

**MODIFIED BERNSTEIN-SCHNABL OPERATORS  
ON CONVEX COMPACT SUBSETS OF LOCALLY CONVEX  
SPACES AND THEIR LIMIT SEMIGROUPS**

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*Dedicated to Professor Espedito de Pascale on the occasion of his retirement*

**ABSTRACT.** In this paper we introduce a sequence  $(M_n)_{n \geq n_0}$  of positive linear operators as a modification of the Bernstein-Schnabl operators associated with a positive projection on  $C(K)$ , where  $K$  is a convex compact subset of a locally convex space; moreover we study its main approximation and qualitative properties.

Furthermore, we establish an asymptotic formula for those operators, and we prove that to the sequence  $(M_n)_{n \geq n_0}$  there corresponds a uniquely determined  $C_0$ -semigroup (in some special case a Feller one) which is representable as a limit of suitable powers of the operators.

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## 1. INTRODUCTION

In the last two decades, starting with the pioneer paper [2], F. Altomare and successively his school have developed a branch of researches which enlightens the interplay among suitable positive projections, sequences of positive linear operators and Feller semigroups. The results of such studies, updated to 1994, are fully expounded in the monograph [4], while more recent developments may be found in several successive papers among which we mention, without any sake of completeness, [8], [9], [10], [13], [14], [23], [24], [25], and the survey [7] with its numerous references.

Those kind of investigations have an application to a wide class of differential problems arising from physics, genetics, financial mathematics and other fields.

In more details, given a positive projection  $T$  on the space  $C(K)$  of all continuous real functions defined on a compact, convex and metrizable subset  $K$  of a locally

convex space, it is possible to construct a related sequence  $(B_n)_{n \geq 1}$  of positive linear operators on  $C(K)$ , namely the Bernstein-Schnabl operators.

That sequence, first introduced by Schnabl in [26], and successively analyzed by Grossmann [16] and Nishishiraho [18], [20], was intensively studied by F. Altomare (see [2] and also [4, Ch. 6]) who showed that, under suitable assumptions on  $T$ , to  $(B_n)_{n \geq 1}$  there corresponds a uniquely determined Feller semigroup which, in its turn, admits a representation in terms of iterates of the Bernstein-Schnabl operators themselves.

Such a representation may be fruitfully employed in the study of several abstract Cauchy problems and, in some finite-dimensional settings, it applies to many concrete diffusion problems; in fact, it allows us to express the solutions by means of Bernstein-Schnabl operators, and hence to infer some of their qualitative properties from the corresponding ones held by the  $B_n$ 's.

In the finite dimensional case the theory now sketched has been deepened with respect to differential problems governed by operators of the form

$$W_T(u)(x) := \sum_{i,j=1}^p \alpha_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \quad (x \in K)$$

for suitable coefficients  $\alpha_{ij} \in C(K)$  depending on the projection  $T$  and, apart from the one-dimensional setting, only in few special cases (see [12], [14], [15]) by "complete" second-order differential operators as

$$V_T(u)(x) := \sum_{i,j=1}^p \alpha_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^p \beta_i(x) \frac{\partial u}{\partial x_i}(x) + \gamma(x)u(x), \quad (x \in K) \quad (1.1)$$

for some continuous function  $\beta : K \rightarrow K$  and  $\gamma \in C(K)$ . In the recent paper [5] the authors, in collaboration with F. Altomare, focused their attention on differential operators of the general type (1.1) in arbitrary compact convex subsets of  $\mathbf{R}^p$ ,  $p \geq 2$ .

In the present work we investigate the more general case in which  $K$  is a convex compact subset of a locally convex space, also because this framework seems to be of some interest for possible applications to a wider class of problems.

The paper is organized as follows: after the basic notation and some preliminaries about the Bernstein-Schnabl operators we introduce the so-called modified Bernstein-Schnabl operators

$$M_n(f) := B_n \left( \left( 1 + \frac{\gamma}{n} \right) \left( f \circ \left( e + \frac{\beta}{n} \right) \right) \right) \quad (n \geq n_0, f \in C(K)),$$

where  $n_0$  is a suitable positive integer,  $e$  denotes the identity on  $K$ ,  $\beta$  and  $\gamma$  are the maps appearing in (1.1), and  $K$  is a convex, compact and metrizable subset of a locally convex space.

We prove that the sequence  $(M_n)_{n \geq n_0}$  is an approximation process on  $C(K)$ , and we estimate the rate of convergence of such approximation. Moreover, a Lipschitz preservation property of the  $M_n$ 's is analyzed.

The fourth section is mainly devoted to establish an asymptotic formula for the sequence  $(M_n)_{n \geq n_0}$  on a suitable dense subspace of  $C(K)$ . Such formula allows us to prove that to the sequence  $(M_n)_{n \geq n_0}$  there corresponds a  $C_0$ -semigroup (in some special case a Feller semigroup) which is representable as a limit of iterates of the  $M_n$ 's.

As a consequence, the solutions to the abstract Cauchy problems related to the generator of the semigroup admit a representation by means of powers of the  $M_n$ 's, so that we infer some regularity properties of the solutions by deriving them from the analogous ones of the operators.

## 2. NOTATION AND PRELIMINARIES

In this section we set the main notation of the paper and we recall the definition of the Bernstein-Schnabl operators associated with a positive projection (see [2], [4, Section 6.1]), together with their main properties.

Let  $K$  be a metrizable convex compact subset of a locally convex space  $X$  such that its interior  $\overset{\circ}{K}$  is nonempty; we denote by  $M^+(K)$  (resp.,  $M_1^+(K)$ ) the cone of all regular Borel measures on  $K$  (resp., the cone of all regular Borel probability measures on  $K$ ). For any  $\mu \in M^+(K)$  the support of  $\mu$ , i.e. the complement of the largest open subset of  $K$  on which  $\mu$  is zero, is denoted by  $\text{supp}(\mu)$ .

$C(K)$  stands for the space of all real-valued continuous functions on  $K$ ; it is a Banach lattice when endowed with the sup-norm  $\|\cdot\|_\infty$  and the natural (pointwise) ordering. Furthermore, by  $C(K, K)$  we mean the space of all continuous functions  $g : K \rightarrow K$ .

From now on,  $A(K)$  denotes the space of all continuous affine functions on  $K$ , and the symbols  $\mathbf{0}$  and  $\mathbf{1}$  stand for the constant functions on  $K$  taking value 0 and 1, respectively.

Finally, denote by  $X'$  the dual space of  $X$  and by

$$L(K) := \{\varphi|_K : \varphi \in X'\}. \tag{2.1}$$

In order to introduce the Bernstein-Schnabl operators associated with a positive projection we preliminarily set for every  $f \in C(K)$ ,  $z \in K$  and  $\alpha \in [0, 1]$

$$f_{z,\alpha}(x) := f(\alpha x + (1 - \alpha)z) \quad (x \in K). \tag{2.2}$$

Consider further a positive projection  $T : C(K) \rightarrow C(K)$ , that is a positive linear operator such that  $T \circ T = T$ , and suppose that its range

$$H := T(C(K))$$

satisfies the assumptions

$$A(K) \subset H, \tag{2.3}$$

i.e.  $T(h) = h$  for every  $h \in A(K)$ , and

$$h_{z,\alpha} \in H \text{ for every } z \in K, \alpha \in [0, 1] \text{ and } h \in H. \tag{2.4}$$

By means of such a projection, it is possible to construct a sequence of positive linear operators, as fully described in [2] (see also [4, Section 6.1]).

In fact, we remind that for every  $x \in K$  there exists a (unique)  $\mu_x^T \in M_1^+(K)$  satisfying

$$T(f)(x) = \int_K f \, d\mu_x^T \quad \text{for every } f \in C(K). \tag{2.5}$$

Therefore, for any  $n \geq 1$ , we consider the positive linear operator  $B_n : C(K) \rightarrow C(K)$  defined by setting for every  $f \in C(K)$  and  $x \in K$

$$B_n(f)(x) := \int_K \dots \int_K f \left( \frac{x_1 + \dots + x_n}{n} \right) d\mu_x^T(x_1) \dots d\mu_x^T(x_n). \tag{2.6}$$

$B_n$  is said to be the  $n$ -th Bernstein-Schnabl operator associated with the projection  $T$ .

It is easy to prove that  $\|B_n\| = 1$  for every  $n \geq 1$ ; moreover, the sequence  $(B_n)_{n \geq 1}$  is an approximation process on  $C(K)$  (for a proof, see [4, Theorem 6.1.10]), i.e. for every  $f \in C(K)$

$$\lim_{n \rightarrow +\infty} B_n(f) = f \tag{2.7}$$

uniformly on  $K$ .

Here we present some examples of projections satisfying (2.3) and (2.4) and their associated Bernstein-Schnabl operators; other examples may be found in [2], [4, Sect. 6.3].

**Examples 2.1.** 1. Let  $K$  be a metrizable Bauer simplex and consider the canonical projection  $T : C(K) \rightarrow C(K)$  associated with  $K$  defined for every  $f \in C(K)$  and  $x \in K$  by

$$T(f)(x) := \int_K f \, d\mu_x, \tag{2.8}$$

where  $\mu_x \in M_1^+(K)$  is the unique probability Borel measure on  $K$  which is concentrated on the set  $\partial_e K$  of all extreme points of  $K$  (i.e. such that  $\text{supp}(\mu_x) \subset \partial_e K$ ), and whose barycenter is  $x$  (see [4, Sect. 1.5]).

We recall that  $T$  is the unique positive projection on  $C(K)$  such that  $T(C(K)) = A(K)$  (see [4, Cor. 1.5.9]) and, consequently, conditions (2.3) and (2.4) are satisfied,

so that it is possible to associate with  $T$  the sequence of Bernstein-Schnabl operators defined as in (2.6).

In particular, if  $K$  is the canonical simplex  $K_p$  in  $\mathbf{R}^p$

$$K_p := \left\{ (x_1, \dots, x_p) \in \mathbf{R}^p : x_i \geq 0 \text{ for every } i = 1, \dots, p \text{ and } \sum_{i=1}^p x_i \leq 1 \right\},$$

then the positive projection in (2.8) is the canonical projection  $T_p : C(K_p) \rightarrow C(K_p)$  associated with  $K_p$  defined by

$$T_p(f)(x) := \left( 1 - \sum_{i=1}^p x_i \right) f(0) + \sum_{i=1}^p x_i f(e_i) \tag{2.9}$$

( $f \in C(K_p), x = (x_1, \dots, x_p) \in K_p$ ) where, for every  $i = 1, \dots, p$ ,  $e_i := (\delta_{i,j})_{1 \leq j \leq p}$ , and  $\delta_{i,j}$  is the Kronecker symbol (see, for instance, [4, Section 6.3.3]).

Finally, the Bernstein-Schnabl operators associated with  $T_p$  are the classical Bernstein operators on  $C(K_p)$ , i.e. for every  $f \in C(K_p)$ ,  $x = (x_1, \dots, x_p) \in K_p$  and  $n \geq 1$

$$B_n(f)(x) := \sum_{\substack{h_1, \dots, h_p=0, \dots, n \\ h_1 + \dots + h_p \leq n}} f\left(\frac{h_1}{n}, \dots, \frac{h_p}{n}\right) \frac{n!}{h_1! \dots h_p! (n - h_1 - \dots - h_p)!} \\ \times x_1^{h_1} \dots x_p^{h_p} \left( 1 - \sum_{i=1}^p x_i \right)^{n - \sum_{i=1}^p h_i}. \tag{2.10}$$

2. Let  $(K_i)_{1 \leq i \leq p}$  be a finite family of metrizable Bauer simplices and set  $K := \prod_{i=1}^p K_i$ . Consider the positive projection  $T : C(K) \rightarrow C(K)$  such that, for every  $f \in C(K)$  and  $x = (x_1, \dots, x_p) \in K$

$$T(f)(x) := \int_{K_1} \dots \int_{K_p} f(y_1, \dots, y_p) d\mu_{x_1}(y_1) \dots d\mu_{x_p}(y_p) \tag{2.11}$$

where, for every  $i = 1, \dots, p$ ,  $\mu_{x_i} \in M_1^+(K_i)$  is the probability Borel measure defined in (2.8).

The range  $H$  of the projection  $T$  is the space of all continuous functions on  $K$  which are affine with respect to each variable (see [1]), so that (2.3) and (2.4) are satisfied. Thus, it is possible to associate with the projection  $T$  the sequence of Bernstein-Schnabl operators as in (2.6).

In particular, if  $K = [0, 1]^p$ , then the projection in (2.11) is the positive projection  $S_p : C(K) \rightarrow C(K)$  defined by

$$S_p(f)(x) := \sum_{h_1, \dots, h_p=0}^1 f(\delta_{h_1,1}, \dots, \delta_{h_p,1}) x_1^{h_1} (1 - x_1)^{1-h_1} \dots x_p^{h_p} (1 - x_p)^{1-h_p}$$

for every  $f \in C(K)$  and  $x = (x_1, \dots, x_p) \in K$ .

Finally, the Bernstein-Schnabl operators associated with  $S_p$  are the Bernstein operators on  $C([0, 1]^p)$  defined by

$$B_n(f)(x) := \sum_{h_1, \dots, h_p=0}^n \binom{n}{h_1} \cdots \binom{n}{h_p} f\left(\frac{h_1}{n}, \dots, \frac{h_p}{n}\right) x_1^{h_1} (1-x_1)^{n-h_1} \cdots x_p^{h_p} (1-x_p)^{n-h_p} \quad (2.12)$$

for every  $f \in C([0, 1]^p)$ ,  $x = (x_1, \dots, x_p) \in [0, 1]^p$  and  $n \geq 1$ .

3. If  $x_0 \in \mathbf{R}^p$  and  $r > 0$ , denote by  $\Omega$  the open ball  $B(x_0, r)$  of center  $x_0$  and radius  $r$  and consider the Dirichlet operator  $T : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ , i.e., for every  $f \in C(\overline{\Omega})$ ,  $T(f)$  denotes the unique solution to the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{on } \Omega, \quad v \in C(\overline{\Omega}) \cap C^2(\Omega); \\ v|_{\partial K} = f|_{\partial K}. \end{cases}$$

More explicitly, for every  $f \in C(\overline{\Omega})$  and  $x \in \overline{\Omega}$

$$T(f)(x) = \begin{cases} \frac{r^2 - \|x - x_0\|_2^2}{r\sigma_p} \int_{\partial\Omega} \frac{f(z)}{\|z - x\|_2^p} d\sigma(z), & \text{if } \|x - x_0\|_2 < r; \\ f(x) & \text{if } \|x - x_0\|_2 = r, \end{cases}$$

where  $\|\cdot\|_2$  is the euclidean norm on  $\mathbf{R}^p$ , and  $\sigma_p$  and  $\sigma$  denote the surface area of the unit sphere in  $\mathbf{R}^p$  and the surface measure of  $\partial\Omega$ , respectively.

$T$  is a positive projection satisfying (2.3) and (2.4) (see [4, Corollary 3.3.6]) and the corresponding Bernstein-Schnabl operators are defined by

$$B_n(f)(x) = \left( \frac{r^2 - \|x - x_0\|_2^2}{r\sigma_p} \right)^n \int_{\partial\Omega} \cdots \int_{\partial\Omega} \frac{f\left(\frac{x_1 + \dots + x_n}{n}\right)}{\|x_1 - x\|_2^p \cdots \|x_n - x\|_2^p} d\sigma(x_1) \cdots d\sigma(x_n),$$

for every  $f \in C(\overline{\Omega})$ ,  $x \in \overline{\Omega}$  and  $n \geq 1$  (for more details, see [2], [4, Section 6.3.9], [11]).

There is a strong interplay between the sequence  $(B_n)_{n \geq 1}$  and certain Feller semigroups, in the sense that, by means of suitable iterates of the Bernstein-Schnabl operators, it is possible to represent the semigroups themselves.

In order to show this link, we need some further notation.

For every  $m \geq 1$ , let  $A_m$  be the linear subspace generated by all products of  $m$  affine functions, i.e.

$$A_m := \text{span} \left( \left\{ \prod_{i=1}^m h_i \mid h_1, \dots, h_m \in A(K) \right\} \right).$$

The sequence  $(A_m)_{m \geq 1}$  is increasing; therefore by the Stone-Weierstrass Theorem

$$A_\infty := \bigcup_{m=1}^{\infty} A_m$$

is a dense subalgebra of  $C(K)$ .

For every  $h_1, \dots, h_m \in A(K)$  set

$$L_T \left( \prod_{i=1}^m h_i \right) := \begin{cases} \mathbf{0} & \text{if } m = 1; \\ T(h_1 h_2) - h_1 h_2 & \text{if } m = 2; \\ \sum_{1 \leq i < j \leq m} (T(h_i h_j) - h_i h_j) \prod_{\substack{k=1 \\ k \neq i, j}}^m h_k & \text{if } m \geq 3. \end{cases} \quad (2.13)$$

The mapping  $L_T$  is involved in an asymptotic formula regarding the Bernstein-Schnabl operators, as the following result states (for a proof, see [4, Th. 6.2.1]).

**Theorem 2.2.** *The sequence  $(n(B_n(f) - f))_{n \geq 1}$  is uniformly convergent on  $K$  to a function  $Z^*(f) \in C(K)$  for every  $f \in A_\infty$ .*

Moreover, if  $h_1, \dots, h_m \in A(K)$ , then

$$Z^* \left( \prod_{i=1}^m h_i \right) = L_T \left( \prod_{i=1}^m h_i \right).$$

The previous theorem also infers that  $L_T$  may be extended to a linear operator from  $A_\infty$  into  $C(K)$  (see [4, Remark to Th. 6.2.1]) which, by an abuse of notation, we shall continue to denote by  $L_T$ .

In [4, Th. 6.2.6] it is shown that such an extension is closable and its closure, which has  $A_\infty$  as a core, is the generator of a Feller semigroup  $(T(t))_{t \geq 0}$  such that, for any  $t \geq 0$  and  $f \in C(K)$

$$T(t)(f) = \lim_{n \rightarrow \infty} B_n^{[nt]}(f)$$

uniformly on  $K$ , where  $[nt]$  stands for the integer part of  $nt$ .

Finally, we remark that if  $K$  is a convex compact subset in  $\mathbf{R}^p$ , then, as a matter of fact,  $A_\infty$  is the space of all polynomials on  $K$ ; in this case, an asymptotic formula for Bernstein-Schnabl operators holds true in the bigger subalgebra  $C^2(K)$  of all continuous real-valued functions on  $K$  which are twice continuously differentiable in  $\overset{\circ}{K}$  and whose partial derivatives up to the order 2 can be continuously extended to  $K$  (see [4, Th. 6.2.5]).

### 3. THE SEQUENCE OF MODIFIED BERNSTEIN-SCHNABL OPERATORS

This section is mainly devoted to the analysis of a modification of Bernstein-Schnabl operators which was first introduced in [3] and successively treated in [5] in finite-dimensional settings.

Under the same assumptions of the previous section, we consider a convex compact subset  $K$  of a locally convex space  $X$  and we fix  $\beta \in C(K, K)$  and  $\gamma \in C(K)$ ; moreover, we assume that there exists  $n_0 \geq 1$  such that

$$x + \frac{\beta(x)}{n} \in K \quad \text{and} \quad 1 + \frac{\gamma(x)}{n} \geq 0 \quad (3.1)$$

for every  $x \in K$  and  $n \geq n_0$ .

In what follows, we shall present some concrete settings in which (3.1) holds true.

**Examples 3.1.** 1. If  $\beta(x) := -\rho x + \rho c$  ( $x \in K$ ), where  $\rho \geq 0$  and  $c \in K$ , then (3.1) is satisfied for  $n \geq \max\{\|\gamma\|_\infty, \rho\}$ .

2. If  $K = [0, 1]^p$  is the hypercube of  $\mathbf{R}^p$ ,  $\beta$  has components  $\beta_1, \dots, \beta_p$  and if for every  $i = 1, \dots, p$

$$\theta_i := \sup_{\substack{x \in K \\ pr_i(x) \neq 0}} \frac{-\beta_i(x)}{pr_i(x)} < +\infty \quad \text{and} \quad \eta_i := \sup_{\substack{x \in K \\ pr_i(x) \neq 1}} \frac{\beta_i(x)}{1 - pr_i(x)} < +\infty,$$

then (3.1) holds true provided that  $n \geq \max\{-\theta_1, \dots, -\theta_p, \eta_1, \dots, \eta_p, \|\gamma\|_\infty\}$ .

3. If  $K$  is the canonical simplex of  $\mathbf{R}^p$  and  $\beta$  has components  $\beta_1, \dots, \beta_p$ , then (3.1) is satisfied for  $n \geq \max\{-\theta_1, \dots, -\theta_p, M, \|\gamma\|_\infty\}$ , where the  $\theta_i$ 's are defined as above and are supposed to be finite and

$$M := \sup_{\substack{x \in K \\ pr_1(x) + \dots + pr_p(x) < 1}} \frac{\beta_1(x) + \dots + \beta_p(x)}{1 - pr_1(x) - \dots - pr_p(x)} < +\infty.$$

For every  $n \geq n_0$  let  $M_n : C(K) \rightarrow C(K)$  be the positive linear operator defined by

$$M_n(f) := B_n \left( \left( 1 + \frac{\gamma}{n} \right) \left( f \circ \left( e + \frac{\beta}{n} \right) \right) \right) \quad (3.2)$$

for every  $f \in C(K)$ , where  $e(y) := y$  for every  $y \in K$  and  $B_n$  is the  $n$ -th Bernstein-Schnabl operator associated with a fixed positive projection  $T$  on  $C(K)$  satisfying (2.3) and (2.4).

More precisely, for every  $f \in C(K)$  and  $x \in K$

$$M_n(f)(x) = \int_K \dots \int_K \left( 1 + \frac{1}{n} \gamma \left( \frac{x_1 + \dots + x_n}{n} \right) \right) \times f \left( \frac{x_1 + \dots + x_n}{n} + \frac{1}{n} \beta \left( \frac{x_1 + \dots + x_n}{n} \right) \right) d\mu_x^T(x_1) \dots d\mu_x^T(x_n).$$

It is also possible to write explicitly the operators  $M_n$ 's in the particular settings of Examples 2.1 by simply replacing the term  $f \left( \frac{h_1}{n}, \dots, \frac{h_p}{n} \right)$  with

$$\left( 1 + \frac{1}{n} \gamma \left( \frac{h_1}{n}, \dots, \frac{h_p}{n} \right) \right) f \left( \frac{h_1}{n} + \frac{1}{n} \beta_1 \left( \frac{h_1}{n}, \dots, \frac{h_p}{n} \right), \dots, \frac{h_p}{n} + \frac{1}{n} \beta_p \left( \frac{h_1}{n}, \dots, \frac{h_p}{n} \right) \right)$$

in (2.10) and (2.12), where  $\beta \in C(K, K)$  has components  $\beta_1, \dots, \beta_p$ .



As an example, we express them for  $p = 1$  and  $p = 2$  (see [3, Section 3] and [10]); in the case of the interval  $[0,1]$ , for every  $f \in C([0, 1])$  and  $0 \leq x \leq 1$ ,

$$M_n(f)(x) = \sum_{k=0}^n \binom{n}{k} \left(1 + \frac{1}{n} \gamma \left(\frac{k}{n}\right)\right) f \left(\frac{k}{n} + \frac{1}{n} \beta_1 \left(\frac{k}{n}\right)\right) x^k (1-x)^{n-k}.$$

On the triangle  $K_2$  in  $\mathbf{R}^2$ , for every  $f \in C(K_2)$  and  $(x, y) \in K_2$

$$M_n(f)(x, y) = \sum_{\substack{h \geq 0, k \geq 0 \\ h+k \leq n}} \frac{n!}{h!k!(n-h-k)!} \left(1 + \frac{1}{n} \gamma \left(\frac{h}{n}, \frac{k}{n}\right)\right) \times f \left(\frac{h}{n} + \frac{1}{n} \beta_1 \left(\frac{h}{n}, \frac{k}{n}\right), \frac{k}{n} + \frac{1}{n} \beta_2 \left(\frac{h}{n}, \frac{k}{n}\right)\right) x^h y^k (1-x-y)^{n-h-k};$$

on the square  $Q_2$  in  $\mathbf{R}^2$ , for every  $f \in C(Q_2)$  and  $(x, y) \in Q_2$

$$M_n(f)(x, y) = \sum_{h,k=0}^n \binom{n}{h} \binom{n}{k} \left(1 + \frac{1}{n} \gamma \left(\frac{h}{n}, \frac{k}{n}\right)\right) \times f \left(\frac{h}{n} + \frac{1}{n} \beta_1 \left(\frac{h}{n}, \frac{k}{n}\right), \frac{k}{n} + \frac{1}{n} \beta_2 \left(\frac{h}{n}, \frac{k}{n}\right)\right) x^h (1-x)^{n-h} y^k (1-y)^{n-k}.$$

Finally, for all  $x_0 \in \mathbf{R}^p$  and  $r > 0$ , if  $\Omega$  denotes the open ball  $B(x_0, r)$  of center  $x_0$  and radius  $r$ , the modified Bernstein-Schnabl operators on  $C(\overline{\Omega})$  associated with the Dirichlet operator are defined as

$$M_n(f)(x) = \left(\frac{r^2 - \|x - x_0\|_2^2}{r\sigma_p}\right)^n \int_{\partial\Omega} \dots \int_{\partial\Omega} \left(1 + \frac{1}{n} \gamma \left(\frac{x_1 + \dots + x_n}{n}\right)\right) \times \frac{f \left(\frac{x_1 + \dots + x_n}{n} + \frac{1}{n} \beta \left(\frac{x_1 + \dots + x_n}{n}\right)\right)}{\|x_1 - x\|_2^p \dots \|x_n - x\|_2^p} d\sigma(x_1) \dots d\sigma(x_n),$$

for every  $f \in C(\overline{\Omega})$  and  $x \in \overline{\Omega}$  (see (1.14)).

We now investigate some qualitative properties of modified Bernstein-Schnabl operators.

First of all we recall the following definitions.

A function  $f \in C(K)$  is said to be  $T$ -convex if

$$f_{z,\alpha} \leq T(f_{z,\alpha}) \quad \text{for every } z \in K \text{ and } \alpha \in [0, 1],$$

where  $f_{z,\alpha}$  is defined as in (2.2) (see [4, p. 404]); it is easy to prove that any convex function  $f \in C(K)$  is  $T$ -convex.

Moreover, for every  $x \in K$  set

$$D_x := \{(u, v) \in K^2 \mid \text{there exist } \beta \geq 0 \text{ and } p, q \in \text{supp}(\mu_x^T) \text{ such that } u-v = \beta(p-q)\},$$

where  $\mu_x^T$  is defined in (2.5); we say that  $f \in C(K)$  is axially convex (see [4, p. 406]) whenever it is convex on  $D_x$  for any  $x \in K$ . In other words, a function is axially convex if it is convex on each segment parallel to a segment joining two points of  $\text{supp}(\mu_x^T)$  for every  $x \in K$ .

We also point out that, if  $K$  a Bauer simplex, then  $f \in C(K)$  is  $T$ -convex with respect to the projection  $T$  defined in (2.8) if and only if it is axially convex (see [4, Th. 6.3.2] or [23]).

For other examples of  $T$ -convex functions on particular finite-dimensional settings see [5, Sect. 3] and the references quoted therein.

In case  $K$  is an arbitrary convex compact subset of a locally convex space  $X$ , the sequence of Bernstein-Schnabl operators  $(B_n)_{n \geq 1}$  satisfies the inequalities

$$f \leq B_n(f) \leq T(f) \quad (n \geq 1)$$

for every  $T$ -convex function  $f \in C(K)$  ([23]; see also [4, Theorem 6.1.13]).

Hence, we obtain the following result, whose proof runs straightforwardly.

**Proposition 3.2.** *Under condition (3.1), assume further that  $\gamma$  is constant. If  $f \in C(K)$  and if  $f \circ \left(e + \frac{\beta}{n}\right)$  is  $T$ -convex for every  $n \geq n_0$ , then*

$$\left(1 + \frac{\gamma}{n}\right) \left(f \circ \left(e + \frac{\beta}{n}\right)\right) \leq M_n(f) \leq \left(1 + \frac{\gamma}{n}\right) T \left(f \circ \left(e + \frac{\beta}{n}\right)\right). \quad (3.3)$$

**Remark 3.3.** In the special case in which  $K$  is a metrizable Bauer simplex, if  $\beta$  is of the form  $\beta(x) = -\rho x + \rho c$  ( $x \in K$ ) for some  $\rho \geq 0$  and  $c \in K$ , for any axially convex function  $f \in C(K)$  also  $f \circ \left(e + \frac{\beta}{n}\right)$  is  $T$ -convex, and hence formula (3.3) holds true for every constant function  $\gamma$ . Moreover, each  $M_n(f)$  is axially convex, too ( $n \geq n_0$ ). The proof of those assertions runs analogously as in [5, Corollary 3.4, (1)], to which we refer for some further results concerning the behavior of the operators  $M_n$ 's on  $T$ -convex functions in finite-dimensional settings.

It is easy to show that the sequence  $(M_n)_{n \geq n_0}$  is equibounded, in fact  $\|M_n\| \leq 1 + \frac{\|\gamma\|_\infty}{n}$  for every  $n \geq n_0$ .

Moreover,  $(M_n)_{n \geq n_0}$  is an approximation process on  $C(K)$ , as the following result shows.

**Proposition 3.4.** *For every  $f \in C(K)$*

$$\lim_{n \rightarrow +\infty} M_n(f) = f$$

*uniformly on  $K$ .*

**Proof.** First, we observe that  $\{\mathbf{1}\} \cup L(K) \cup L^2(K)$  (see (2.1)) is a Korovkin set for  $C(K)$ , where  $L^2(K) := \{h^2 : h \in L(K)\}$ , (see [6, Th. 5.3 and the comments before it]). Then, it is sufficient to show that  $M_n(\mathbf{1}) \rightarrow \mathbf{1}$  uniformly on  $K$  and that

$$\lim_{n \rightarrow +\infty} M_n(h) = h \quad \text{and} \quad \lim_{n \rightarrow +\infty} M_n(h^2) = h^2$$

uniformly on  $K$  for every  $h \in L(K)$ .

Indeed, taking (2.7) into account, and since

$$M_n(\mathbf{1}) = \mathbf{1} + \frac{1}{n}B_n(\gamma),$$

and, for every  $h \in L(K)$

$$M_n(h) = B_n(h) + \frac{1}{n^2}B_n(\gamma(h \circ \beta)) + \frac{1}{n}B_n(h \circ \beta + \gamma h)$$

and

$$\begin{aligned} M_n(h^2) &= B_n(h^2) + \frac{1}{n}B_n(2h(h \circ \beta) + \gamma h^2) + \frac{1}{n^2}B_n(h^2 \circ \beta + 2\gamma h(h \circ \beta)) \\ &\quad + \frac{1}{n^3}B_n(\gamma(h^2 \circ \beta)), \end{aligned}$$

the previous assertion easily follows. □

In order to provide a rate of the pointwise as well as the uniform convergence of the approximating sequence  $(M_n)_{n \geq n_0}$ , we first introduce the following notation and assumptions.

For every  $\delta > 0$  and  $f \in C(K)$

$$\omega(f, \delta) := \sup\{|f(x) - f(y)| : x, y \in K, d(x, y) \leq \delta\} \tag{3.4}$$

stands for the first modulus of continuity of  $f$  with argument  $\delta > 0$ , where  $d$  is a distance on  $K$  inducing its topology.

Moreover, up to the end of this section we shall assume that

$$\begin{aligned} &d \text{ is induced by some distance } d_0 \text{ on } X \text{ such that} \\ &\text{for every } x, y, z \in X \quad \text{and} \quad k \in [0, 1] \\ &d_0(x + z, y + z) = d_0(x, y) \text{ and } d_0(kx, 0) \leq kd_0(x, 0). \end{aligned} \tag{3.5}$$

We explicitly notice that from the previous properties we deduce

$$d_0(kx, ky) \leq kd_0(x, y) \tag{3.6}$$

for every  $k \in [0, 1]$  and  $x, y \in X$ .

For some details and examples of locally convex spaces satisfying such properties we refer the interested reader to [19] and [22].

Finally, we set

$$M_\beta := \max_{x \in K} d_0(\beta(x), 0),$$

where by 0 we mean the neutral element of  $X$ .

Thus we state the following result.

**Proposition 3.5.** *Let  $f \in C(K)$ ; then, for every  $n \geq n_0$  and  $x \in K$*

$$|M_n(f)(x) - f(x)| \leq 2\Omega \left( \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right), \sqrt{\frac{1}{n}} \right) + \omega \left( f, \frac{M_\beta}{n} \right) + \frac{1}{n} \left| \gamma(x) f \left( x + \frac{\beta(x)}{n} \right) \right|,$$

where  $\Omega(g, \delta)$  is defined as in [4, (5.1.10)] for every  $\delta > 0$  and  $g \in C(K)$ . In particular, for every  $n \geq n_0$

$$\|M_n(f) - f\|_\infty \leq 2\Omega \left( \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right), \sqrt{\frac{1}{n}} \right) + \omega \left( f, \frac{M_\beta}{n} \right) + \frac{1}{n} \|\gamma\|_\infty \|f\|_\infty.$$

**Proof.** We show the pointwise estimation. Fix  $f \in C(K)$  and preliminarily notice that, by virtue of (3.5), for any  $n \geq n_0$  and  $x \in K$

$$\left| f \circ \left( e + \frac{\beta}{n} \right) (x) - f(x) \right| \leq \omega \left( f, \frac{M_\beta}{n} \right);$$

thus, for every  $n \geq n_0$  and  $x \in K$ ,

$$\begin{aligned} & |M_n(f)(x) - f(x)| \\ & \leq \left| B_n \left( \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right) \right) (x) - \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right) (x) \right| \\ & \quad + \left| \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right) (x) - f(x) \right| \\ & \leq \left| B_n \left( \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right) \right) (x) - \left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right) (x) \right| \\ & \quad + \omega \left( f, \frac{M_\beta}{n} \right) + \frac{1}{n} |\gamma(x)| \left| f \left( x + \frac{\beta(x)}{n} \right) \right|. \end{aligned}$$

Hence, taking estimate (6.1.58) of [4] into account, the result easily follows.  $\square$

We now pass to analyze some Lipschitz preservation properties of the  $M_n$ 's, and to this purpose we set, for any  $M \geq 0$  and  $0 < \alpha \leq 1$

$$Lip(M, \alpha) := \{f \in C(K) : |f(x) - f(y)| \leq Md(x, y)^\alpha \text{ for every } x, y \in K\}$$

and, similarly,

$$Lip_K(M, \alpha) := \{g \in C(K, K) : d(g(x), g(y)) \leq Md(x, y)^\alpha \text{ for every } x, y \in K\}.$$

In what follows we shall assume that the distance  $d$  verifies (3.5), and that for some  $c \geq 0$

$$T(f) \in Lip(c, 1) \quad \text{for every } f \in Lip(1, 1). \quad (3.7)$$

We recall that, under condition (3.7), it was shown in [23] (see also [4, Theorem 6.1.21]) that

$$B_n(f) \in Lip(cM, 1) \quad (3.8)$$

for every  $f \in Lip(M, 1)$  and  $n \geq 1$ . Moreover (see [23] or [4, Corollary 6.1.22]),

$$\omega(B_n(f), \delta) \leq (1 + c)\omega(f, \delta) \tag{3.9}$$

for every  $n \geq 1$ ,  $f \in C(K)$  and  $\delta > 0$  (see (3.4)).

We are now ready to state next result.

**Proposition 3.6.** *Suppose that condition (3.7) is satisfied, and that  $\beta \in Lip_K(C, 1)$  and  $\gamma \in Lip(N, 1)$  for some  $C, N \geq 0$ . Then, for every  $f \in Lip(M, 1)$  and  $n \geq n_0$*

$$M_n(f) \in Lip\left(cM\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\left(1 + \frac{C}{n}\right) + c\|f\|_\infty\frac{N}{n}, 1\right).$$

Moreover, if  $\gamma$  is constant, for any  $n \geq n_0$

$$M_n(f) \in Lip\left(cM\left(1 + \frac{|\gamma|}{n}\right)\left(1 + \frac{C}{n}\right), 1\right).$$

In particular, if  $\gamma = 0$  and  $\beta$  is constant, then for every  $n \geq n_0$

$$M_n(f) \in Lip(cM, 1).$$

**Proof.** By an easy calculation, and also taking (3.6) into account, we obtain for any  $n \geq n_0$

$$\left(1 + \frac{\gamma}{n}\right) f \circ \left(e + \frac{\beta}{n}\right) \in Lip\left(M\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\left(1 + \frac{C}{n}\right) + \|f\|_\infty\frac{N}{n}, 1\right)$$

and hence, by applying (3.8), the statement follows. □

Finally, by (3.9) we easily deduce next proposition.

**Proposition 3.7.** *Under assumption (3.7), suppose further that  $\beta \in Lip_K(C, 1)$  for some  $C \geq 0$ . Then, for every  $f \in C(K)$ ,  $\delta > 0$  and  $n \geq n_0$*

$$\omega(M_n(f), \delta) \leq (1 + c)\left(1 + \frac{\|\gamma\|_\infty}{n}\right)\omega\left(f, \left(1 + \frac{C}{n}\right)\delta\right) + \frac{(1 + c)\|f\|_\infty}{n}\omega(\gamma, \delta).$$

Moreover, if  $\gamma$  is constant, for any  $n \geq n_0$

$$\omega(M_n(f), \delta) \leq (1 + c)\left(1 + \frac{|\gamma|}{n}\right)\omega\left(f, \left(1 + \frac{C}{n}\right)\delta\right).$$

In particular, if  $\gamma = 0$  and  $\beta$  is a constant function, then for every  $n \geq n_0$

$$\omega(M_n(f), \delta) \leq (1 + c)\omega(f, \delta).$$

#### 4. AN ASYMPTOTIC FORMULA FOR MODIFIED BERNSTEIN-SCHNABL OPERATORS

In this section we establish an asymptotic formula for modified Bernstein-Schnabl operators which extends Theorem 2.2. To this end, we prove some preliminary results.

**Proposition 4.1.** *For any  $f \in C(K)$*

$$\lim_{n \rightarrow +\infty} f \circ \left( e + \frac{\beta}{n} \right) = f \quad \text{uniformly on } K.$$

**Proof.** For any  $n \geq n_0$ , consider the positive linear operator  $C_n : C(K) \rightarrow C(K)$  defined by setting, for every  $f \in C(K)$ ,

$$C_n(f) := f \circ \left( e + \frac{\beta}{n} \right).$$

As  $C_n(\mathbf{1}) = \mathbf{1}$ ,  $C_n(h) = h + \frac{1}{n}(h \circ \beta)$  and  $C_n(h^2) = h^2 + \frac{2}{n}h(h \circ \beta) + \frac{1}{n^2}h^2 \circ \beta$  for every  $h \in L(K)$ , and since  $\{\mathbf{1}\} \cup L(K) \cup L^2(K)$  is a Korovkin set for  $C(K)$  (see the proof of Proposition 3.4), we obtain the required assertion.  $\square$

From now on, for every  $m \geq 1$  we shall denote by

$$L_1(K) := L(K) \cup \{\mathbf{1}\},$$

and

$$L_m(K) := \left\{ \prod_{i=1}^m h_i : h_1, \dots, h_m \in L_1(K) \right\}.$$

Moreover, set

$$\mathfrak{L}_m(K) := \text{span}(L_m(K))$$

the linear space generated by  $L_m(K)$  and

$$\mathfrak{L}_\infty(K) := \bigcup_{m=1}^{\infty} \mathfrak{L}_m(K).$$

Observe that, since for every  $m \geq 1$   $\mathfrak{L}_m(K) \subset \mathfrak{L}_{m+1}(K)$ ,  $\mathfrak{L}_\infty(K)$  is a linear subspace of  $C(K)$ ; moreover it is also a subalgebra which, by the Stone-Weierstrass theorem, is dense in  $C(K)$ .

We now pass to define the following mapping

$$B(f) := \begin{cases} \mathbf{0} & \text{if } f = \mathbf{1}; \\ f \circ \beta & \text{if } f \in L(K); \\ \sum_{i=1}^m (h_i \circ \beta) \prod_{\substack{j=1 \\ j \neq i}}^m h_j & \text{if } f = \prod_{i=1}^m h_i, m \geq 2, h_1, \dots, h_m \in L(K), \end{cases} \quad (4.1)$$

and we introduce some further notation.

For every  $m, p \geq 1$ ,  $1 \leq p \leq m$ , set

$$N_m(p) := \{(i_1, \dots, i_p) \in \{1, \dots, m\}^p \mid i_r \neq i_s \text{ for } r \neq s\}$$

and

$$\begin{aligned} \tilde{N}_m := \{ & ((i_1, \dots, i_p), (j_1, \dots, j_{m-p})) \in N_m(p) \times N_m(m-p) \mid i_h \neq j_k \\ & \text{for every } h = 1, \dots, p, \text{ and } k = 1, \dots, m-p\}. \end{aligned}$$

Then, the following result holds true.

**Proposition 4.2.** *The sequence  $(n(f \circ (e + \frac{\beta}{n}) - f))_{n \geq n_0}$  is uniformly convergent on  $K$  for every  $f \in \mathfrak{L}_\infty(K)$ . In particular, if  $m \geq 1$  is an integer such that  $f \in L_m(K)$ , then*

$$\lim_{n \rightarrow +\infty} n \left( f \circ \left( e + \frac{\beta}{n} \right) - f \right) = B(f).$$

**Proof.** Without loss of generality, we may assume that  $f \in L_m(K)$ ,  $m \geq 1$ . The result is trivial if  $f \in L_1(K)$  as well as if  $f = h_1 h_2$  with  $h_1, h_2 \in L(K)$ ; otherwise, suppose that, for some  $m \geq 3$  and  $h_1, \dots, h_m \in L(K)$ ,  $f = \prod_{i=1}^m h_i$ .

Since for any  $n \geq n_0$

$$\begin{aligned} \prod_{i=1}^m h_i \circ \left( e + \frac{\beta}{n} \right) &= \prod_{i=1}^m \left( h_i + \frac{h_i \circ \beta}{n} \right) = \prod_{i=1}^m h_i + \frac{1}{n} \sum_{i=1}^m (h_i \circ \beta) \prod_{\substack{i=1 \\ i \neq j}}^m h_j \\ &+ \sum_{p=1}^{m-2} \frac{1}{n^{m-p}} \sum_{((i_1, \dots, i_p), (j_1, \dots, j_{m-p})) \in \tilde{N}_m} h_{i_1} \dots h_{i_p} (h_{j_1} \circ \beta) \dots (h_{j_{m-p}} \circ \beta) \\ &+ \frac{1}{n^m} \prod_{i=1}^m (h_i \circ \beta), \end{aligned}$$

we obtain

$$\begin{aligned} n \left[ \left( \prod_{i=1}^m h_i \right) \circ \left( e + \frac{\beta}{n} \right) - \prod_{i=1}^m h_i \right] &= B \left( \prod_{i=1}^m h_i \right) \\ &+ \sum_{p=1}^{m-2} \frac{1}{n^{m-p-1}} \sum_{((i_1, \dots, i_p), (j_1, \dots, j_{m-p})) \in \tilde{N}_m} h_{i_1} \dots h_{i_p} (h_{j_1} \circ \beta) \dots (h_{j_{m-p}} \circ \beta) \\ &+ \frac{1}{n^{m-1}} \prod_{i=1}^m (h_i \circ \beta). \end{aligned}$$

Therefore, the result easily follows. □

**Remark 4.3.** By virtue of the previous proposition, the mapping  $B$  defined in (4.1) may be extended to a linear operator from  $\mathfrak{L}_\infty(K)$  into  $C(K)$ . Accordingly, by an abuse of notation, we shall still denote such an extension by  $B$ .

We are now ready to state the preannounced asymptotic formula on  $\mathfrak{L}_\infty(K)$  for the modified Bernstein-Schnabl operators.

**Theorem 4.4.** *The sequence  $(n(M_n(f) - f))_{n \geq n_0}$  converges uniformly on  $K$  for every  $f \in \mathfrak{L}_\infty(K)$ .*

*In particular, if  $f \in L_m(K)$  for some  $m \geq 1$ , then*

$$\lim_{n \rightarrow +\infty} n(M_n(f) - f) = L_T(f) + B(f) + \gamma f,$$

where  $L_T$  is defined in (2.13) and  $B$  in (4.1).

**Proof.** Without loss of generality we limit ourselves to prove that for any  $f \in L_m(K)$  ( $m \geq 1$ ) the sequence  $(n(M_n(f) - f))_{n \geq n_0}$  is uniformly convergent on  $K$ .

The result is straightforward if  $f \in L_1(K)$ . Assume, instead, that  $f = \prod_{i=1}^m h_i$ , for some  $m \geq 2$  and  $h_1, \dots, h_m \in L(K)$ . We preliminarily show that

$$\lim_{n \rightarrow +\infty} n \left( B_n \left( f \circ \left( e + \frac{\beta}{n} \right) \right) - f \circ \left( e + \frac{\beta}{n} \right) \right) = L_T(f), \quad (1)$$

$$\lim_{n \rightarrow +\infty} \left( B_n \left( \gamma f \circ \left( e + \frac{\beta}{n} \right) \right) - \gamma f \circ \left( e + \frac{\beta}{n} \right) \right) = 0, \quad (2)$$

and

$$\lim_{n \rightarrow +\infty} n \left( \left( 1 + \frac{\gamma}{n} \right) f \circ \left( e + \frac{\beta}{n} \right) - f \right) = B(f) + \gamma f, \quad (3)$$

the three convergences being uniform on  $K$ .

In fact, starting with (1), the calculation made for the previous proposition leads to

$$\begin{aligned} & n \left( B_n \left( f \circ \left( e + \frac{\beta}{n} \right) \right) - f \circ \left( e + \frac{\beta}{n} \right) \right) = n \left[ B_n \left( \prod_{i=1}^m h_i \right) - \prod_{i=1}^m h_i \right] \\ & + \sum_{p=1}^{m-1} \frac{1}{n^{m-p-1}} \sum_{((i_1, \dots, i_p), (j_1, \dots, j_{m-p})) \in \tilde{N}_m} [B_n(h_{i_1} \dots h_{i_p}(h_{j_1} \circ \beta) \dots (h_{j_{m-p}} \circ \beta)) \\ & - h_{i_1} \dots h_{i_p}(h_{j_1} \circ \beta) \dots (h_{j_{m-p}} \circ \beta)] + \frac{1}{n^{m-1}} \left[ B_n \left( \prod_{i=1}^m (h_i \circ \beta) \right) - \prod_{i=1}^m (h_i \circ \beta) \right]; \end{aligned}$$

taking Theorem 2.2 and (2.7) into account, the result easily follows.

To prove (2), simply observe that

$$\begin{aligned} & B_n \left( \gamma f \circ \left( e + \frac{\beta}{n} \right) \right) - \gamma f \circ \left( e + \frac{\beta}{n} \right) \\ & = B_n \left( \gamma f \circ \left( e + \frac{\beta}{n} \right) - \gamma f \right) + (B_n(\gamma f) - \gamma f) + \left( \gamma f - \gamma f \circ \left( e + \frac{\beta}{n} \right) \right) \end{aligned}$$

where each term tends uniformly to zero by virtue of Proposition 4.1 and (2.7).

Finally from Propositions 4.1 and 4.2 we obtain (3).



Hence, by writing

$$\begin{aligned} n(M_n(f) - f) &= n \left( B_n \left( \left( 1 + \frac{\gamma}{n} \right) \left( f \circ \left( e + \frac{\beta}{n} \right) \right) \right) - f \right) \\ &= n \left( B_n \left( f \circ \left( e + \frac{\beta}{n} \right) \right) - f \circ \left( e + \frac{\beta}{n} \right) \right) \\ &\quad + \left( B_n \left( \gamma f \circ \left( e + \frac{\beta}{n} \right) \right) - \gamma f \circ \left( e + \frac{\beta}{n} \right) \right) \\ &\quad + n \left( \left( 1 + \frac{\gamma}{n} \right) f \circ \left( e + \frac{\beta}{n} \right) - f \right) \end{aligned}$$

and by using (1), (2) and (3), we get the required assertion. □

**Remark 4.5.** In the following section we shall often refer to the mapping  $f \in \mathfrak{L}_\infty \mapsto L_T(f) + B(f) + \gamma f \in C(K)$  as to the linear operator defined by means of the extensions of the functions  $L_T$  and  $B$  to  $\mathfrak{L}_\infty(K)$  (to this regard see comments after formula (2.13) and Remark 4.3).

We finally point out that if  $K$  is a convex compact subset of  $\mathbf{R}^p$ , then an asymptotic formula involving a complete second-order differential operator holds for functions belonging to a bigger space, namely  $C^2(K)$ . More precisely, if  $\beta = (\beta_1, \dots, \beta_p)$ , for every  $u \in C^2(K)$  and  $x = (x_1, \dots, x_p) \in K$

$$\lim_{n \rightarrow +\infty} n(M_n(u) - u)(x) = \frac{1}{2} \sum_{i,j=1}^p \alpha_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^p \beta_i(x) \frac{\partial u}{\partial x_i}(x) + \gamma(x)u(x)$$

uniformly w.r.t.  $x \in K$ , where

$$\alpha_{i,j}(x) = T(pr_i pr_j)(x) - x_i x_j,$$

$pr_i$  being the  $i^{th}$  coordinate function on  $K$ .

Hence, if  $K$  is a convex compact subset of  $\mathbf{R}^p$ , for every  $f \in \mathfrak{L}_\infty(K)$

$$L_T(f) + B(f) + \gamma f = \frac{1}{2} \sum_{i,j=1}^p \alpha_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^p \beta_i \frac{\partial f}{\partial x_i} + \gamma f.$$

To this regard, we refer the interested reader to [5, Th. 3.7] for more details.

### 5. $C_0$ -SEMIGROUPS ASSOCIATED WITH MODIFIED BERNSTEIN-SCHNABL OPERATORS

We show that, under suitable assumptions on the projection  $T$  and the function  $\beta$ , an operator arising from the asymptotic formula of Theorem 4.4 is the generator of a  $C_0$ -semigroup on  $C(K)$ . In its turn, that semigroup may be approximated by iterates of the modified Bernstein-Schnabl operators  $M_n$ 's.

To this purpose we remark that, by the iterate of  $M_n$  of order  $k \geq 1$  we mean

$$M_n^k := \begin{cases} M_n & \text{if } k = 1 \\ M_n \circ M_n^{k-1} & \text{if } k \geq 2. \end{cases}$$

From now on, we assume that there exist  $a, b \in \mathbf{R}$  and  $z \in K$  such that

$$ax + bz \in K \quad \text{for every } x \in K \quad (5.1)$$

and we define  $\beta : K \rightarrow K$  by setting

$$\beta(x) := ax + bz \quad (x \in K). \quad (5.2)$$

To this regard we observe that (5.1) occurs for any  $z \in K$  whenever  $a, b \geq 0$  and  $a + b \leq 1$ .

We also recall that a *core* for a linear operator  $L : D(L) \subset C(K) \rightarrow C(K)$  is a linear subspace  $D_0$  of  $D(L)$  which is dense in  $D(L)$  with respect to the graph norm  $\|u\|_L := \|L(u)\|_\infty + \|u\|_\infty$  ( $u \in D(L)$ ). If  $M$  is a bounded operator on  $C(K)$  and  $D_0$  is a core for  $L$ , then  $D_0$  is a core also for  $L + M$ , since  $\|\cdot\|_{L+M} \leq (1 + \|M\|)\|\cdot\|_L$ .

Denote by  $I$  the identity operator on  $C(K)$ ; if  $(L, D(L))$  is closed and  $\lambda I - L$  is invertible for some  $\lambda \in \mathbf{R}$ , then a subspace  $D_0$  of  $D(L)$  is a core for  $L$  if and only if  $(\lambda I - L)(D_0)$  is dense in  $C(K)$ .

We now state the following

**Theorem 5.1.** *Let  $T : C(K) \rightarrow C(K)$  be a positive projection such that (2.3) and (2.4) hold true, and assume that*

$$T(h_1 h_2) \in L(K) \quad \text{for every } h_1, h_2 \in L(K). \quad (5.3)$$

*Consider  $\beta \in C(K, K)$  defined as in (5.2),  $\gamma \in C(K)$  satisfying (3.1), and the sequence  $(M_n)_{n \geq n_0}$  of modified Bernstein-Schnabl operators associated with  $T$  introduced in (3.2). Then*

(1) *there exists a positive  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $C(K)$  such that*

$$\|T(t)\| \leq \exp(\|\gamma\|_\infty t) \quad \text{for every } t \geq 0; \quad (5.4)$$

*moreover, if  $\gamma \leq 0$ ,  $(T(t))_{t \geq 0}$  is a Feller semigroup, i.e.*

$$\|T(t)\| \leq 1 \quad \text{for every } t \geq 0.$$

(2) *If  $t \geq 0$  and  $(\rho_n)_{n \geq 1}$  is a sequence of positive integers such that  $\lim_{n \rightarrow +\infty} \frac{\rho_n}{n} = t$ , then*

$$\lim_{n \rightarrow +\infty} M_n^{\rho_n}(f) = T(t)(f) \quad (f \in C(K)) \quad (5.5)$$

*uniformly on  $K$ , where each  $M_n^{\rho_n}$  denotes the iterate of  $M_n$  of order  $\rho_n$ . In particular, for every  $f \in C(K)$ ,*

$$\lim_{n \rightarrow +\infty} M_n^{[nt]}(f) = T(t)(f)$$

uniformly on  $K$ , where  $[nt]$  denotes the integer part of  $nt$ .

- (3) The generator  $(A, D(A))$  of the semigroup  $(T(t))_{t \geq 0}$  is the closure of the linear operator  $Z : D(Z) \rightarrow C(K)$  defined by

$$Z(f) := \lim_{n \rightarrow +\infty} n(M_n(f) - f)$$

for every  $f \in D(Z)$ , where

$$D(Z) := \left\{ g \in C(K) \mid \lim_{n \rightarrow +\infty} n(M_n(g) - g) \text{ exists in } C(K) \right\};$$

- (4)  $\mathfrak{L}_\infty(K)$  is a core for  $(A, D(A))$  and

$$A(f) = L_T(f) + B(f) + \gamma f$$

for every  $f \in \mathfrak{L}_\infty(K)$ , where  $L_T$  and  $B$  are the extensions to  $\mathfrak{L}_\infty$  of the operators defined by (2.13) and (4.1), respectively (see also Remark (4.5)).

**Proof.** Let us consider the positive linear contractions  $L_n : C(K) \rightarrow C(K)$  defined by setting

$$L_n(f) := B_n \left( f \circ \left( e + \frac{\beta}{n} \right) \right) \quad (f \in C(K), n \geq n_0).$$

Moreover we introduce the linear operator  $(V, D(V))$  with domain

$$D(V) := \left\{ g \in C(K) \mid \lim_{n \rightarrow +\infty} n(L_n(g) - g) \text{ exists in } C(K) \right\},$$

and such that

$$V(f) := \lim_{n \rightarrow +\infty} n(L_n(f) - f)$$

for every  $f \in D(V)$ .

By virtue of Theorem 4.4,  $\mathfrak{L}_\infty(K) \subset D(V)$ .

We pass to prove that there exists  $\lambda > 0$  such that the range  $R(\lambda I - V)$  of  $\lambda I - V$  is dense in  $C(K)$ ; since  $\mathfrak{L}_\infty(K)$  is dense in  $C(K)$  it suffices to show that

$$\overline{(\lambda I - V)(\mathfrak{L}_\infty(K))} = C(K). \tag{1}$$

To this purpose consider  $\lambda > 0$  such that  $\lambda \neq -\frac{m(m-1)}{2} + am$  for every  $m \geq 1$ , and fix an arbitrary continuous linear functional  $\mu : C(K) \rightarrow \mathbf{R}$  such that  $\mu = 0$  on  $(\lambda I - V)(\mathfrak{L}_\infty(K))$ , i.e.

$$\mu(f) = \frac{1}{\lambda} \mu(V(f)) = \frac{1}{\lambda} (\mu(L_T(f)) + \mu(B(f)))$$

for every  $f \in \mathfrak{L}_\infty(K)$ ; by a consequence of Hahn-Banach theorem, (1) will be proved if we show that  $\mu = 0$ .

If  $f = \mathbf{1}$ , then

$$\mu(\mathbf{1}) = \frac{1}{\lambda} (\mu(L_T(\mathbf{1})) + \mu(B(\mathbf{1}))) = 0$$

(see (2.13) and (4.1)).

Observe that, if  $f \in L(K)$ , then

$$f \circ \beta = af + bf(z)\mathbf{1},$$

and thus

$$\mu(f) = \frac{1}{\lambda}(\mu(L_T(f)) + \mu(B(f))) = \frac{1}{\lambda}(a\mu(f) + bf(z)\mu(\mathbf{1})) = \frac{a}{\lambda}\mu(f),$$

so that, also in this case,  $\mu(f) = 0$ .

In order to examine the case in which  $f$  is the finite product of continuous linear functions, observe that for every  $m \geq 2$  and  $h_1, \dots, h_m \in L(K)$ ,

$$B\left(\prod_{i=1}^m h_i\right) = \sum_{i=1}^m (ah_i + bh_i(z)\mathbf{1}) \prod_{\substack{j=1 \\ j \neq i}}^m h_j = ma \prod_{i=1}^m h_i + b \sum_{i=1}^m h_i(z) \prod_{\substack{j=1 \\ j \neq i}}^m h_j. \quad (2)$$

Hence, if  $f = h_1 h_2$ , with  $h_1, h_2 \in L(K)$  then, taking (2.13), (5.3) and (2) into account, we have that

$$\begin{aligned} \mu(f) &= \frac{1}{\lambda}(\mu(T(f) - \mu(f) + \mu(B(f)))) \\ &= \frac{1}{\lambda}(-\mu(f) + 2a\mu(f) + bh_1(z)\mu(h_2) + bh_2(z)\mu(h_1)) \\ &= \frac{1}{\lambda}(-\mu(f) + 2a\mu(f)), \end{aligned}$$

and therefore  $\mu(f) = 0$ .

Let us now fix  $m > 2$  and suppose that  $\mu = 0$  on  $L_m(K)$ ; we shall prove that  $\mu = 0$  on  $L_{m+1}(K)$ . To this end, consider  $h_1, \dots, h_{m+1} \in L(K)$  and set  $f = \prod_{i=1}^{m+1} h_i$ ; then, by virtue of (2.13), (5.3) and (2),

$$\begin{aligned} \mu(f) &= \frac{1}{\lambda}(\mu(L_T(f)) + \mu(B(f))) \\ &= \frac{1}{\lambda}\mu\left(\sum_{1 \leq i < j \leq m+1} T(h_i h_j) \prod_{\substack{k=1 \\ k \neq i, j}}^{m+1} h_k - \binom{m+1}{2} f\right) \\ &\quad + \frac{1}{\lambda}\left((m+1)a\mu(f) + b \sum_{i=1}^{m+1} h_i(z)\mu\left(\prod_{\substack{j=1 \\ j \neq i}}^{m+1} h_j\right)\right) \\ &= \frac{1}{\lambda}\left(-\frac{m(m+1)}{2} + a(m+1)\right)\mu(f). \end{aligned}$$

Accordingly,  $\mu(f) = 0$ ; hence, by induction,  $\mu = 0$  on each  $L_m(K)$ ,  $m \geq 1$ , and thus, by the linearity of  $\mu$ , on every  $\mathfrak{L}_m(K)$  and therefore  $\mu = 0$  on  $\mathfrak{L}_\infty(K)$ . Since  $\overline{\mathfrak{L}_\infty(K)} = C(K)$ , we conclude that  $\mu = 0$ , so that (1) holds true.

By virtue of a Trotter's theorem ([28]; see also [21, Chapter 3, Theorem 6.7]) there exists a strongly continuous positive contraction semigroup  $(S(t))_{t \geq 0}$  on  $C(K)$

such that for every  $t \geq 0$  and  $f \in C(K)$

$$S(t)(f) = \lim_{n \rightarrow +\infty} L_n^{\rho_n}(f)$$

uniformly on  $K$  for every sequence  $(\rho_n)_{n \geq 1}$  of positive integers such that  $\lim_{n \rightarrow +\infty} \frac{\rho_n}{n} = t$ , and the generator  $(W, D(W))$  of  $(S(t))_{t \geq 0}$  is the closure of  $(V, D(V))$ .

Moreover,  $W = V = L_T + B$  on  $\mathfrak{L}_\infty(K)$ .

Consequently, by (1) it follows that

$$\overline{(I - \lambda W)(\mathfrak{L}_\infty(K))} = \overline{(I - \lambda V)(\mathfrak{L}_\infty(K))} = C(K),$$

and thus  $\mathfrak{L}_\infty(K)$  is a core for  $(W, D(W))$ .

Consider now the bounded operator  $\Gamma$  on  $C(K)$  defined by  $\Gamma(f) := \gamma f$  ( $f \in C(K)$ ) and set

$$A := W + \Gamma$$

defined on  $D(A) := D(W)$ . Then, by virtue of a perturbation result (see [21, Corollary 1.3]) the operator  $A$  generates a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  of operators such that

$$\|T(t)\| \leq \exp(\|\gamma\|_\infty t) \quad (t \geq 0).$$

If  $\gamma \leq 0$ , then  $\Gamma$ , as well as  $W$ , satisfies the positive maximum principle (see, e.g., [27, Theorem 9.3.3]). Hence also  $A$  satisfies the positive maximum principle, so that  $\|T(t)\| \leq 1$  for every  $t \geq 0$  (see, e.g., [27, Corollary 9.3.6]). Moreover  $\mathfrak{L}_\infty(K)$  is a core for  $A$  and hence  $(\lambda I - A)(\mathfrak{L}_\infty(K))$  is dense in  $C(K)$  for any  $\lambda > \|\gamma\|_\infty$ .

Therefore, as

$$Z(f) = V(f) + \gamma f = W(f) + \gamma f = A(f) \tag{3}$$

for every  $f \in \mathfrak{L}_\infty(K)$ , also  $(\lambda I - Z)(\mathfrak{L}_\infty(K))$  is dense in  $C(K)$  for any  $\lambda > \|\gamma\|_\infty$ .

Furthermore, as  $\|M_n\| \leq 1 + \frac{\|\gamma\|_\infty}{n} \leq \exp\left(\frac{\|\gamma\|_\infty}{n}\right)$  for every  $n \geq n_0$ ,

$$\|M_n^k\| \leq \exp\left(\frac{\|\gamma\|_\infty k}{n}\right) \quad (k \geq 1).$$

By applying again Trotter's theorem we deduce that the operator  $(Z, D(Z))$  is closable and its closure  $(\tilde{A}, D(\tilde{A}))$  generates a  $C_0$ -semigroup  $(\tilde{T}(t))_{t \geq 0}$  on  $C(K)$  satisfying (5.4) and (5.5).

Finally, the statement will be completely proved once we recognize that  $\tilde{T}(t) = T(t)$  for every  $t \geq 0$  or, equivalently, that  $(\tilde{A}, D(\tilde{A})) = (A, D(A))$ .

From (3) it follows that  $D(\tilde{A}) \subset D(A)$  and  $A = \tilde{A}$  on  $D(\tilde{A})$ . Conversely, since  $\mathfrak{L}_\infty(K)$  is a core for  $A$ , for every  $u \in D(A)$  there exists a sequence  $(u_n)_{n \geq 1}$  in  $\mathfrak{L}_\infty(K) \subset D(\tilde{A})$  such that  $u_n \rightarrow u$  and  $A(u_n) \rightarrow A(u)$ . Since  $A(u_n) = \tilde{A}(u_n)$  ( $n \geq 1$ ) and  $\tilde{A}$  is closed,  $u \in D(\tilde{A})$  and  $\tilde{A}(u) = A(u)$ . □

We remark that an example of projection satisfying (5.3) is provided by the canonical projection on the  $p$ -dimensional simplex introduced by (2.9).

Moreover, other generation and approximation results obtained in finite dimensional settings, together with some applications of theirs, may be found in [5, Sections 4 and 5].

As a consequence of Theorem 5.1 let us now consider the abstract Cauchy problem related to  $(A, D(A))$

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = A(u(\cdot, t))(x) & x \in K, \quad t \geq 0, \\ u(x, 0) = u_0(x) & u_0 \in D(A), \quad x \in K. \end{cases}$$

As  $(A, D(A))$  is the generator of a  $C_0$ -semigroup, the Cauchy problem admits a unique solution  $u : K \times [0, +\infty[ \rightarrow \mathbf{R}$  given by  $u(x, t) = T(t)(u_0)(x)$  for every  $x \in K$  and  $t \geq 0$  (see, e.g., [17, Chapter A-II]). Hence, it is possible to approximate such solution by means of iterates of the modified Bernstein-Schnabl operators as follows:

$$u(x, t) = T(t)(u_0)(x) = \lim_{n \rightarrow +\infty} M_n^{[nt]}(u_0)(x), \quad (5.6)$$

the limit being uniform with respect to  $x \in K$ .

This latter allows us to infer some regularity properties for the solution  $u(x, t)$ ; in fact, assuming that  $\gamma$  is constant,  $\beta \in Lip_K(C, 1)$ , for some  $C \geq 0$  and (3.5) holds true, if  $T(Lip(1, 1)) \subset Lip(1, 1)$  and  $u_0 \in Lip(M, 1)$  for some  $M \geq 0$ , then

$$u(\cdot, t) \in Lip(M \exp(|\gamma| + C)t, 1) \quad \text{for every } t \geq 0. \quad (5.7)$$

To show (5.7) it suffices to make use of Proposition 3.6, and argue as done below formula (4.16) in [5].

Finally we observe that  $\beta$  is Lipschitz-continuous in case (3.5) holds true and  $a \in [0, 1]$ .

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