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PERTURBATIONS OF BERNSTEIN-DURRMEYER OPERATORS ON THE SIMPLEX AND BEST APPROXIMATION PROPERTIES

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ABSTRACT. We study some modifications of Bernstein-Durrmeyer operators on a d-dimensional simplex in order to satisfy a best approximation property and an integro-differential Voronovskaja-type formula. Some applications concerning the representation of the solution of suitable evolution problems are also considered.

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1. INTRODUCTION

We consider the following evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = Au(t) \ , \\ u(0) = u_0 \ , \end{cases} \tag{1.1}$$

where $u(t)$ is a function with values in a Hilbert space H, and $A : D(A) \to H$ is a densely defined linear operator on H.

We consider a closed linear subspace W of H and the following evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = P_W Au(t) \ , \\ u(0) = u_0 \ , \end{cases} \tag{1.2}$$

where P_W is the orthogonal projection onto W. Starting with an approximation process connected with the problem (1.1) through a Voronovkaja-type formula, we consider some perturbations consisting in adding suitable functionals in order to obtain a sequence of linear operators L_n connected via a Voronovskaja-type formula with the operator $P_W A$, which gives a better approximation on the subspace W and a representation of the solution of the problem (1.2) in terms of its iterates. This

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method is based on Trotter's theorem on the approximation of semigroups [23]; for quantitative estimates see also [11].

We shall consider the Bernstein-Durrmeyer operators and we shall give the representation of the solution of an evolution problem coming from population genetics, projected on a suitable subspace.

2. PERTURBATIONS OF BERNSTEIN-DURRMEYER OPERATORS AND VORONOVSKAJA'S FORMULAS

First we briefly recall the definition and some spectral properties of Bernstein-Durrmeyer operators which are used in the sequel. Consider the d-dimensional simplex

$$S^d := \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1, \ldots, x_d \geq 0 \ , \ \sum_{i=1}^{d} x_i \leq 1 \right\} .$$

For every $(x_1, \ldots, x_d) \in S^d$ we set $x_0 = 1 - x_1 - \cdots - x_d$. If $\mu \in]-1, \infty[^{d+1}$, we consider the Jacobi weight $w_\mu : S^d \to \mathbb{R}$ defined by setting, for every $x = (x_1, \ldots, x_d) \in S^d$ (see [4, 6, 16]),

$$w_\mu(x) := x_0^{\mu_0}\, x_1^{\mu_1} \cdots x_d^{\mu_d} .$$

We denote by $L^2_\mu(S^d)$ the space of all measurable functions $f : S^d \to \mathbb{R}$ such that

$$\int_{S^d} |f(x)|^2 w_\mu(x)\, dx < \infty ,$$

endowed with the weighted inner product $\langle \cdot, \cdot \rangle_\mu$ defined by setting

$$\langle f, g \rangle_\mu := \int_{S^d} f(x) g(x) w_\mu(x) dx , \qquad f, g \in L^2_\mu(S^d) .$$

We shall denote by $\| \cdot \|_\mu$ the norm associated with the above inner product.

Now, let $\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$ and denote by $|\alpha| := \alpha_0 + \cdots + \alpha_d$ the length of α. The Bernstein polynomial $B_\alpha : S^d \to \mathbb{R}$ of total degree $|\alpha|$ is defined by putting, for every $x = (x_1, \ldots, x_d) \in S^d$,

$$B_\alpha(x) := \binom{|\alpha|}{\alpha} w_\alpha(x_1, \ldots, x_d) = \frac{(|\alpha|)!}{\alpha_0! \cdots \alpha_d!}\, x_0^{\alpha_0} \cdots x_d^{\alpha_d} .$$

We observe that the length of α can be considered even if $\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{Z}^{d+1}$ and in this case we can use the convention $\binom{|\alpha|}{\alpha} = 0$ if $\alpha_i < 0$ for some $i = 0, \ldots, d$.

Now we can define the n-th Bernstein-Durrmeyer operator $M_{n,\mu} \colon L^2_\mu(S^d) \to L^2_\mu(S^d)$ by setting

$$M_{n,\mu} f(x) := \sum_{|\alpha| = n} \frac{\langle f, B_\alpha \rangle_\mu}{\langle \mathbf{1}, B_\alpha \rangle_\mu}\, B_\alpha(x) , \qquad f \in L^2_\mu(S^d) ,$$

where $\mathbf{1}$ denotes the function defined by setting $\mathbf{1}(x) = 1$.
Bernstein-Durrmeyer operators have been introduced by Durrmeyer [18] and have

been studied by Derriennic [13] (see also [14, 15, 25]). With respect to the weight w_μ, they were introduced by Păltănea [19] and studied by Berens and Xu [6, 7], Ditzian [16] and also by Berdysheva, Jetter and Stöckler [4], Waldron [24] and other authors.

We recall that $M_{n,\mu}$ is a positive contraction on $L^2_\mu(S^d)$ and is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_\mu$, i.e., for every $f, g \in L^2_\mu(S^d)$,

$$\langle M_{n,\mu} f, g\rangle_\mu = \langle f, M_{n,\mu} g\rangle_\mu \ .$$

Many other properties are listed in [16] (see also [4]); here, we only mention that $M_{n,\mu}\mathbf{1} = \mathbf{1}$ and that $\|M_{n,\mu}f - f\|_\mu$ is equivalent to the K-functional $K_\mu(f, 1/n)$ for every $f \in L^2_\mu(S^d)$ (see [16, (1.10)] for the definition and more details). It follows in particular that

$$\lim_{n\to+\infty} \|M_{n,\mu}f - f\|_\mu = 0 \ , \qquad f \in L^2_\mu(S^d) \ .$$

Moreover, the following Voronovskaja formula was established in [13, 14] (see also [4, Theorem B]) for every $f \in C^2(S^d)$, with respect to the L^2_μ convergence

$$\lim_{n\to+\infty} n(M_{n,\mu}f - f) = A_\mu f \ , \tag{2.1}$$

where $A_\mu : C^2(S^d) \to C(S^d)$ is defined by setting, for every $f \in C^2(S^d)$ and $x = (x_1, \ldots, x_d) \in S^d$ (see [4, (1.4)]),

$$\begin{aligned} A_\mu f(x) \ := \ & \sum_{0\le i<j\le d} \frac{1}{w_\mu(x_1,\ldots,x_d)} \times \\ & \times \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i}\right)\left(w_\mu(x_1,\ldots,x_d)\, x_i\, x_j \left(\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}\right)\right)(x) \ . \end{aligned}$$

It can be readily seen that A_μ has the following expression, for every $f \in C^2(S^d)$ and $x = (x_1, \ldots, x_d) \in S^d$,

$$A_\mu f(x) = \sum_{i,j=1}^d x_i(\delta_{ij} - x_j)\, \frac{\partial^2 f}{\partial x_i\, \partial x_j} + \sum_{i=1}^d b_i(x)\, \frac{\partial f}{\partial x_i} \ ,$$

where

$$b_i(x) := \mu_i + 1 - \sum_{j=0}^d (\mu_j + 1)\, x_i \ .$$

In particular we have

$$A_0 f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(x_i(\delta_{ij} - x_j)\frac{\partial f}{\partial x_j}(x)\right) \ .$$

Observe that the operator A_μ is densely defined, symmetric in $L^2_\mu(S^d)$ and commutes with the Bernstein-Durrmeyer operators (see [4, Lemma 2]). Moreover, the closure of $(A_\mu, C^\infty(S^d))$ in $L^2_\mu(S^d)$ generates a bounded analytic symmetric positive C_0-semigroup $(T_\mu(t))_{t\ge 0}$ (see [9]) which also satisfies the ultra-contractivity property [1]. For other properties of the operators A_μ we refer again to [4] and [15, 5, 12, 6, 7, 16, 8, 2].

Here, we are interested in some connections between the differential operator A_μ and some diffusion models in population genetics [21, 22].
We consider a linear subspace W of $L^2_\mu(S^d)$ and the projection of the operator A_μ onto W, obtaining the following evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = P_W A_\mu u(t,x) =: B_\mu u(t,x) \;, & t \ge 0 \;,\; x \in S^d \;, \\ u(0,x) = u_0(x) \;, & u_0 \in L^2_\mu(S^d) \;, \end{cases} \tag{2.2}$$

where P_W denotes the orthogonal projection onto W.
Our aim is to find suitable perturbations of the operators $M_{n,\mu}$ which allows us to connect them to the operator B_μ. The perturbation is performed by adding to $M_{n,\mu}$ suitable functionals.

We denote by V the subspace generated by the functions $x_i(\delta_{ij} - x_j)$, $i, j = 1, \dots, d$, and by W its orthogonal complement.

The choice of the subspace V is motivated by the fact that different models in population genetics lead to a differential operator where the coefficients of the first-order derivatives belong to V (some examples concerned with Wright-Fisher models can be found in [20, Examples 5.3–5.5]). In this way we consider a sequence of linear operators with a better approximation on the subspace W and presumably a better behavior of the Voronovskaja's formula which approximates the differential operator.

For every $n \ge 1$ we define the operator $L_n : L^2_\mu(S^d) \to L^2_\mu(S^d)$ by setting, for every $f \in L^2_\mu(S^d)$ and $x = (x_1, \dots, x_d) \in S^d$,

$$L_n f(x) := M_{n,\mu} f(x) + \sum_{1 \le i \le j \le d} x_i(\delta_{ij} - x_j)\, \mu_{ij}(f) \;,$$

where $\mu_{ij} : L^2_\mu(S^d) \to \mathbb{R}$, $1 \le i \le j \le d$, are suitable functionals.

We have the following result.

Theorem 2.1. *For every $1 \le i \le j \le d$, consider the functional $\mu_{ij} : L^2_\mu(S^d) \to \mathbb{R}$ defined by*

$$\begin{aligned} \mu_{ij}(f) &= \sum_{1 \le h \le k \le d} (\beta^{-1})_{ij,hk} \int_{S^d} x_h(\delta_{hk} - x_k)(f(x) - M_{n,\mu} f(x))\, w_\mu(x)\, dx \\ &= \sum_{1 \le h \le k \le d} (\beta^{-1})_{ij,hk} \,\langle x_h(\delta_{hk} - x_k), f - M_{n,\mu}(f) \rangle_\mu \;, \qquad f \in L^2_\mu(S^d) \;, \end{aligned}$$

where β^{-1} *denotes the inverse matrix of* $\beta := (\beta_{ij,hk})_{\substack{1\le i\le j\le d\\ 1\le h\le k\le d}}$ *where the* $d(d+1)/2$ *rows are indexed by* (i,j) *and the* $d(d+1)/2$ *columns by* (h,k) *and further*

$$\begin{aligned}\beta_{ij,hk} &:= \int_{S^d} x_i(\delta_{ij}-x_j)\, x_h(\delta_{hk}-x_k)\, w_\mu(x)\, dx \\ &= \langle \mathrm{pr}_i(\delta_{ij}-\mathrm{pr}_j)\, , \mathrm{pr}_h(\delta_{hk}-\mathrm{pr}_k)\rangle\mu\ .\end{aligned}$$

Then

1) L_n *is a approximation process;*
2) *The perturbations* $v := \sum_{1\le i\le j\le d} x_i(\delta_{ij}-x_j)\,\mu_{ij}(f)$ *solve the minimum problem* $\min_{v\in V} \|M_{n,\mu}(f)+v-f\|_\mu$ *;*
3) *For every* $f \in C^2(S^d)$ *we have*

$$\lim_{n\to+\infty} n(L_n f(x) - f(x)) = B_\mu f(x)$$

with respect to the L^2_μ*-norm.*

Proof. First we observe that the functions $x_i(\delta_{ij}-x_j)$, $i,j=1,\dots,d$, are linearly independent and the matrix β is invertible.
Now, let $f \in L^2_\mu(S^d)$ and define $u := \sum_{1\le i\le j\le d} x_i(\delta_{ij}-x_j)\,\mu_{ij}(f)$. Observe that $u = P_V(f - M_{n,\mu}(f))$; indeed for every $i,j = 1,\dots d$,

$$\begin{aligned}&\langle (f-M_{n,\mu}(f)) - u, x_i(\delta_{ij}-x_j)\rangle_\mu = \langle (f-M_{n,\mu}(f)), x_i(\delta_{ij}-x_j)\rangle_\mu \\ &\quad - \langle \sum_{1\le l\le m\le d} x_l(\delta_{lm}-x_m)\,\mu_{lm}(f), x_i(\delta_{ij}-x_j)\rangle_\mu \\ &= \langle (f-M_{n,\mu}(f)), x_i(\delta_{ij}-x_j)\rangle_\mu - \sum_{1\le l\le m\le d}\ \sum_{1\le h\le k\le d} (\beta^{-1})_{lm,hk}\times \\ &\quad \times \langle x_l(\delta_{lm}-x_m), x_i(\delta_{ij}-x_j)\rangle_\mu\, \langle x_h(\delta_{hk}-x_k), f-M_{n,\mu}(f)\rangle_\mu \\ &= \langle (f-M_{n,\mu}(f)), x_i(\delta_{ij}-x_j)\rangle_\mu \\ &\quad - \sum_{1\le l\le m\le d}\ \sum_{1\le h\le k\le d} (\beta^{-1})_{hk,lm}\beta_{lm,ij}\, \langle x_h(\delta_{hk}-x_k), f-M_{n,\mu}(f)\rangle_\mu \\ &= \langle (f-M_{n,\mu}(f)), x_i(\delta_{ij}-x_j)\rangle_\mu - \langle (f-M_{n,\mu}(f)), x_i(\delta_{ij}-x_j)\rangle_\mu = 0\ .\end{aligned}$$

Since V is a finite dimensional subspace of the Hilbert space $L^2_\mu(S^d)$, the previous properties characterize the orthogonal projection onto V, consequently the operator L_n becomes

$$L_n f = M_{n,\mu} f + P_V(f - M_{n,\mu}(f))\ .$$

Since $M_{n,\mu}$ is an approximation process, $\|P_V(f - M_{n,\mu} f)\|_\mu$ tends to zero as $n\to\infty$ and then also L_n is an approximation process.

From the properties of the orthogonal projection it follows that the perturbation u is such that

$$\begin{aligned}\|f - L_n(f)\|_\mu &= \|f - M_{n,\mu}(f) - u\|_\mu = \|f - M_{n,\mu}(f) - P_V(f - M_{n,\mu}(f))\|_\mu \\ &= \min_{v\in V} \|f - M_{n,\mu}(f) - v\|_\mu \ .\end{aligned}$$

As regards the statement 3), for every $f \in C^2(S^d)$ we have

$$\begin{aligned}\|n(L_n(f) - f)) - B_\mu f\|_\mu &= \|n(M_{n,\mu}f - f - P_V(M_{n,\mu}(f) - f)) - P_W(A_\mu f)\|_\mu \\ &= \|P_W(n(M_{n,\mu}f - f) - A_\mu f)\|_\mu \\ &\leq \|n(M_{n,\mu}f - f) - A_\mu f\|_\mu \ ,\end{aligned}$$

which converges to zero from (2.1). □

The operator $(B_\mu, C^2(S^d))$ is a A_μ-bounded perturbation of the operator A_μ and therefore its closure generates a C_0-semigroup $(S(t))_{t\geq 0}$ of positive operators on $L^2_\mu(S^d)$ and $C^2(S^d)$ is a core for this closure. Using Trotter's approximation theorem (see e.g. [3, Theorem 6.2.6, p. 436]) we obtain the following representation of the solution of the perturbed evolution problem.

Theorem 2.2. *For every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$\lim_{n\to+\infty} L_n^{k(n)} = S(t) \qquad \textit{strongly on } L^2_\mu(S^d) \ .$$

From the above representation it also follows that $(S(t))_{t\geq 0}$ is a contractions semigroup and the solution $u(t,x)$ of problem (2.2) can be expressed as follows

$$u(x,t) = S(t)u_0(x) = \lim_{n\to+\infty} L_n^{[nt]} u_0(x) \ ,$$

in the $L^2_\mu(S^d)$-norm with respect to $x \in S^d$ and uniformly in compact intervals with respect to $t \geq 0$.

In the case $d = 1$ we find the results of [10].

Using the above technique, we have the possibility of taking into account some external effects in the corresponding diffusion model in population genetics which produces a perturbation of the operator A_μ similar to that considered in the approximating operators L_n.

3. MODIFYING FUNCTIONALS TO BERNSTEIN-DURRMEYER OPERATORS

A different way of perturbing Bernstein-Durrmeyer operators consists in modifying some functionals in the standard decomposition of the Hilbert space $L^2_\mu(S^d)$

by means of spaces of orthogonal polynomials. If we denote by P_m the space of all polynomials of total degree less or equal to m, we recall that

$$L^2_\mu(S^d) = \sum_{m=0}^{+\infty} E_{m,\mu} \ , \tag{3.1}$$

where $E_{0,\mu} := P_0$ and, for every $m \geq 1$, $E_{m,\mu} := P_m \cap P^\perp_{m-1}$.

Theorem A of [4] states that $E_{m,\mu}$ is an eigenspace of $M_{n,\mu}$ for every $n, m \in \mathbb{N}$ and $M_{n,\mu} p_m = \gamma_{n,m,\mu} p_m$ for every polynomial $p_m \in E_{m,\mu}$, where

$$\gamma_{n,m,\mu} := \begin{cases} \dfrac{n!}{(n-m)!} \dfrac{\Gamma(n+d+|\mu|+1)}{\Gamma(n+d+|\mu|+m+1)} \ , & m \leq n \ , \\ 0 \ , & m > n \ . \end{cases}$$

In particular, for every $f = \sum_{m=0}^{+\infty} p_m$ with $p_m \in E_{m,\mu}$ depending on f (see (3.1)), we have

$$M_{n,\mu} f = \sum_{m=0}^{n} \gamma_{n,m,\mu} p_m = \sum_{m=0}^{n} \gamma_{n,m,\mu} P_{E_{m,\mu}}(f) \ ,$$

where $P_{E_{m,\mu}}$ denotes the canonical projection onto $E_{m,\mu}$ and consequently, for every $k \geq 1$,

$$M^k_{n,\mu} f = \sum_{m=0}^{n} \gamma^k_{n,m,\mu} p_m = \sum_{m=0}^{n} \gamma^k_{n,m,\mu} P_{E_{m,\mu}}(f) \ .$$

Now, we split the space $L^2_\mu(S^d)$ into the direct sum of two orthogonal subspaces and consider a combination of $M_{n,\mu}$ and the identity operator on these subspaces.

Namely, we fix a subset $J \subset \mathbb{N}$ and consider the subspace $E_{J,\mu} := \bigoplus_{m\in J} E_{m,\mu}$ and the operators $L_{n,\mu,J} : L^2_\mu(S^d) \to L^2_\mu(S^d)$ defined by

$$L_{n,\mu,J} f := M_{n,\mu} f - P_{E_{J,\mu}}(M_{n,\mu} f - f) \ , \qquad f \in L^2_\mu(S^d) \ .$$

We have

$$\begin{aligned} L_{n,\mu,J} f &= P_{E^\perp_{J,\mu}}(M_{n,\mu} f) + P_{E_{J,\mu}}(f) = \sum_{\substack{m=0 \\ m\notin J}}^{n} \gamma_{n,m,\mu} P_{E_{m,\mu}}(f) + \sum_{\substack{m=0 \\ m\in J}}^{n} P_{E_{m,\mu}}(f) \\ &= \sum_{m=0}^{n} \gamma^*_{n,m,\mu} P_{E_{m,\mu}}(f) \ , \end{aligned}$$

where

$$\gamma^*_{n,m,\mu} := \begin{cases} \gamma_{n,m,\mu} \ , & m \notin J \ , \\ 1 \ , & m \in J \ . \end{cases}$$

It is clear that the new sequence $(L_{n,\mu,J})_{n\geq 1}$ is an approximation process on $L^2_\mu(S^d)$. We are interested in a Voronovskaja-type formula for the operators $L_{n,\mu,J}$.

From [1] it follows that $\lim_{n\to\infty} n(1-\gamma_{n,m,\mu}) = 1$ and therefore for every $f \in C^2(S^d)$, we have

$$\begin{aligned}\lim_{n\to\infty} n(L_{n,\mu,J}f - f) &= \lim_{n\to\infty}\left(n(M_{n,\mu}f - f) + \sum_{m\in J} n(1-\gamma_{n,m,\mu})P_{E_{m,\mu}}(f)\right)\\ &= A_\mu f + P_{E_{J,\mu}}(f) =: \tilde{A}_\mu f\ .\end{aligned}$$

Hence, the differential operator $\tilde{A}_\mu$ arising from the Voronovskaja formula associated with the operators $L_{n,\mu,J}$ is a bounded perturbation of the operator A_μ. It follows that the closure of $(\tilde{A}_\mu, C^2(S^d))$ generates as well a C_0-semigroup $(U(t))_{t\geq 0}$ on $L^2_\mu(S^d)$ which can be represented by iterates of the operators $L_{n,\mu,J}$ as stated in the following result.

Theorem 3.1. *For every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim_{n\to+\infty} k(n)/n = t$, we have*

$$\lim_{n\to+\infty} L^{k(n)}_{n,\mu,J} = U(t) \qquad \textit{strongly on } L^2_\mu(S^d)\ .$$

Also in this case we have that $(U(t))_{t\geq 0}$ is a contractions semigroup.

As a particular case we consider for simplicity the projection onto $E_{1,\mu}$ which is generated by the polynomials

$$p_i(x) = x_i - \frac{\langle x_i, \mathbf{1}\rangle_\mu}{\langle \mathbf{1}, \mathbf{1}\rangle_\mu}\ , \qquad i = 1, \dots, d\ .$$

The general case of the projection onto $E_{m,\mu}$ can be treated following [17].

Taking into account the expression of the Beta functions given in [24]

$$B(\alpha) = \int_{S^d} w_{\alpha-1}(x)\,dx = \frac{\Gamma(\alpha)}{\Gamma(|\alpha|)} = \frac{\Gamma(\alpha_0)\cdots\Gamma(\alpha_d)}{\Gamma(\alpha_0+\alpha_1+\cdots+\alpha_d)}$$

for any multi-index $\alpha > 0$ and letting $\beta = (\beta_0, \beta_1, \dots, \beta_d)$ be the multi-index such that $\beta_i = 1$ and $\beta_j = 0$ for $j \neq i$, we obtain

$$\begin{aligned}p_i(x) &= x_i - \frac{B(\mu+1+\beta)}{B(\mu+1)}\\ &= x_i - \frac{\Gamma(\mu_0+1)\cdots\Gamma(\mu_{i-1}+1)\Gamma(\mu_i+2)\Gamma(\mu_{i+1}+1)\cdots\Gamma(\mu_d+1)}{\Gamma(\mu_0+\cdots+\mu_d+d+2)}\\ &\quad\times\frac{\Gamma(\mu_0+\cdots+\mu_d+d+1)}{\Gamma(\mu_0+1)\cdots\Gamma(\mu_d+1)}\\ &= x_i - \frac{\Gamma(\mu_i+2)}{\Gamma(\mu_i+1)}\frac{\Gamma(\mu_0+\cdots+\mu_d+d+1)}{\Gamma(\mu_0+\cdots+\mu_d+d+2)}\\ &= x_i - \frac{\mu_i+1}{\mu_0+\cdots\mu_d+d+1} =: x_i - b_i\ .\end{aligned}$$

Hence in this case we get the differential operator

$$\tilde{A}_\mu(f) = A_\mu(f) + \sum_{i=1}^{d}(x_i - b_i)\sum_{j=1}^{d}\left(B^{-1}\right)_{ij}\langle x_j - b_j, f\rangle_\mu$$

where B^{-1} denotes the inverse matrix of $B := \left(\langle x_i - b_i, x_j - b_j\rangle_\mu\right)_{\substack{i=1,\ldots,d\\ j=1,\ldots,d}}$.

In the case $d = 1$ we explicitly have

$$\tilde{A}_\mu f = A_\mu f + \frac{(x-b)}{\|\mathrm{id} - b\|_\mu^2}\int_0^1 (t-b)t^{\mu_1}(1-t)^{\mu_0} f(t)dt\ ,$$

with $b = \frac{\mu_0+1}{\mu_0+\mu_1+2}$, while in the case $d = 2$ we get

$$B^{-1} = \frac{1}{\|p_1\|_\mu^2\|p_2\|_\mu^2 - \langle p_1, p_2\rangle_\mu^2}\begin{pmatrix} \|p_2\|_\mu^2 & -\langle p_1, p_2\rangle_\mu \\ -\langle p_1, p_2\rangle_\mu & \|p_1\|_\mu^2 \end{pmatrix}$$

and

$$\begin{aligned}\tilde{A}_\mu(f) =& A_\mu(f) + Q_1(x)\int_{S_2}(t_1 - b_1)t_1^{\mu_1}t_2^{\mu_2}(1 - t_1 - t_2)^{\mu_0} f(t)\,dt \\ &+ Q_2(x)\int_{S_2}(t_2 - b_2)t_1^{\mu_1}t_2^{\mu_2}(1 - t_1 - t_2)^{\mu_0} f(t)\,dt\ ,\end{aligned}$$

where

$$Q_1(x) = (x_1 - b_1)B^{-1}_{1,1} + (x_2 - b_2)B^{-1}_{2,1}\ , \quad Q_2(x) = (x_1 - b_1)B^{-1}_{1,2} + (x_2 - b_2)B^{-1}_{2,2}\ .$$

If $\mu = 0$ we can write more explicitly

$$B^{-1} = \begin{bmatrix} 12 & 72/5 \\ 72/5 & 12 \end{bmatrix}$$

and $p_1(x) = x_1 - 1/3$, $p_2(x) = x_2 - 1/3$ so we have

$$\tilde{A}_0(f) = A_0(f) + Q_1(x)\int_{S_2}\left(t_1 - \frac{1}{3}\right) f(t)\,dt + Q_2(x)\int_{S_2}\left(t_2 - \frac{1}{3}\right) f(t)\,dt\ ,$$

where

$$Q_1(x) = 12\left(x_1 - \frac{1}{3}\right) + \frac{72}{5}\left(x_2 - \frac{1}{3}\right)\ , \quad Q_2(x) = \frac{72}{5}\left(x_1 - \frac{1}{3}\right) + 12\left(x_2 - \frac{1}{3}\right)\ .$$

All the above examples show how it is possible to add or modify the functionals in the definition of Bernstein-Durrmeyer operators in order to obtain a Voronovskaja formula which takes into account an integral term which may arise from suitable perturbations of the diffusion process associated with some evolution problems in population genetics. In every case we can represent the solution of the perturbed evolution problem by means of iterates of modified Bernstein-Durrmeyer operators.

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