)$ . In particular, if  $w \in C_0(J)$ , then  $C_b(J) \subset C_0^w(J)$ . Moreover, the space  $C_0(J)$  is dense in  $C_0^w(J)$  and, if  $w \in C_0(J)$ , then  $C_*(J)$  is dense in  $C_0^w(J)$  as well.

Given a linear operator  $A : D(A) \rightarrow E$  acting on a linear subspace  $D(A)$  of a Banach space  $(E, \|\cdot\|)$ , a linear subspace  $D$  of  $D(A)$  is called a *core* for  $(A, D(A))$  if it is dense in  $D(A)$  for the graph norm

$$\|u\|_A := \|u\| + \|Au\| \quad (u \in D(A)),$$

i.e., for every  $u \in D(A)$  there exists a sequence  $(u_n)_{n \geq 1}$  in  $D$  such that  $u_n \rightarrow u$  and  $Au_n \rightarrow Au$  in  $E$ .

If  $(A, D(A))$  is closed, then a linear subspace  $D$  of  $D(A)$  is a core for  $(A, D(A))$  if and only if the restriction  $A|_D$  of  $A$  to  $D$  is closable and its closure  $\overline{A|_D}$  coincides with  $A$ .

If in addition the resolvent set  $\rho(A)$  of  $A$  is non empty, then  $D$  is a core for  $(A, D(A))$  if and only if  $(\lambda I - A)D$  is dense in  $E$  for one/all  $\lambda \in \rho(A)$  (here  $I$  stands for the identity operator).

The next result shows the usefulness of cores for the approximation of semigroups. Although it is a simple consequence of Trotter's theorem (see [34, Theorem 5.2]; see also [30, Chapter 3, Theorem 6.7]), we present here a proof for the reader's convenience.

In the sequel, given a linear operator  $L$  on  $E$  and  $m \geq 1$ , the symbol  $L^m$  denotes the  $m$ -th iterate of  $L$ , i.e.,

$$L^m = \begin{cases} L & \text{if } m = 1, \\ L^{m-1} \circ L & \text{if } m \geq 2. \end{cases}$$

**Theorem 2.1.** *Let  $(A, D(A))$  be the generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  and suppose that there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|T(t)\| \leq Me^{\omega t}$  ( $t \geq 0$ ). Moreover, let  $D$  be a core for  $(A, D(A))$  and  $(L_n)_{n \geq 1}$  a sequence of bounded linear operators on  $E$  such that*

- (i)  $\|L_n^k\| \leq Me^{\omega \rho_n k}$  for every  $n \geq 1, k \geq 1$ ,
- (ii)  $\lim_{n \rightarrow \infty} \frac{L_n u - u}{\rho_n} = Au$  for every  $u \in D$ ,

where  $(\rho_n)_{n \geq 1}$  is a null sequence of positive real numbers.

If  $t \geq 0$  and if  $(k(n))_{n \geq 1}$  is a sequence of positive integers such that  $k(n)\rho_n \rightarrow t$  then for every  $f \in E$

$$T(t)f = \lim_{n \rightarrow \infty} L_n^{k(n)} f.$$

*Proof.* Let  $B : D(B) \subset E \rightarrow E$  be the linear operator defined by

$$Bu := \lim_{n \rightarrow \infty} \frac{L_n u - u}{\rho_n},$$

for every  $u \in D(B) := \left\{ u \in E \mid \text{there exists } \lim_{n \rightarrow \infty} \frac{L_n u - u}{\rho_n} \in E \right\}$ .

By (ii),  $D \subset D(B)$  and  $D$  is dense in  $E$  because  $D(A)$  is dense in  $E$ . Therefore  $D(B)$  is dense in  $E$  too. Moreover, since  $(A, D(A))$  is a generator, there exists  $\lambda > \omega$  such that  $\lambda I - A$  is invertible, so that  $(\lambda I - B)(D) = (\lambda I - A)(D)$  is dense in  $E$ . Then, from Trotter's theorem (see [34, Theorem 5.2]; see also [30, Chapter 3, Theorem 6.7]) it follows that the operator  $(B, D(B))$  is closable and its closure  $(\overline{B}, D(\overline{B}))$  generates a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  on  $E$  such that

$$(1) \|S(t)\| \leq Me^{\omega t} \text{ for every } t \geq 0$$

and

$$(2) S(t)f = \lim_{n \rightarrow \infty} L_n^{k(n)} f,$$

for every  $f \in E, t \geq 0$  and for every sequence  $(k(n))_{n \geq 1}$  of positive integers such that  $k(n)\rho_n \rightarrow t$  as  $n \rightarrow \infty$ .

The result will be proved once we show that  $S(t) = T(t)$  ( $t \geq 0$ ). To this end it suffices to prove that  $(\overline{B}, D(\overline{B})) = (A, D(A))$ .

First observe that, for  $\lambda > \omega$ , the operator  $\lambda I - \overline{B}$  is invertible as well and  $(\lambda I - \overline{B})(D) = (\lambda I - B)(D) = (\lambda I - A)(D)$  is dense in  $E$ ; hence  $D$  is a core for  $(\overline{B}, D(\overline{B}))$ . Therefore, according to the previous remark,  $A = \overline{A|_D} = \overline{B|_D} = \overline{B}$ .  $\square$

**Corollary 2.2.** *Let  $(A, D(A))$  be the generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $E$  and suppose that  $\|T(t)\| \leq Me^{\omega t}$  for some  $M \geq 1$  and  $\omega \in \mathbb{R}$  and for all  $t \geq 0$ . Let  $D$  be a core for  $(A, D(A))$  and consider a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  on  $E$  satisfying*

- (i)  $\|S(\rho_n)\| \leq Me^{\omega \rho_n}$  ( $n \geq 1$ ),

$$(ii) \lim_{n \rightarrow \infty} \frac{S(\rho_n)(u) - u}{\rho_n} = Au \text{ for every } u \in D,$$

where  $(\rho_n)_{n \geq 1}$  is a null sequence of positive real numbers.

Then  $S(t) = T(t)$  for every  $t \geq 0$ .

*Proof.* It suffices to apply Theorem 2.1 to the sequence  $L_n = S(\rho_n)$  ( $n \geq 1$ ). □

### 3. SECOND-ORDER DIFFERENTIAL OPERATORS ON REAL INTERVALS

As in the previous section let  $J$  be an arbitrary noncompact real interval and set  $r_1 := \inf J \in \mathbb{R} \cup \{-\infty\}$  and  $r_2 := \sup J \in \mathbb{R} \cup \{+\infty\}$ .

Consider  $\alpha \in C(\overset{\circ}{J})$  and assume that

$$\alpha(x) > 0 \text{ for every } x \in \overset{\circ}{J} \tag{3.1}$$

and

$$\alpha(x) = O(x^2) \text{ as } x \rightarrow r_i \text{ whenever } r_i \in \{-\infty, +\infty\}, i = 1, 2. \tag{3.2}$$

Furthermore consider a weight  $w$  on  $J$  and assume that it is twice differentiable on  $\overset{\circ}{J}$  and that

$$\omega := \sup_{\overset{\circ}{J}} \frac{|\alpha(2(w')^2 - ww'')|}{w^2} < +\infty. \tag{3.3}$$

For every  $u \in C^2(\overset{\circ}{J})$  and  $x \in \overset{\circ}{J}$ , set

$$Au(x) := \alpha(x)u''(x).$$

Note that, if  $u \in C^2(\overset{\circ}{J})$  and if  $r_i \in J$  for some  $i = 1, 2$ , then

$$\lim_{x \rightarrow r_i} w(x)\alpha(x)u''(x) = 0 \text{ if and only if } \lim_{x \rightarrow r_i} \alpha(x)u''(x) = 0.$$

Therefore, considering the linear subspace

$$D_w(A) := \left\{ u \in C_0^w(J) \cap C^2(\overset{\circ}{J}) \mid \lim_{x \rightarrow r_i} w(x)\alpha(x)u''(x) = 0 \text{ for every } i = 1, 2 \right\},$$

for  $u \in D_w(A)$  the function  $Au$  can be continuously extended to the whole  $J$  and its extension, which we continue to denote by  $Au$ , belongs to  $C_0^w(J)$ .

From now on the symbol  $(A, D_w(A))$  stands for the extended operator  $A : D_w(A) \rightarrow C_0^w(J)$ . Thus, if  $u \in D_w(A)$ ,  $Au = \alpha u''$  on  $\overset{\circ}{J}$  and  $Au = 0$  on  $J \setminus \overset{\circ}{J}$  whenever  $J \setminus \overset{\circ}{J} \neq \emptyset$ .

Analogously we may consider a similar extension  $\tilde{A} : D(\tilde{A}) \rightarrow C_*(J)$  of  $A$  where

$$D(\tilde{A}) := \left\{ u \in C_*(J) \cap C^2(\overset{\circ}{J}) \mid \lim_{x \rightarrow r_i} \alpha(x)u''(x) = 0 \text{ for every } i = 1, 2 \right\}.$$

Again, if  $u \in D(\tilde{A})$ ,  $\tilde{A}u = \alpha u''$  on  $\overset{\circ}{J}$  and, if  $J \setminus \overset{\circ}{J} \neq \emptyset$ ,  $\tilde{A}u = 0$  on  $J \setminus \overset{\circ}{J}$ .

Clearly,  $D(\tilde{A}) \cap C_0(J) \subset D_w(A)$  and, if  $w \in C_0(J)$ , then  $D(\tilde{A}) \subset D_w(A)$ .

Note that, if  $r_i \in \{-\infty, +\infty\}$  for some  $i = 1, 2$ , then, after choosing an arbitrary  $x_0 \in \overset{\circ}{J}$ , from (3.2) it follows that

$$\int_{r_i}^{x_0} \int_{r_i}^x \frac{1}{\alpha(t)} dt dx = +\infty.$$

Therefore, from [6, Theorem 3.2], the following result can be easily deduced, the detailed verification being left to the reader.

**Theorem 3.1.** *Under assumptions (3.1), (3.2) and (3.3), the operator  $(A, D_w(A))$  generates a positive  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $C_0^w(J)$  such that*

$$\|T(t)\| \leq \exp(\omega t) \quad \text{for every } t \geq 0.$$

*Moreover, the restriction of  $(T(t))_{t \geq 0}$  to  $C_0(J)$  is a Feller semigroup on  $C_0(J)$  whose generator is  $(\tilde{A}, D(\tilde{A}) \cap C_0(J))$ .*

*Finally, if  $w \in C_0(J)$ , the restriction of  $(T(t))_{t \geq 0}$  to  $C_*(J)$  is a Feller semigroup whose generator is  $(\tilde{A}, D(\tilde{A}))$ .*

We recall here that a Feller semigroup on  $C_0(J)$  or on  $C_*(J)$  is a  $C_0$ -semigroup of positive linear contractions on  $C_0(J)$  or on  $C_*(J)$ .

We also point out that, according to [6, Theorem 2.6], the above semigroup is also the transition semigroup of a suitable right-continuous Markov process.

#### 4. CORES FOR SECOND-ORDER DIFFERENTIAL OPERATORS

This section contains the main results of the paper and it is devoted to the investigation of cores for the differential operators  $(A, D_w(A))$ . Without no further mention we shall assume that (3.1), (3.2) and (3.3) hold true.

We begin by proving the following preliminary result.

**Proposition 4.1.** *Consider a linear subspace  $D$  of  $C_*(J) \cap C^2(\overset{\circ}{J})$ . Then the following statements hold true:*

- (1) *If  $D$  is a core for  $(\tilde{A}, D(\tilde{A}) \cap C_0(J))$ , then  $D$  is a core of  $(A, D_w(A))$ ;*
- (2) *If  $w \in C_0(J)$  and  $D$  is a core for  $(\tilde{A}, D(\tilde{A}))$ , then  $D$  is a core of  $(A, D_w(A))$ .*

*Proof.* Let  $\lambda > \omega$  such that both  $\lambda I - \tilde{A} : D(\tilde{A}) \cap C_0(J) \rightarrow C_0(J)$  (resp.,  $\lambda I - \tilde{A} : D(\tilde{A}) \rightarrow C_*(J)$ ) and  $\lambda I - A : D_w(A) \rightarrow C_0^w(J)$  are invertible.

If  $D$  is a core for  $(\tilde{A}, D(\tilde{A}) \cap C_0(J))$  (resp.,  $(\tilde{A}, D(\tilde{A}))$ ), then  $(\lambda I - \tilde{A})(D) = (\lambda I - A)(D)$  is dense in  $C_0(J)$  (resp.,  $C_*(J)$ ).

This yields that  $(\lambda I - A)(D)$  is dense in  $C_0^w(J)$  and so  $D$  is a core for  $(A, D_w(A))$  as well. □

According to the previous result, we may restrict our investigation simply on the existence of cores for  $(\tilde{A}, D(\tilde{A}) \cap C_0(J))$  and  $(\tilde{A}, D(\tilde{A}))$ .

When  $\alpha$  is bounded, some results to this respect have been already stated in [22, Proposition 3.1]. Next we shall address this question without no boundedness assumptions.

To this aim it is useful to look at the behaviour at the endpoints of the first derivative of the function  $u \in D(\tilde{A})$ . Arguing as in Propositions 2.8 and 2.9 in [16], we obtain the following lemmas.

**Lemma 4.2.** *Let  $\delta \in \overset{\circ}{J}$ ,  $h \in C([\delta, r_2])$  and let  $\varphi : [\delta, r_2[ \rightarrow \mathbb{R}$  be a differentiable function such that  $\varphi(x) \neq 0$  for every  $x \in [\delta, r_2[$ . Assume that one of the following conditions (a), (b) or (c) is satisfied:*

- (a) i)  $\frac{1}{h} \in L^1([\delta, r_2])$  or  
 i')  $(\varphi'h)(x) \neq 0$  for  $x \in [\delta, r_2[$  and  $\frac{\varphi^2}{\varphi'h} = O(1)$  as  $x \rightarrow r_2$ ,  
 and  
 ii)  $\lim_{x \rightarrow r_2} \varphi(x) = 0$ ;
- (b) i)  $\frac{1}{h} \in L^1([\delta, r_2])$ ,  
 ii)  $\frac{1}{\varphi} \notin L^1([\delta, r_2])$  and  $\lim_{x \rightarrow r_2} \varphi(x) \in \mathbb{R} \setminus \{0\}$ ;
- (c) if  $r_2 = +\infty$ ,  
 i)  $\frac{1}{h} \in L^1([\delta, +\infty])$ ,  
 ii)  $\lim_{x \rightarrow +\infty} \varphi(x) \in \{-\infty, +\infty\}$ ,  
 iii)  $(\varphi'h)(x) \neq 0$  for  $x \in [\delta, +\infty[$  and  $\frac{\varphi^2}{\varphi'h} = O(1)$  as  $x \rightarrow +\infty$ .

Then

$$\lim_{x \rightarrow r_2} \varphi(x)u'(x) = 0,$$

for every  $u \in C^2(\overset{\circ}{J})$  such that  $\lim_{x \rightarrow r_2} u(x) \in \mathbb{R}$  and  $\lim_{x \rightarrow r_2} h(x)u''(x) = 0$ .

*Proof.* The proof is the same as that one of Propositions 2.8 and 2.9 in [16], by replacing there  $W$  with 1 and  $\alpha$  with  $h$ . We leave out the details for the sake of brevity. □

**Lemma 4.3.** *Let  $\delta \in \overset{\circ}{J}$ ,  $h \in C^2([\delta, r_2])$  such that  $h(x) \neq 0$  for every  $x \in [\delta, r_2[$  and  $\varphi : [\delta, r_2[ \rightarrow \mathbb{R}$ . Assume that:*

- (i)  $h''(x) = O(1)$  as  $x \rightarrow r_2$ ;
- (ii)  $\lim_{x \rightarrow r_2} \frac{\varphi(x)}{h(x)} = 0$  if  $r_2 \in \mathbb{R}$  or  $\frac{\varphi(x)}{h(x)} = O\left(\frac{1}{x}\right)$  as  $x \rightarrow +\infty$  if  $r_2 = +\infty$ .

If  $u \in C^2(\overset{\circ}{J})$  and if  $\lim_{x \rightarrow r_2} u(x) = \lim_{x \rightarrow r_2} h(x)u''(x) = 0$ , then

$$\lim_{x \rightarrow r_2} \varphi(x)u'(x) = 0.$$

*Proof.* Let  $u \in C^2(\overset{\circ}{J})$  such that  $\lim_{x \rightarrow r_2} u(x) = \lim_{x \rightarrow r_2} h(x)u''(x) = 0$  and fix  $\varepsilon > 0$ ; observe that, by (i),  $h'(x) = O(x)$  as  $x \rightarrow r_2$ , so there exist  $\delta_1 \in ]\delta, r_2[$  and  $M > 0$  such that, for  $x \in [\delta_1, r_2[$ ,

$$|u(x)| \leq \varepsilon, \quad |(hu'')(x)| \leq \varepsilon, \quad |h'(x)| \leq M|x|.$$

Since  $h''$  is bounded on  $[\delta, r_2[$  and since

$$\begin{aligned} \int_{\delta_1}^x (hu'')(s) ds &= h(x)u'(x) - h(\delta_1)u'(\delta_1) - h'(x)u(x) + h'(\delta_1)u(\delta_1) \\ &\quad + \int_{\delta_1}^x h''(s)u(s) ds, \end{aligned}$$

we get

$$\begin{aligned} |\varphi(x)u'(x)| &\leq \left| \frac{\varphi(x)}{h(x)} \right| \left[ \int_{\delta_1}^x |(hu'')(s)| ds + |h'(x)||u(x)| \right. \\ &\quad \left. + |h'(\delta_1)u(\delta_1) - h(\delta_1)u'(\delta_1)| + \|h''\|_\infty \int_{\delta_1}^x |u(s)| ds \right] \\ &\leq \left| \frac{\varphi(x)}{h(x)} \right| \varepsilon [(1 + \|h''\|_\infty)|x - \delta_1| + M|x|] \\ &\quad + \left| \frac{\varphi(x)}{h(x)} \right| |h'(\delta_1)u(\delta_1) - h(\delta_1)u'(\delta_1)|. \end{aligned}$$

The result now easily follows on account of condition (ii). □

Similar results can be stated for the endpoint  $r_1$ .

Next we discuss separately the cases where  $J$  is bounded and where  $J$  is unbounded. For the case  $J = \mathbb{R}$  we refer the reader to [16, Section 4].

#### 4.1. BOUNDED INTERVALS.

Let  $r_1, r_2 \in \mathbb{R}$  and assume that

$$0 < \alpha(x) \leq \frac{(x - r_1)(r_2 - x)}{2} \quad \text{for every } x \in \overset{\circ}{J}. \tag{4.1}$$

Setting

$$\lambda(x) := \frac{2\alpha(x)}{(x - r_1)(r_2 - x)} \quad (x \in \overset{\circ}{J}) \tag{4.2}$$

we get  $\lambda \in C(\overset{\circ}{J})$ ,  $0 < \lambda(x) \leq 1$  for every  $x \in \overset{\circ}{J}$  and

$$\alpha(x) = \frac{(x - r_1)(r_2 - x)}{2} \lambda(x) \quad \text{for every } x \in \overset{\circ}{J}.$$

For every  $u \in C^2(J)$  and  $x \in \overset{\circ}{J}$ , set

$$Bu(x) := \frac{(x - r_1)(r_2 - x)}{2} u''(x).$$

Then the operator  $B$  can be extended to a linear operator  $\tilde{B} : D(\tilde{B}) \rightarrow C_*(J)$  where

$$D(\tilde{B}) := \left\{ u \in C_*(J) \cap C^2(\overset{\circ}{J}) \mid \lim_{x \rightarrow r_i} (x - r_i)u''(x) = 0 \text{ for every } i = 1, 2 \right\}.$$

If  $f \in D(\tilde{B})$ ,  $\tilde{B}u = Bu$  on  $\overset{\circ}{J}$  and, if  $J \setminus \overset{\circ}{J} \neq \emptyset$ ,  $\tilde{B}u = 0$  on  $J \setminus \overset{\circ}{J}$ .

Clearly  $D(\tilde{B}) \subset D(\tilde{A})$ ; moreover,  $(\tilde{B}, D(\tilde{B}))$  generates a Feller semigroup on  $C_*(J)$  (see [24, Theorem 2]).

**Proposition 4.4.** *The subspace  $D$  of  $D(\tilde{B})$ , defined by*

$$D := \left\{ u \in C_*(J) \cap C^2(\overset{\circ}{J}) \mid u'' \in C_*(J) \right\}, \tag{4.3}$$

*is a core for the operator  $(\tilde{B}, D(\tilde{B}))$ .*

*Proof.* Let  $u \in D(\tilde{B})$  and  $\varepsilon > 0$ . The function  $u$  is uniformly continuous and for every  $i = 1, 2$ , by applying Lemma 4.2-(a) to  $\varphi_i(x) = h_i(x) := r_i - x$  ( $x \in \overset{\circ}{J}$ ), it follows that  $\lim_{x \rightarrow r_i} (r_i - x)u'(x) = 0$ .

Then there exists  $\delta_i \in \overset{\circ}{J}$ ,  $|r_i - \delta_i| \leq 1$ , such that, for every  $x, y \in I(\delta_i, r_i)$ ,

$$|u(x) - u(y)| \leq \frac{\varepsilon}{3}, \quad |u'(x)(r_i - x)| \leq \frac{\varepsilon}{3}, \quad |u''(x)(r_i - x)| \leq \frac{\varepsilon}{3},$$

where  $I(\delta_i, r_i) = ]r_1, \delta_1]$  if  $i = 1$  and  $I(\delta_i, r_i) = [\delta_2, r_2[$  if  $i = 2$ .

Now, fixed  $x_1 \in ]r_1, \delta_1]$  and  $x_2 \in [\delta_2, r_2[$ , define the function  $v \in D$  as

$$v(x) := \begin{cases} u(x_1) + u'(x_1)(x - x_1) + \frac{u''(x_1)}{2}(x - x_1)^2 & \text{if } x \in ]r_1, x_1], \\ u(x) & \text{if } x \in ]x_1, x_2[, \\ u(x_2) + u'(x_2)(x - x_2) + \frac{u''(x_2)}{2}(x - x_2)^2 & \text{if } x \in [x_2, r_2[. \end{cases}$$

Then we have that  $\|u - v\|_{\tilde{B}} \leq \varepsilon$ .

Indeed, for  $x \in ]x_1, x_2[$ ,  $|u(x) - v(x)| = |\tilde{B}u(x) - \tilde{B}v(x)| = 0$ .

Otherwise, for  $x \in I(x_i, r_i)$  and  $i = 1, 2$ ,

$$\begin{aligned} |u(x) - v(x)| &\leq |u(x) - u(x_i)| + |u'(x_i)||x - x_i| + \frac{|u''(x_i)|}{2}|x - x_i|^2 \\ &\leq |u(x) - u(x_i)| + |u'(x_i)(r_i - x_i)| + |u''(x_i)(r_i - x_i)| \leq \varepsilon \end{aligned}$$

and

$$\begin{aligned} |\tilde{B}u(x) - \tilde{B}v(x)| &= \left| \frac{(x - r_1)(r_2 - x)}{2}u''(x) - \frac{(x - r_1)(r_2 - x)}{2}u''(x_i) \right| \\ &\leq \left| \frac{(x - r_1)(r_2 - x)}{2}u''(x) \right| + \left| \frac{(x_i - r_1)(r_2 - x_i)}{2}u''(x_i) \right| \leq \frac{(r_2 - r_1)\varepsilon}{3}. \square \end{aligned}$$

**Proposition 4.5.** *The operators  $(\tilde{B}, D(\tilde{B}) \cap C_0(\overset{\circ}{J}))$  and  $(\tilde{A}, D(\tilde{A}) \cap C_0(\overset{\circ}{J}))$  generate some Feller semigroups on  $C_0(\overset{\circ}{J})$ . Moreover,  $D \cap C_0(\overset{\circ}{J})$  is a core for  $(\tilde{B}, D(\tilde{B}) \cap C_0(\overset{\circ}{J}))$ .*

*Proof.* The first part of the assertion follows from Proposition 2.2 in [6].

In order to prove the last part, fix  $u \in D(\tilde{B}) \cap C_0(\overset{\circ}{J})$  and  $\varepsilon > 0$ . Since  $D$  is a core for  $(\tilde{B}, D(\tilde{B}))$ , there exists  $v \in D$  such that  $\|u - v\|_{\tilde{B}} < \varepsilon/3$ .

Observe that  $u(r_1) = u(r_2) = 0$ , so  $|v(r_i)| \leq \|u - v\|_{\infty} < \varepsilon/3$  for  $i = 1, 2$ .

Consider the function

$$v_0(x) := v(x) - \frac{x - r_1}{r_2 - r_1}v(r_2) - \frac{r_2 - x}{r_2 - r_1}v(r_1) \quad (x \in J),$$

then  $v_0 \in D \cap C_0(\overset{\circ}{J})$ . Moreover

$$\|u - v_0\|_{\infty} \leq \|u - v\|_{\infty} + |v(r_1)| + |v(r_2)| < \varepsilon$$

and

$$\|\tilde{B}u - \tilde{B}v_0\|_{\infty} = \|\tilde{B}u - \tilde{B}v\|_{\infty} < \varepsilon. \quad \square$$

We can describe the domain  $D(\tilde{A})$  by the elements of  $D(\tilde{B})$ . This allows us to find cores for  $(\tilde{A}, D(\tilde{A}) \cap C_0(\overset{\circ}{J}))$  and  $(\tilde{A}, D(\tilde{A}))$ .

**Proposition 4.6.** *Under assumption (4.1) the following equalities hold true:*

$$D(\tilde{A}) \cap C_0(\overset{\circ}{J}) = \{u \in D(\tilde{A}) \mid \text{there exists } (u_n)_{n \geq 1} \text{ in } D(\tilde{B}) \cap C_0(\overset{\circ}{J}) \text{ such that } u_n \rightarrow u \text{ and } \tilde{A}u_n \rightarrow \tilde{A}u \text{ uniformly on } J\},$$

and

$$D(\tilde{A}) = \{u \in D(\tilde{A}) \mid \text{there exists } (u_n)_{n \geq 1} \text{ in } D(\tilde{B}) \text{ such that } u_n \rightarrow u \text{ and } \tilde{A}u_n \rightarrow \tilde{A}u \text{ uniformly on } J\}.$$

*Proof.* Consider the auxiliary function  $\lambda$  defined by (4.2). By Proposition 4.5 and Corollary 2.7 in [5], the perturbed operator  $(\lambda\tilde{B}, D(\tilde{B}) \cap C_0(\overset{\circ}{J}))$  is closable and its closure  $(C, D(C))$  generates a Feller semigroup on  $C_0(\overset{\circ}{J})$ .

Since  $\lambda\tilde{B} = \tilde{A}$  and  $D(\tilde{B}) \cap C_0(\overset{\circ}{J}) \subset D(\tilde{A}) \cap C_0(\overset{\circ}{J})$ , then  $(\tilde{A}, D(\tilde{A}) \cap C_0(\overset{\circ}{J}))$  is a closed extension of  $(C, D(C))$ . Moreover  $(\tilde{A}, D(\tilde{A}) \cap C_0(\overset{\circ}{J}))$  generates a Feller semigroup on  $C_0(\overset{\circ}{J})$  and hence

$$D(C) = D(\tilde{A}) \cap C_0(\overset{\circ}{J}) \quad \text{and} \quad \tilde{A}|_{D(\tilde{A}) \cap C_0(\overset{\circ}{J})} = C.$$

Accordingly, the first equality follows taking the characterization of the closure  $D(C)$  into account ([25, Proposition B.4, p. 516]).

Now, let  $u \in D(\tilde{A})$  and set  $v(x) := u(x) - \frac{x-r_1}{r_2-r_1}u(r_2) - \frac{r_2-x}{r_2-r_1}u(r_1)$  ( $x \in J$ ).

Then  $v \in D(\tilde{A}) \cap C_0(\overset{\circ}{J})$  and there exists a sequence  $(v_n)_{n \geq 1} \subset D(\tilde{B}) \cap C_0(\overset{\circ}{J})$  such that  $v_n \rightarrow v$  and  $\tilde{A}v_n \rightarrow \tilde{A}v$  uniformly on  $J$ .

It is easy to check that  $u_n := v_n + u - v \in D(\tilde{B})$  and that  $u_n \rightarrow u$  and  $\tilde{A}u_n \rightarrow \tilde{A}u$  uniformly on  $J$ .

So the assertion is completely proved. □

**Theorem 4.7.** *Let  $D$  be the subspace of  $C_*(J) \cap C^2(\overset{\circ}{J})$  defined by (4.3). Under assumption (4.1) the following assertions hold true:*

- (1)  $D \cap C_0(\overset{\circ}{J})$  is a core for  $(\tilde{A}, D(\tilde{A}) \cap C_0(\overset{\circ}{J}))$ ;
- (2)  $D$  is a core for  $(\tilde{A}, D(\tilde{A}))$ .

Therefore  $D \cap C_0(\overset{\circ}{J})$  (resp.,  $D$  whenever  $w \in C_0(J)$ ) is a core for  $(A, D_w(A))$ .

*Proof.* Let  $u \in D(\tilde{A}) \cap C_0(\overset{\circ}{J})$  (resp.  $u \in D(\tilde{A})$ ) and  $\varepsilon > 0$ ; by Proposition 4.6 there exists  $v \in D(\tilde{B}) \cap C_0(\overset{\circ}{J})$  (resp.,  $v \in D(\tilde{B})$ ) such that  $\|u - v\|_{\tilde{A}} < \varepsilon/2$ .

On the other hand, Proposition 4.5 (resp., Proposition 4.4) ensures that  $D \cap C_0(\overset{\circ}{J})$  (resp.,  $D$ ) is a core for the operator  $(\tilde{B}, D(\tilde{B}) \cap C_0(\overset{\circ}{J}))$  (resp.,  $(\tilde{B}, D(\tilde{B}))$ ); then there exists  $h \in D \cap C_0(\overset{\circ}{J})$  (resp.,  $h \in D$ ) such that  $\|v - h\|_{\tilde{B}} < \varepsilon/2$ .

Therefore

$$\|u - h\|_{\infty} \leq \|u - v\|_{\infty} + \|v - h\|_{\infty} < \varepsilon$$

and

$$\|\tilde{A}u - \tilde{A}h\|_{\infty} \leq \|\tilde{A}u - \tilde{A}v\|_{\infty} + \|\lambda\tilde{B}v - \lambda\tilde{B}h\|_{\infty} < \varepsilon. \quad \square$$

## 4.2. UNBOUNDED INTERVALS.

We restrict our analysis only to the cases  $r_i \in \mathbb{R}$  and  $r_j \in \{-\infty, +\infty\}$ . For the case  $J = \mathbb{R}$  we refer the reader to [16, Section 4].

Consider the following subspaces:

$$D_1 := \left\{ u \in C_*(J) \cap C^2(\overset{\circ}{J}) \mid u \text{ is constant on a neighborhood of } r_j \right. \\ \left. \text{and } \lim_{x \rightarrow r_i} u''(x) \in \mathbb{R} \right\}$$

and

$$D_2 := \left\{ u \in C_*(J) \cap C^2(\overset{\circ}{J}) \mid u \text{ is constant on a neighborhood of } r_j \right. \\ \left. \text{and } \lim_{x \rightarrow r_i} \alpha(x)u''(x) = 0 \right\}.$$

Observe that, denoted by  $UC_b^2(J)$  the space of all functions  $f \in C^2(J)$  whose second derivative is uniformly continuous and bounded, we have  $D_1 \subset UC_b^2(J)$ . Moreover, if there exist  $C > 0$  and  $a \in \overset{\circ}{J}$  such that

$$\alpha(x) \geq C > 0 \quad \text{for every } x \in I(r_i, a),$$

then  $D_2 \subset \{u \in C_*(J) \cap C^2(\overset{\circ}{J}) \mid \lim_{x \rightarrow r_i} u''(x) = 0, \text{ for } i = 1, 2\} \subset UC_b^2(J)$ .

We discuss some cases where  $D_1$  or  $D_2$  are cores for the operator  $(\tilde{A}, D(\tilde{A}))$ .

**Theorem 4.8.** *Assume that*

(a) there exist  $C_0, C_1, C_2 > 0$ ,  $a \in \overset{\circ}{J}$  and a function  $p \in C(I(r_i, a))$ , increasing if  $i = 1$  or decreasing if  $i = 2$ , such that

$$C_0(x - r_i)^2 \leq \alpha(x) \quad \text{for every } x \in I(r_i, a), \tag{4.4}$$

$$C_1p(x) \leq \alpha(x) \leq C_2p(x) \quad \text{for every } x \in I(r_i, a), \tag{4.5}$$

$$\lim_{x \rightarrow r_i} \alpha(x) = 0 \tag{4.6}$$

and

(b) there exist  $b \in \overset{\circ}{J}$ ,  $K_1, K_2 > 0$  and  $q \in C^1(I(r_j, b))$  such that

$$K_1q(x) \leq \alpha(x) \leq K_2q(x) \quad \text{for } x \in I(r_j, b). \tag{4.7}$$

Then, in each of the following cases:

(1)  $q \in C^2(I(r_j, b))$  and  $q''(x) = O(1)$  as  $x \rightarrow r_j$ ;

(2) i)  $\frac{1}{q} \in L^1(I(r_j, b))$  and  $\lim_{x \rightarrow r_j} \frac{q(x)}{|x|} = +\infty$ ,

or

i')  $\lim_{x \rightarrow r_j} \frac{q(x)}{|x|} = 0$ ;

and

ii)  $xq'(x) - q(x) \neq 0$  for  $x \in I(r_j, b)$  and  $\frac{q(x)}{xq'(x) - q(x)} = O(1)$  as  $x \rightarrow r_j$ ;

the space  $D_1$  is a core for  $(\tilde{A}, D(\tilde{A}))$  and hence for  $(A, D_w(A))$ , provided that  $w \in C_0(J)$ .

*Proof.* Assume  $r_1 \in \mathbb{R}$  and  $r_2 = +\infty$ .

We need some preliminary remarks. First, note that  $D_1 \subset D(\tilde{A})$  because of condition (4.6).

Now, let  $u \in D(\tilde{A})$  and  $\varepsilon > 0$ ; assume preliminary that  $\lim_{x \rightarrow +\infty} u(x) = 0$ .

By (4.4), we have  $\lim_{x \rightarrow r_1} (x - r_1)^2 u''(x) = 0$ . Accordingly, by Lemma 4.2-(a) with  $\varphi(x) = x - r_1$  and  $h(x) = (x - r_1)^2 (x \in ]r_1, a])$ ,

$$\lim_{x \rightarrow r_1} (x - r_1)u'(x) = 0.$$

Moreover, by (4.7) and by Lemma 4.3 in case (1) and Lemma 4.2 in case (2) with  $\varphi(x) = \frac{q(x)}{x}$  and  $h(x) = q(x)$  ( $x \in [b, +\infty[$ ), it follows that  $\lim_{x \rightarrow +\infty} \frac{q(x)u'(x)}{x} = 0$ , which means

$$\lim_{x \rightarrow +\infty} \frac{\alpha(x)u'(x)}{x} = 0.$$

Hence, there exists  $\delta_1 > r_1$  such that, for every  $x, y \in ]r_1, \delta_1[$ ,

$$|u(x) - u(y)| \leq \varepsilon/3, \quad |(x - r_1)u'(x)| \leq \varepsilon/3, \quad |(x - r_1)^2 u''(x)| \leq \varepsilon/3, \quad |\alpha(x)u''(x)| \leq \varepsilon,$$

and there exists  $\delta_2 > |\delta_1|$  such that, for every  $x \geq \delta_2$ ,

$$|u(x)| \leq \varepsilon, \quad |\alpha(x)u(x)| \leq \varepsilon x^2, \quad |\alpha(x)u'(x)| \leq \varepsilon x, \quad |\alpha(x)u''(x)| \leq \varepsilon.$$

Consider now a function  $\varphi \in K^2(\mathbb{R})$  such that

$$0 \leq \varphi \leq 1, \quad \varphi(x) = 1 \text{ for } |x| \leq 1, \quad \varphi(x) = 0 \text{ for } |x| \geq 2.$$

Fixed  $x_1 \in ]r_1, \delta_1[$  and  $x_2 \geq \delta_2$ , set

$$v(x) := \begin{cases} u(x_1) + u'(x_1)(x - x_1) + \frac{u''(x_1)}{2}(x - x_1)^2 & \text{if } r_1 \leq x \leq x_1, \\ u(x) & \text{if } x_1 < x < x_2, \\ u(x)\varphi\left(\frac{x}{x_2}\right) & \text{if } x \geq x_2. \end{cases} \quad (4.8)$$

Clearly  $v \in D_1$ . We proceed to prove that  $\|u - v\|_{\tilde{A}} \leq \varepsilon$ .

Indeed, if  $r_1 \leq x \leq x_1$ , then

$$\begin{aligned} |u(x) - v(x)| &= |u(x) - u(x_1) - u'(x_1)(x - x_1) - \frac{u''(x_1)}{2}(x - x_1)^2| \\ &\leq |u(x) - u(x_1)| + |u'(x_1)(r_1 - x_1)| + |u''(x_1)|(r_1 - x_1)^2 \leq \varepsilon \end{aligned}$$

and

$$\begin{aligned} |\tilde{A}u(x) - \tilde{A}v(x)| &= |\alpha(x)u''(x) - \alpha(x)u''(x_1)| \leq |\alpha(x)u''(x)| + C_2|p(x)u''(x_1)| \\ &\leq |\alpha(x)u''(x)| + C_2|p(x_1)u''(x_1)| \leq \left(1 + \frac{C_2}{C_1}\right)\varepsilon. \end{aligned}$$

If  $x_1 < x < x_2$ ,  $|u(x) - v(x)| = |\tilde{A}u(x) - \tilde{A}v(x)| = 0$ .

If  $x_2 \leq x \leq 2x_2$ ,

$$|u(x) - v(x)| = \left| \left(1 - \varphi\left(\frac{x}{x_2}\right)\right) u(x) \right| \leq |u(x)| \leq \varepsilon$$

and

$$\begin{aligned} \tilde{A}u(x) - \tilde{A}v(x) &= |\alpha(x)u''(x) - \alpha(x)v''(x)| \\ &\leq |\alpha(x)u''(x)| + \frac{2\|\varphi'\|_\infty}{x_2}|\alpha(x)u'(x)| + \frac{\|\varphi''\|_\infty}{x_2^2}|\alpha(x)u(x)| \\ &\leq (1 + 4\|\varphi'\|_\infty + 4\|\varphi''\|_\infty)\varepsilon. \end{aligned}$$

Finally, for  $x > 2x_2$ ,

$$|u(x) - v(x)| = \left| \left(1 - \varphi\left(\frac{x}{x_1}\right)\right) u(x) \right| = |u(x)| \leq \varepsilon$$

and

$$|\tilde{A}u(x) - \tilde{A}v(x)| = |\alpha(x)u''(x)| \leq \varepsilon.$$

Consider now the general case  $\lim_{x \rightarrow +\infty} u(x) = l \in \mathbb{R}$ . Then the function  $h := u - l\mathbf{1}$  belongs to  $D(\tilde{A})$  and  $\lim_{x \rightarrow +\infty} h(x) = 0$ . Hence, there exists  $v \in D$  such that  $\|h - v\|_{\tilde{A}} \leq \varepsilon$ , that is  $\|u - l - v\|_{\tilde{A}} = \|u - (l + v)\|_{\tilde{A}} \leq \varepsilon$  and  $v + l \in D$ .

As regards the case where  $r_1 = -\infty$  and  $r_2 \in \mathbb{R}$ , we can prove the statement in the same way, by replacing the function  $v$ , defined by (4.8), with the function

$$g(x) := \begin{cases} u(x_2) + u'(x_2)(x - x_2) + \frac{u''(x_2)}{2}(x - x_2)^2 & \text{if } x_2 \leq x \leq r_2, \\ u(x) & \text{if } x_1 < x < x_2, \\ u(x)\varphi\left(\frac{x}{x_1}\right) & \text{if } x \leq x_1, \end{cases} \quad (4.9)$$

for suitable  $x_1$  and  $x_2$  in  $J$ .

So the proof is complete. □

**Theorem 4.9.** *Assume that  $\alpha$  satisfies condition (4.7). Then, in each of the following cases:*

(1)  $q \in C^2(I(r_j, b))$  and  $q''(x) = O(1)$  as  $x \rightarrow r_j$ ;

(2) i)  $\frac{1}{q} \in L^1(I(r_j, b))$  and  $\lim_{x \rightarrow r_j} \frac{q(x)}{|x|} = +\infty$ ,

or

i')  $\lim_{x \rightarrow r_j} \frac{q(x)}{|x|} = 0$ ;

and

ii)  $xq'(x) - q(x) \neq 0$  for  $x \in I(r_j, b)$  and  $\frac{q(x)}{xq'(x) - q(x)} = O(1)$  as  $x \rightarrow r_j$ ;

the space  $D_2$  is a core for  $(\tilde{A}, D(\tilde{A}))$  and hence for  $(A, D_w(A))$ , provided that  $w \in C_0(J)$ .

*Proof.* The proof is similar to that one of Theorem 4.8, by replacing the function  $v$ , defined by (4.8), with

$$v(x) := \begin{cases} u(x) & \text{if } r_1 \leq x < x_2, \\ u(x)\varphi\left(\frac{x}{x_2}\right) & \text{if } x \geq x_2 \end{cases}$$

and the function  $g$ , defined by (4.9), with

$$g(x) := \begin{cases} u(x) & \text{if } x \leq x_1, \\ u(x)\varphi\left(\frac{x}{x_1}\right) & \text{if } x_1 < x \leq r_2. \end{cases} \quad \square$$

### 5. AN APPLICATION: THE LIMIT SEMIGROUP GENERATED BY MODIFIED SZÁSZ-MIRAKJAN OPERATORS

In this section we discuss a simple application of the previous results concerning a degenerate differential operator on  $[0, +\infty[$  and the approximation of the corresponding semigroup in terms of iterates of an integral modification of Szász-Mirakjan operators which are defined by

$$S_n(f)(x) := \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) p_{n,k}(x) \quad (x \geq 0, n \geq 1)$$

for every  $f \in C([0, +\infty[)$  having a polynomial growth at infinity, where

$$p_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!} \quad (x \geq 0, n \geq 1, k \geq 0).$$

For every  $m \geq 0$  and  $x \geq 0$ , set  $e_m(t) := t^m$  and  $\psi_x(t) := t - x$  ( $t \geq 0$ ). From [21, Lemma 3] it follows that, for every  $n \geq 1$  and  $x \geq 0$ ,

$$\begin{aligned} S_n(\mathbf{1})(x) &= \mathbf{1}, & S_n(e_1)(x) &= x, & S_n(e_2)(x) &= x^2 + \frac{x}{n}, \\ S_n(e_3)(x) &= x^3 + \frac{3x^2}{n} + \frac{x}{n^2}, & S_n(e_4)(x) &= x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}, \end{aligned}$$

and hence

$$\begin{aligned} S_n(\psi_x)(x) &= 0, & S_n(\psi_x^2)(x) &= \frac{x}{n}, \\ S_n(\psi_x^3)(x) &= \frac{x}{n^2}, & S_n(\psi_x^4)(x) &= \frac{x}{n^3} + \frac{3x^2}{n^2}. \end{aligned}$$

Now consider the sequence of positive linear operators  $(L_n)_{n \geq 1}$ , introduced in [28] and defined by

$$L_n(f)(x) := e^{-nx} f(0) + \sum_{k=1}^{+\infty} n \left( \int_0^{+\infty} f(t) p_{n,k-1}(t) dt \right) p_{n,k}(x)$$

for every  $x \geq 0, n \geq 1$  and  $f \in C([0, +\infty[)$  having a polynomial growth at infinity.

Note that, for every  $n \geq 1$  and  $p \geq 0$ ,

$$\int_0^{+\infty} e^{-nt} t^p dt = \frac{p!}{n^{p+1}}$$

and that, in particular, for every  $n, k \geq 1$ ,

$$\int_0^{+\infty} p_{n,k}(t) dt = \frac{1}{n}.$$

Moreover, for  $n, m \geq 1$  and  $x \geq 0$ ,

$$\begin{aligned} L_n(e_m)(x) &= \sum_{k=1}^{+\infty} \frac{n^k}{(k-1)!} \left( \int_0^{+\infty} e^{-nt} t^{k+m-1} dt \right) p_{n,k}(x) \\ &= \sum_{k=1}^{+\infty} \frac{k(k+1) \cdot \dots \cdot (k+m-1)}{n^m} p_{n,k}(x). \end{aligned}$$

Thus

$$L_n(\mathbf{1})(x) = \mathbf{1}, \quad L_n(e_1)(x) = x, \tag{5.1}$$

$$\begin{aligned} L_n(e_2)(x) &= \sum_{k=1}^{+\infty} \frac{k(k+1)}{n^2} p_{n,k}(x) = \sum_{k=1}^{+\infty} \frac{k^2}{n^2} p_{n,k}(x) + \frac{1}{n} \sum_{k=1}^{+\infty} \frac{k}{n} p_{n,k}(x) \\ &= S_n(e_2)(x) + \frac{1}{n} S_n(e_1)(x) = x^2 + \frac{2x}{n}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} L_n(e_3)(x) &= \sum_{k=1}^{+\infty} \frac{k(k+1)(k+2)}{n^3} p_{n,k}(x) \\ &= \sum_{k=1}^{+\infty} \frac{k^3 + k^2}{n^3} p_{n,k}(x) + \frac{2}{n} L_n(e_2)(x) = x^3 + \frac{6x^2}{n} + \frac{6x}{n^2}, \end{aligned}$$

$$\begin{aligned} L_n(e_4)(x) &= \sum_{k=1}^{+\infty} \frac{k(k+1)(k+2)(k+3)}{n^4} p_{n,k}(x) \\ &= \sum_{k=1}^{+\infty} \frac{k^4 + 3k^3 + 2k^2}{n^4} p_{n,k}(x) + \frac{3}{n} L_n(e_3)(x) = x^4 + \frac{12x^3}{n} + \frac{36x^2}{n^2} + \frac{24x}{n^3}. \end{aligned}$$

By the previous formulae it also follows that, for every  $n \geq 1$  and  $x \geq 0$ ,

$$L_n(\psi_x)(x) = 0, \quad L_n(\psi_x^2)(x) = \frac{2x}{n}, \quad L_n(\psi_x^4)(x) = \frac{12x^2}{n^2} + \frac{24x}{n^3}. \tag{5.3}$$

Consider now the Banach lattice

$$E_2^0 := \left\{ f \in C([0, +\infty[) \mid \lim_{x \rightarrow +\infty} \frac{f(x)}{1+x^2} = 0 \right\}$$

endowed with natural pointwise order and the weighted norm

$$\|f\|_2 := \sup_{x \geq 0} \frac{|f(x)|}{1+x^2} \quad (f \in E_2^0).$$

**Theorem 5.1.** *The following properties hold true:*

- (1) For every  $n \geq 1$ ,  $L_n(E_2^0) \subset E_2^0$ ,  $L_n$  is continuous and  $\|L_n\| \leq 1 + \frac{1}{n}$ ;
- (2) For every  $f \in E_2^0$ ,  $\lim_{n \rightarrow \infty} L_n(f) = f$  in  $(E_2^0, \|\cdot\|_2)$  and the convergence is uniform on compact subsets of  $[0, +\infty[$ ;
- (3) For every  $f \in E_2^0 \cap C^2([0, +\infty[)$  such that  $f''$  is uniformly continuous and bounded

$$\lim_{n \rightarrow \infty} n(L_n(f) - f) = Af \quad \text{in } (E_2^0, \|\cdot\|_2)$$

and the convergence is uniform on compact subsets of  $[0, +\infty[$ , where

$$Af(x) := xf''(x) \quad (x \geq 0).$$

*Proof.* (1) First observe that, because of (5.2), for every  $x \geq 0$ ,

$$\frac{L_n(\mathbf{1} + e_2)(x)}{1+x^2} \leq 1 + \frac{1}{n}. \tag{5.4}$$

Let  $n \geq 1$  and  $f \in E_2^0$ ; for  $\varepsilon > 0$  there exists  $t_0 \geq 0$  such that

$$|f(t)| \leq \frac{\varepsilon}{4}(1+t^2) \quad (t \geq t_0).$$

Then, set  $M := \sup_{0 \leq t \leq t_0} |f(t)|$ , we have

$$|f(t)| \leq M + \frac{\varepsilon}{4}(1+t^2) \quad (t \geq 0).$$

Accordingly, for every  $x \geq 0$ ,

$$|L_n(f)(x)| \leq L_n(|f|)(x) \leq M + \frac{\varepsilon}{4}L_n(\mathbf{1} + e_2)(x)$$

whence, choosing  $x_0 \geq 0$  such that  $\frac{M}{1+x^2} \leq \frac{\varepsilon}{2}$ , for  $x \geq x_0$  we get

$$\frac{|L_n(f)(x)|}{1+x^2} \leq \varepsilon$$

which means  $L_n(f) \in E_2^0$ .

Finally, note that, if  $f \in E_2^0$ , then  $|f| \leq \|f\|_2(1 + e_2)$ ; accordingly, taking (5.4) into account,

$$\|L_n(f)\|_2 \leq \|f\|_2\|L_n(1 + e_2)\|_2 \leq \|f\|_2 \left(1 + \frac{1}{n}\right)$$

and so  $\|L_n\| \leq 1 + \frac{1}{n}$ .

(2) For every  $\lambda > 0$ , consider the function  $f_\lambda(x) := e^{-\lambda x}$  ( $x \geq 0$ ). By simple calculation we obtain, for every  $n \geq 1$  and  $x \geq 0$ ,

$$L_n(f_\lambda)(x) = e^{-\frac{n}{n+\lambda}\lambda x} \longrightarrow f_\lambda(x) \quad \text{as } n \rightarrow \infty.$$

Since the sequence  $(L_n(f_\lambda))_{n \geq 1}$  is decreasing, from Dini's theorem it follows that  $L_n(f_\lambda) \rightarrow f_\lambda$  uniformly on  $[0, +\infty[$  and then with respect to  $\|\cdot\|_2$ .

On the other hand, by Stone-Weierstrass theorem, the subspace generated by  $\{f_\lambda \mid \lambda > 0\}$  is dense in  $C_0([0, +\infty[)$  and then in  $(E_2^0, \|\cdot\|_2)$ . Thus, by the equicontinuity of  $(L_n)_{n \geq 1}$ , we obtain (2).

(3) It suffices to take formulae (5.1) and (5.3) into account and to apply [4, Theorem 1] with  $\alpha(x) = 2x$ ,  $\beta(x) = \gamma(x) = 0$ ,  $w(x) = (1 + x^2)^{-1}$  ( $x \geq 0$ ),  $q = 4$  and  $E := \left\{ f \in C([0, +\infty[) \mid \sup_{x \geq 0} \frac{|f(x)|}{1+x^4} \right\}$ . □

Now consider the differential operator  $A : D(A) \rightarrow E_2^0$ , defined by

$$Au(x) := \begin{cases} xu''(x) & x > 0, \\ 0 & x = 0, \end{cases}$$

for every  $u \in D(A)$  and  $x \geq 0$ , where

$$D(A) := \left\{ u \in E_2^0 \cap C^2(]0, +\infty[) \mid \lim_{x \rightarrow 0^+} xu''(x) = \lim_{x \rightarrow +\infty} \frac{x}{1+x^2}u''(x) = 0 \right\}.$$

**Theorem 5.2.** *The operator  $(A, D(A))$  generates a positive  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $E_2^0$  such that*

$$\|T(t)\| \leq e^t \quad \text{for every } t \geq 0$$

and the subspace

$$D_1 := \left\{ u \in C_*([0, +\infty[) \cap C^2(]0, +\infty[) \mid \lim_{x \rightarrow 0^+} u''(x) \in \mathbb{R} \text{ and } u \text{ is constant on a neighborhood of } +\infty \right\}$$

is a core for  $(A, D(A))$ .

Moreover, if  $t \geq 0$  and  $(k(n))_{n \geq 1}$  is a sequence of positive integers such that  $k(n)/n \rightarrow t$ , then for every  $f \in E_2^0$

$$T(t)f = \lim_{n \rightarrow \infty} L_n^{k(n)} f \quad \text{in } E_2^0 \quad (5.5)$$

and the convergence is uniform on compact subsets of  $[0, +\infty[$ .

*Proof.* The first part of the statement follows from Theorem 3.1, with  $J = [0, +\infty[$ ,  $\alpha(x) = x$  and  $w(x) = (1 + x^2)^{-1}$  ( $x \geq 0$ ) (here  $\omega = 1$  (see formula (3.3)).

By Theorem 4.8, with  $p(x) = q(x) = x$  ( $x \geq 0$ ), the subspace  $D_1$  is a core for  $(A, D(A))$ .

Since  $D_1 \subset UC_b^2([0, +\infty[)$ , by Theorem 5.1-(3) we obtain

$$\lim_{n \rightarrow \infty} n(L_n u) - u = Au \quad \text{in } E_2^0 \quad (u \in D_1).$$

Finally part (1) of Theorem 5.1 yields  $\|L_n^k\| \leq (1 + \frac{1}{n})^k \leq e^{\frac{k}{n}}$  ( $n, k \geq 1$ ).

Therefore, by Theorem 2.1 we get the representation formula (5.5). □

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## REFERENCES

- [1] F. Altomare, *Limit semigroups of Bernstein-Schnabl operators associated with positive projections*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 16 (1989), no.2, 259-279.
- [2] F. Altomare, *Lototsky-Schnabl operators on compact convex sets and their associated limit semigroups*, Mh. Math. 114 (1992), 1-13.
- [3] F. Altomare, *Approximation theory methods for the study of diffusion equations*, Approximation theory (Witten, 1995), 9-26, Math. Res., 86, Akademie Verlag, Berlin, 1995.
- [4] F. Altomare and R. Amiar, *Asymptotic formulae for positive linear operators*, dedicated to the 70th anniversary of Academician Blagovest Sendov, Math. Balkanica (N.S.) 16 (2002), no. 1-4, 283-304.
- [5] F. Altomare and A. Attalienti, *Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black-Scholes equation - part I*, Results Math. 42 (2002), no. 3-4, 193-211.
- [6] F. Altomare and A. Attalienti, *Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black-Scholes equation - part II*, Results Math. 42 (2002), no. 3-4, 212-228.
- [7] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, de Gruyter Studies in Mathematics, 17, Walter de Gruyter & Co., Berlin, 1994.
- [8] F. Altomare, M. Cappelletti Montano and S. Diomede, *Degenerate elliptic operators, Feller semigroups and modified Bernstein-Schnabl operators*, preprint (2008).

- [9] F. Altomare and I. Carbone, *On some degenerate differential operators on weighted function spaces*, J. Math. Anal. Appl. 213 (1997), no. 1, 308-333.
- [10] F. Altomare and V. Leonessa, *Continuous selection of Borel measures, positive operators and degenerate evolution problems*, Rev. Anal. Numér. Théor. Approx., 36 (2007), no. 1, 9-23.
- [11] F. Altomare, V. Leonessa and S. Milella, *Continuous selections of Borel measures and Bernstein-Schnabl operators*, Numerical analysis and approximation theory, 1-26, Casa Cărții de Știință, Cluj-Napoca, 2006.
- [12] F. Altomare, V. Leonessa and I. Raşa, *On Bernstein-Schnabl operators on the unit interval*, Z. Anal. Anwendungen 27 (2008), no. 3, 353-379.
- [13] F. Altomare and E. Mangino, *On a generalization of Baskakov operators*, Rev. Roumaine Math. Pures Appl. 44 (1999), no. 5-6, 683-705.
- [14] F. Altomare and E.M. Mangino, *On a class of elliptic-parabolic equations on unbounded intervals*, Positivity 5 (2001), no. 3, 239-257
- [15] F. Altomare and S. Milella, *Integral-type operators on weighted spaces of continuous functions*, J. Approx. Theory 152 (2008), 107-124.
- [16] F. Altomare and S. Milella, *On the  $C_0$ -semigroups generated by second-order differential operators on the real line*, to appear in Taiwanese J. Math. (2009).
- [17] F. Altomare and I. Raşa, *On some classes of diffusion equations and related approximation problems*, Trends and applications in constructive approximation, 13-26, Internat. Ser. Numer. Math., 151, Birkhäuser, Basel, 2005.
- [18] F. Altomare and I. Raşa, *On a class of exponential-type operators and their limit semigroups*, J. Approx. Theory 135 (2005), no. 2, 258-275.
- [19] A. Attalienti, *Generalized Bernstein-Durrmeyer operators and the associated limit semigroup*, J. Approx. Theory 99 (1999), no. 2, 289-309.
- [20] A. Attalienti and M. Campiti, *Degenerate evolution problems and Beta-type operators*, Studia Math. 140 (2000), no. 2, 117-139.
- [21] M. Becker, *Global approximation theorem for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. 27 (1978), no. 1, 127-142.
- [22] M. Campiti, G. Metafune and D. Pallara, *General Voronovskaja formula and solutions of second-order degenerate differential equations*, Rev. Romaine Math. Pures Appl. 44 (1999), no. 5-6, 755-766.
- [23] I. Carbone, *Trotter's type theorems for generalized Szász-Mirakjan operators in polynomial weighted function spaces*, Indian J. Math. 44 (2002), no. 1, 1-20.
- [24] Ph. Clément and C.A. Timmermans, *On  $C_0$ -semigroups generated by differential operators satisfying Ventcel's boundary conditions*, Nederl. Akad. Wetensch. Indag. Math. 48 (1986), no. 4, 379-387.
- [25] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Text in Mathematics, 194, Springer, New York-Berlin, 2000.
- [26] E.M. Mangino, *Differential operators with second-order degeneracy and positive approximation processes*, Constr. Approx. 18 (2002), no. 3, 443-466.
- [27] E.M. Mangino, *A positive approximation sequence related to Black and Scholes equation*, Proceedings of the Fourth International Conference on Functional Analysis and Approximation Theory, Vol. I (Potenza, 2000), Rend. Circ. Mat. Palermo (2) Suppl. 2002, no. 68, part I, 359-372.
- [28] S.M. Mazhar and V. Totik, *Approximation by modified Szász operators*, Acta Sci. Math. 49 (1985), 257-269.

- [29] S. Milella, *On a class of second-order differential operator on the real line*, Rev. Roumaine Math. Pures Appl. 53 (2008), no. 2-3, 189-207.
- [30] A.Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Science, 44. Springer Verlag, New York, 1983.
- [31] I. Raşa, *Semigroups associated to Mache operators*, Advanced Problems in Constructive Approximation (M.D. Buhmann and D.H. Mache, eds.), ISNM, Vol.142, pp. 143-152, Birkhäuser-Verlag, Basel, 2002.
- [32] I. Raşa, *Semigroups associated to Mache operators, II*, Trends and applications in constructive approximation, 225-228, Internat. Ser. Numer. Math., 151, Birkhäuser, Basel, 2005.
- [33] I. Raşa, *One-dimensional diffusions and approximation*, Mediterr. j. math. 2 (2005), 153-169.
- [34] H. Trotter, *Approximation of semigroups of operators*, Pacific J. Math. 8 (1958), 887-919.