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BOUNDARY VALUE PROBLEMS WITH VANISHING GREEN'S FUNCTION

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Dedicated to Espedito De Pascale
on the occasion of his retirement.

ABSTRACT. We study positive solutions of integral equations in $C[0,1]$ where the kernel (Green's function of the corresponding boundary value problem) is supposed to be non-negative on $[0,1]\times[0,1]$ but may vanish at some interior points which prevents use of some standard cones. We prove existence of one or two positive solution under some conditions which can be sharp.

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1. INTRODUCTION

In recent years there has been an extensive study of the existence of positive solutions of boundary value problems (BVPs) for differential equations involving both local and nonlocal boundary conditions (BCs). A typical example of local BCs is the weakly singular second order problem

$$u''(t) + g(t)f(t,u(t)) = 0,\ t \in (0,1), \quad au(0) - bu'(0) = 0,\ cu(1) + du'(1) = 0. \quad (1.1)$$

A standard method used to treat (1.1) is to find fixed points of the integral equation

$$Tu(t) = \int_0^1 G(t,s)g(s)f(s,u(s))\,ds$$

in the space $C[0,1]$ of continuous functions, where G is the Green's function of the differential equation with the given BCs. To show existence of a *positive* solution (when $g, f \geq 0$), it is required that $G(t,s) \geq 0$ and one seeks fixed points of T in the cone $P := \{u \in C[0,1] : u(t) \geq 0\}$. For similar problems with periodic BCs, see for example [6, 11], for nonlocal BCs, see for example [7, 12].

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It is convenient, especially when seeking *multiple* positive solutions, to work in a smaller cone, namely the cone

$$K = \{u \in P : \min_{t\in[a,b]} u(t) \geq c\|u\|\},$$

where $[a, b]$ is some subset of $[0, 1]$ and $c > 0$. The cone K is well-known, and may be found in the books Krasnosel'skiĭ, [4], and Guo and Lakshmikantham, [2], and has been used by many authors in the study of multiple solutions of BVPs.

For the simple second order boundary value problem (with g, f non-negative),

$$u''(t) + g(t)f(t, u(t)) = 0, \;\; u(0) = 0, \; u(1) = 0,$$

the property $\min_{t\in[a,b]} u(t) \geq c\|u\|$ is an immediate consequence of concavity of possible solutions. However, the property holds for more general equations and BCs and can be shown provided the Green's function G satisfies a somewhat stronger positivity condition. A particulary suitable such assumption was given in [5], namely that there exist a subinterval $[a, b] \subseteq [0, 1]$, a measurable function Φ, and a constant $c = c(a, b) \in (0, 1]$ such that

$$\begin{aligned} G(t, s) &\leq \Phi(s) \text{ for } t \in [0, 1] \text{ and } \; s \in [0, 1], \\ G(t, s) &\geq c\Phi(s) \text{ for } t \in [a, b] \text{ and } \; s \in [0, 1]. \end{aligned} \tag{1.2}$$

This is often shown by proving that there exists a continuous function $c \geq 0$ such that

$$c(t)\Phi(s) \leq G(t, s) \leq \Phi(s), \text{ for } 0 \leq t, s \leq 1, \tag{1.3}$$

which establishes the required inequality when $c(t) \geq c > 0$ on $[a, b]$. This can be shown for very many problems, for example, rather general nonlocal BVPs for second order equations are discussed in [12] and some fourth order problems are discussed in [14]. The cone K, with suitable modifications, has been used in the study of some singular periodic BVPs in [6].

In a recent paper [1], Graef, Kong and Wang studied a periodic BVP for

$$u''(t) + a(t)u(t) = g(t)f(u(t)), \, t \in (0, 1), \tag{1.4}$$

where the Green's function is assumed to be non-negative on the square $[0, 1] \times [0, 1]$, but can be zero at some interior points of the square, for example along the diagonal, that is, $G(s, s) = 0$. (In fact they worked on the interval $[0, 2\pi]$ but, since nothing essential is changed, we consider $[0, 1]$.) In such a case it is not possible to work in the cone K. In fact, in [1], another larger cone was used and the authors proved the existence of one positive solution under a sub-linear condition on f and also under a super-linear condition on f provided that f was convex.

A key assumption made by the authors of [1] is that $\min_{0\leq s\leq 1} \int_0^1 G(t, s)\, dt > 0$. It was also assumed that the functions f, g are continuous and non-negative and that $\min_{t\in[0,1]} g(t) > 0$.

In the present paper we use fixed point index theory to prove the existence of at least one positive solutions under conditions weaker than sub- and super-linearity, and another result on the existence of at least two positive solutions. The important new tool is the use of an open set which allows the idea of [1] to be fully exploited. Our conditions can be sharp, as we show in an example. We allow g to possibly be 0 at some points and we also treat weakly singular problems by allowing g to be an L^1 function which may have some pointwise singularities.

2. EXISTENCE RESULTS

We shall give results for weakly singular (singular in the variable t) problems; we do not cover some singular (in u) periodic problems which have been studied recently by Torres [10]. We shall consider the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds. \tag{2.1}$$

We make the following assumptions throughout the paper.

- (H_1) The kernel G is non-negative and is continuous on $[0,1]\times[0,1]$, with $G(t,s) \le G_0$ for all t and s.
- (H_2) The function g is non-negative almost everywhere, $g \in L^1[0,1]$, and satisfies $g_1 := \int_0^1 g(t)\,dt > 0$.
- (H_3) There is a constant $\alpha > 0$ such that $\int_0^1 G(t,s)g(t)\,dt \ge \alpha$ for all s.
- (H_4) The nonlinearity $f : [0,\infty) \to [0,\infty)$ is continuous.

The assumption (H_3) allows $G(t,s)$ to vanish on part of $[0,1]\times[0,1]$ and can hold when (1.2) is not satisfied, see Example 2.11 below.

Under these assumptions, it is well known that the integral operator T defined by

$$Tu(t) := \int_0^1 G(t,s)g(s)f(u(s))\,ds$$

is a compact map from P to $C[0,1]$, see for example Proposition V.3.1 of [8].

A closed subset K of a Banach space X is called a cone if $x, y \in K$ and $\alpha \ge 0$ imply that $x+y \in K$ and $\alpha x \in K$, and $K \cap (-K) = \{0\}$. A cone K defines a partial order by $x \le y \iff y - x \in K$. The cone is called *normal* if there exists $\sigma > 0$ such that for all $0 \le x \le y$ it follows that $\|x\| \le \sigma\,\|y\|$. The cone is said to be *reproducing* if $X = K - K$ and to be *total* if $X = \overline{K-K}$. It is well known that P is normal and reproducing.

We let

$$\widetilde{K} := \Big\{u \in P : \int_0^1 u(t)g(t)\,dt \ge \frac{\alpha}{G_0}\,\|u\|\Big\}. \tag{2.2}$$

It is easily seen that this is a cone; it is a modification of the one used in [1] which did not include the term g but assumed $\min_{t\in[0,1]} g(t) > 0$.

Lemma 2.1. *Under the above assumptions, $T : P \to \widetilde{K}$.*

Proof. Let $u \in P$. Then we have

$$\begin{aligned}\int_0^1 Tu(t)g(t)\,dt &= \int_0^1 \Big(\int_0^1 G(t,s)g(s)f(u(s))\,ds\Big)g(t)\,dt\\ &= \int_0^1 \Big(\int_0^1 G(t,s)g(t)\,dt\Big)g(s)f(u(s))\,ds\\ &\geq \int_0^1 \alpha g(s)f(u(s))\,ds\\ &\geq \frac{\alpha}{G_0}\int_0^1 G(\tau,s)g(s)f(u(s))\,ds\end{aligned}$$

for arbitrary $\tau \in [0,1]$. Thus we have $\int_0^1 Tu(t)g(t)\,dt \geq \frac{\alpha}{G_0}\|Tu\|$, that is $Tu \in \widetilde{K}$. □

In particular this shows that $\widetilde{K} \neq \{0\}$ unless T is the zero operator on P.

We shall base our proof on some fixed point index results, see for example [2, 4] for its properties. Three of these results relate to the principal eigenvalue of the related linear operator

$$Lu(t) := \int_0^1 G(t,s)g(s)u(s)\,ds. \tag{2.3}$$

Then L is a compact linear operator in $C[0,1]$ and, by the same arguments as applied to T, $L(P) \subset \widetilde{K}$. We suppose that the radius of its spectrum $r(L)$ satisfies $r(L) > 0$. By the Krein-Rutman theorem, L has an eigenfunction $\varphi \in P \setminus \{0\}$ corresponding to the principal eigenvalue $r(L)$; we suppose that $\|\varphi\| = 1$. Since $L : P \to \widetilde{K}$, $\varphi \in \widetilde{K}$. We set $\mu_1 := 1/r(L)$, and call it the *principal characteristic value* of L; it is often called the principal eigenvalue of the corresponding BVP.

NOTATION.

$$f^0 = \limsup_{u\to 0+} f(u)/u, \quad f_0 = \liminf_{u\to 0+} f(u)/u;$$
$$f^\infty = \limsup_{u\to\infty} f(u)/u, \quad f_\infty = \liminf_{u\to\infty} f(u)/u.$$

We will prove the following result which is a sharper version of the result of [1].

Theorem 2.2. *Assume that $(H_1) - (H_4)$ hold and that $r(L) > 0$. Then the integral equation*

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \tag{2.4}$$

has at least one positive solution, that is, a nonzero solution in the cone $\widetilde{K}$ if either of the following conditions (S_1), (S_2) hold.

(S_1) $f_0 > \mu_1$ *and* $f^\infty < \mu_1$.

(S_2) *$f^0 < \mu_1$ and there exists $R > 0$ such that $f(R)/R > 1/\alpha$ and f is convex on* $\left[0, \frac{G_0 g_1 R}{\alpha}\right]$.

The equation (2.4) *has at least two positive solutions if*

(D) *$f^0 < \mu_1$, there exists $R > 0$ such that $f(R)/R > 1/\alpha$ and f is convex on* $\left[0, \frac{G_0 g_1 R}{\alpha}\right]$, *and* $f^\infty < \mu_1$.

Remark 2.3. This results extends Theorem 2.1 of [1] who ask that $f_0 = \infty$ and $f^\infty = 0$ (sub-linear case) be satisfied in place of (S_1), and require $f^0 = 0$ and $f_\infty = \infty$ (super-linear case) and that f be convex on $[0, \infty)$ in place of (S_2). They also assume g is continuous and $\min_{t\in[0,1]} g(t) > 0$. Our result using ($S_1$) is sharp and the result using (S_2) can be sharp (see Example 2.11 below). The result (D) for the existence of two nonzero solutions in $\widetilde{K}$ is new. In fact, when (S_2) holds there are two solutions in $\widetilde{K}$ but the second one is zero; when (D) holds there are three solutions in $\widetilde{K}$ including the zero solution. In case (D) it is important not to assume that f is convex on $[0, \infty)$ since $f(0) = 0$ and $f(u)/u$ is then increasing for all $u \geq 0$ and the third condition cannot be satisfied.

We will prove this result using the well known fixed point index theory for compact maps, see for example [2, 4]. For the proofs of some of the fixed point index lemmas it is convenient to recall the following well known result, whose simple proof is given for completeness.

Lemma 2.4. *Let K be a cone in a Banach space X and let $L : X \to X$ be a bounded linear operator with $L(K) \subseteq K$ and $r(L) < 1$. Then $(I-L)^{-1}(K) \subseteq K$. In particular, if $u, w \in X$ and $u \leq Lu + w$, then $u \leq (I - L)^{-1}w$.*

Proof. Since $r(L) < 1$, $(I - L)^{-1}$ exists and is given by the Neumann series

$$(I - L)^{-1} = I + L + L^2 + \dots .$$

Since K is closed and $L(K) \subseteq K$, it follows that $(I-L)^{-1}(K) \subseteq K$. Then $u \leq Lu+w$ is equivalent to $w - (I - L)u \in K$ which gives $(I - L)^{-1}w - u \in K$. □

In fact, it is known ([9], Proposition 2), that if K is a *normal*, *total* cone and L is a bounded linear operator with $L(K) \subseteq K$ then $(I-L)^{-1}$ is bounded and maps K into K *if and only if* $r(L) < 1$. We only use the easy part of this result given above, which is valid in all cones.

For $r > 0$ we will use the following open (relative to $\widetilde{K}$) subsets of $\widetilde{K}$; use of the second open set is the key to obtaining the sharp results of this paper.

$$\widetilde{K}_r := \{u \in \widetilde{K} : \|u\| < r\}, \quad W_r := \{u \in \widetilde{K} : \overline{u}_g := \int_0^1 u(t)g(t)/g_1 \, dt < r\}.$$

We write $\partial \widetilde{K}_r$ and ∂W_r for the boundaries of these sets relative to $\widetilde{K}$. Here, $\overline{u}_g$ is a weighted average of u. Note that if $u \in \overline{W}_r$, then $rg_1 \geq \int_0^1 u(t)g(t)\,dt \geq \frac{\alpha}{G_0}\|u\|$ which shows that W_r is *bounded.*

Lemma 2.5. *Suppose there exists $r_0 > 0$ such that $f(u) < \mu_1 u$ for $0 < u \leq r_0$. Then $i_{\widetilde{K}}(T, \widetilde{K}_{r_0}) = 1$.*

Proof. Let $\varepsilon > 0$ be chosen so that $f(u) \leq (\mu_1 - \varepsilon)u$ for $0 \leq u \leq r_0$. We show that $Tu \neq \beta u$ for all $\beta \geq 1$ and all $u \in \partial \widetilde{K}_{r_0}$ which implies the result. In fact, in the contrary case, there exist $u \in \widetilde{K}$ with $\|u\| = r_0$ and $\beta \geq 1$ such that

$$\beta u = Tu \leq (\mu_1 - \varepsilon)Lu; \text{ (here the ordering is that of } P).$$

Since $r((\mu_1 - \varepsilon)L) < 1$, Lemma 2.4 implies that $u \leq 0$, contradicting $u \in \widetilde{K}$ with $\|u\| = r_0$. □

Note that this result applies when $f^0 < \mu_1$.

Lemma 2.6. *Suppose there exist $\varepsilon > 0$ and $R_0 > 0$ such that $f(u) \leq (\mu_1 - \varepsilon)\,u$ for all $u \geq R_0$. Then there exists $R_1 \geq R_0$ such that $i_{\widetilde{K}}(T, \widetilde{K}_{R_1}) = 1$.*

Proof. We note that P is a normal cone with normality constant $\sigma = 1$ so that $u \leq v$ implies $\|u\| \leq \|v\|$. As f is continuous on $[0, R_0]$, there exists $C_0 > 0$ such that $f(u) \leq (\mu_1 - \varepsilon)\,u + C_0$ for all $u \geq 0$. Let $w(t) := \int_0^1 G(t,s)g(s)C_0\,ds$, hence $\|w\| \leq G_0 C_0 g_1$. Let $R_1 \geq R_0$ be so large that $R_1 > \|(I - (\mu_1 - \varepsilon)L)^{-1}\|\,\|w\|$. We show that $Tu \neq \beta u$ for all $\beta \geq 1$ and all $u \in \partial \widetilde{K}_{R_1}$ which yields the result. Indeed, if $\beta u = Tu$ for some $\beta \geq 1$ and some $u \in \partial \widetilde{K}_{R_1}$, then $u \leq \beta u \leq (\mu_1 - \varepsilon)Lu + w$ (the ordering of P) and by Lemma 2.4 this yields $u \leq (I - (\mu_1 - \varepsilon)L)^{-1}w$. Hence we obtain

$$R_1 = \|u\| \leq \left\|(I - (\mu_1 - \varepsilon)L)^{-1}w\right\| \leq \left\|(I - (\mu_1 - \varepsilon)L)^{-1}\right\|\,\|w\| < R_1,$$

a contradiction. □

Note that this result applies when $f^\infty < \mu_1$.

The above two results are essentially well known. They are valid in any sub-cone of P. The use of Lemma 2.4 in the proof of Lemma 2.6 is essentially the same as would be the use of Lemma 33.2 of [4].

The next two lemmas are new because we work in the cone $\widetilde{K}$ and use the open set W_r. We refer to [13] for cases when we can use the cone K and other open sets to obtain stronger conclusions.

Lemma 2.7. *Suppose there exists $r_1 > 0$ such that $f(u) > \mu_1 u$ for $0 < u \leq r_1$. Then if $u \neq Tu$ on ∂W_{r_1} we have $i_{\widetilde{K}}(T, W_{r_1}) = 0$.*

Proof. Let φ be the eigenfunction of L in P of norm 1, so that $\mu_1 L\varphi = \varphi$ and note that $\varphi \in \widetilde{K}$ since $L : P \to \widetilde{K}$. We show that $u \neq Tu + \beta\varphi$ for all $\beta \geq 0$ and $u \in \partial W_{r_1}$, which shows that the required index is zero. If there exist u with $u = Tu + \beta\varphi$ then $\beta > 0$ and we have (the ordering is that of P),

$$u \geq \mu_1 Lu + \beta\varphi. \tag{2.5}$$

Since $u \geq \beta\varphi$ and $L : P \to P$, it follows that $Lu \geq \beta L\varphi = (\beta/\mu_1)\varphi$. Substituting into (2.5) gives $u \geq 2\beta\varphi$. Repeating this argument gives $u \geq n\beta\varphi$ for arbitrary $n \in \mathbb{N}$. This yields

$$\int_0^1 u(t)g(t)\,dt \geq n\beta \int_0^1 \varphi(t)g(t)\,dt \geq n\beta \frac{\alpha}{G_0} \|\varphi\|.$$

Taking n sufficiently large, the preceding inequality contradicts $\int_0^1 u(t)g(t)\,dt = r_1 g_1$. $\square$

We recall Jensen's inequality in the form we shall use here, see for example [3].

Lemma 2.8 (Jensen's inequality)**.** *Let m be a (positive) measure and let Ω be a measurable set with $m(\Omega) = 1$. Let I be an interval and suppose that u is a real function in $L^1(dm)$ with $u(t) \in I$ for all $t \in \Omega$. If f is convex on I, then*

$$f\Big(\int_\Omega u(t)\,dm(t)\Big) \leq \int_\Omega f(u(t))\,dm(t). \tag{2.6}$$

The fourth fixed point index result is the one where we use convexity of f, in a more precise form than in [1].

Lemma 2.9. *Suppose there exists $R_2 > 0$ such that $f(R_2)/R_2 > 1/\alpha$ and that f is convex on $\big[0, \frac{G_0 g_1 R_2}{\alpha}\big]$. Then we have $i_{\widetilde{K}}(T, W_{R_2}) = 0$.*

Proof. We note that $u \in \overline{W}_{R_2}$ implies that $\overline{u}_g := \int_0^1 u(t)g(t)/g_1\,dt \leq R_2$ which gives $\|u\| \leq \frac{G_0 g_1 R_2}{\alpha}$. We show that $u \neq Tu + \beta e$ for all $\beta \geq 0$ and $u \in \partial W_{R_2}$, where $e \in \widetilde{K} \setminus \{0\}$. In fact, if we have $u(t) = Tu(t) + \beta e(t)$, then

$$\begin{aligned}
\int_0^1 u(t)g(t)\,dt &\geq \int_0^1 Tu(t)g(t)\,dt \\
&\geq \alpha \int_0^1 f(u(s))g(s)\,ds \quad \text{(as in Lemma 2.1)} \\
&= \alpha g_1 \int_0^1 f(u(s))g(s)/g_1\,ds \\
&\geq \alpha g_1 f\Big(\int_0^1 u(s)g(s)/g_1\,ds\Big),
\end{aligned}$$

using convexity of f and Jensens's inequality with the measure $dm(s) := \frac{g(s)ds}{g_1}$ so that $m([0,1]) = 1$. We have shown that $\overline{u}_g \geq \alpha f(\overline{u}_g)$. Since $\overline{u}_g = R_2$ on ∂W_{R_2} this contradicts $f(R_2)/R_2 > 1/\alpha$. $\square$

Remark 2.10. In the proof of the corresponding result in [1] some factors 2π were omitted in the application of Jensen's inequality. Since they assume $f(u)/u \to \infty$ as $u \to \infty$, this does not affect their conclusion.

Proof of Theorem 2.2. This is a standard application of the additivity property of fixed point index. In case (S_1), we take r_1 small and R_1 sufficiently large, with $R_1 > G_0 r_1 g_1/\alpha$ so $W_{r_1} \subset \widetilde{K}_{R_1}$. If there is a fixed point in ∂W_{r_1} we are finished. Otherwise we apply Lemmas 2.6, 2.7 and the additivity property to get

$$i_{\widetilde{K}}(T, \widetilde{K}_{R_1} \setminus \overline{W}_{r_1}) = i_{\widetilde{K}}(T, \widetilde{K}_{R_1}) - i_{\widetilde{K}}(T, W_{r_1}) = 1 - 0 = 1,$$

so T has a fixed point in $\widetilde{K}_{R_1} \setminus \overline{W}_{r_1}$.

In case (S_2) we take r_0 sufficiently small and $r_0 < R$ so $\widetilde{K}_{r_0} \subset W_R$; we apply Lemmas 2.5, 2.9 and similarly we obtain a fixed point of T in $W_R \setminus \widetilde{K}_{r_0}$, (and a fixed point in $\widetilde{K}_{r_0}$, which may be zero).

For the case (D), we take r_0 small with $r_0 < R$, and $R_1 > G_0 R g_1/\alpha$ sufficiently large so that Lemma 2.6 applies. This gives one solution in $W_R \setminus \widetilde{K}_{r_0}$ and another in $\widetilde{K}_{R_1} \setminus \overline{W}_R$. □

Example 2.11. *We consider the periodic BVP given as a motivation in* [1]*,*

$$u''(t) + \omega^2 u(t) = f(u(t)),\ t \in (0,1);\ \ u(0) = u(1),\ u'(0) = u'(1), \tag{2.7}$$

where $\omega > 0$ *is a constant, and, for simplicity, we take* $g(t) \equiv 1$.

It is known that if $\omega \neq 2n\pi$ (n a positive integer) then the Green's function for (2.7) is given by

$$G(t,s) = \begin{cases} \dfrac{\sin(\omega(t-s)) + \sin(\omega(1-t+s))}{2\omega(1-\cos(\omega))}, & s \le t, \\[2ex] \dfrac{\sin(\omega(s-t)) + \sin(\omega(1-s+t))}{2\omega(1-\cos(\omega))}, & s > t. \end{cases}$$

By some standard trigonometric formulae we have the equivalent form

$$G(t,s) = \begin{cases} \dfrac{\cos(\omega(1/2-t+s))}{2\omega\sin(\omega/2)}, & s \le t, \\[2ex] \dfrac{\cos(\omega(1/2-s+t))}{2\omega\sin(\omega/2)}, & s > t. \end{cases}$$

We see that to have $G(t,s) \ge 0$ we must have $\omega \le \pi$. Also when $\omega = \pi$ we see that $G(s,s) = 0$ for all $s \in [0,1]$, which therefore fits the case studied in this paper but (1.2) does not hold. In the case $\omega = \pi$, we see that $G(t,s) \le 1/2\pi$ and, by a calculation, we obtain $\int_0^1 G(t,s)\,dt = 1/\pi^2$; thus $\alpha = 1/\pi^2$ and $G_0 = 1/2\pi$. Also, $\mu_1 = \pi^2$ with a constant eigenfunction. Therefore Theorem 2.2 gives the following conclusions, in which the constants are sharp.

When $\omega = \pi$, the BVP (2.7) *has at least one positive solution (that is, a nonzero solution in $\widetilde{K}$) if*

$$\text{either} \quad f_0 > \pi^2 \text{ and } f^\infty < \pi^2, \quad \text{or} \quad f \text{ is convex, } f^0 < \pi^2 \text{ and } f_\infty > \pi^2.$$

The BVP (2.7) *has at least two positive solutions if*
$f^0 < \pi^2$, there exists $R > 0$ such that $f(R)/R > \pi^2$ and f is convex on $[0, \pi R/2]$, and $f^\infty < \pi^2$.

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