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# OSCILLATION THEORY FOR IMPULSIVE PARTIAL DIFFERENCE EQUATIONS

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**ABSTRACT.** In this paper, we consider impulsive partial difference equation with continuous variables of the form

$$\begin{aligned} p_1 z(x+a, y+b) + p_2 z(x+a, y) + p_3 z(x, y+b) - p_4 z(x, y) \\ + P(x, y) z(x-\tau, y) + Q(x, y) z(x, y-\sigma) \\ + R(x, y) z(x-\tau, y-\sigma) = 0, \qquad (x, y) \in (\mathbb{R}^+ \times \mathbb{R}^+) \backslash J, \\ z(x_n^+, y) - z(x_n^-, y) = L_n z(x_n^-, y), \quad (x_n, y) \in J. \end{aligned}$$

Sufficient conditions for all solutions of this equation to be oscillatory are established.

**Key Words:** Oscillation; Partial difference equation; Impulsive differential equation; Continuous variable.

**AMS Subject Classification:** 34K11, 34K45

## 1. INTRODUCTION

Partial difference equations arise in applications involving population dynamics with spatial migrations, chemical reactions, mathematical physics, as well as finite difference schemes [1, 8, 10, 12]. The qualitative theory of partial difference equations has received much attention in the past few years (see the survey paper [13] and the references cited therein). In particular, oscillation of partial difference equations with continuous variables has been investigated in the papers [2, 7, 14–16]. On the other hand, impulsive differential equations appear as a natural description of observed evolution phenomena of several real world problems [9,11]. But, only a few papers have been published on the oscillation of impulsive partial differential-difference equations [3–6]. The purpose of this paper is to establish sufficient conditions for the oscillation

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of solutions of a certain impulsive partial difference equation with continuous variables. We note that our results generalize some known theorems for partial difference equation to impulsive partial difference equations [2, 7].

Let $0 < x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots$ be fixed points with $\lim_{n\to\infty} x_n = \infty$. Define $J_{imp} = \{x_n\}_{n=1}^{\infty}$, $\mathbb{R}^+ = [0,\infty)$, $J = \left\{(x,y) : x \in J_{imp},\ y \in \mathbb{R}^+\right\}$. In this paper we shall consider the following impulsive partial difference equation with continuous variables

$$\begin{aligned} p_1 z(x+a,y+b) + p_2 z(x+a,y) + p_3 z(x,y+b) - p_4 z(x,y) \\ + P(x,y)z(x-\tau,y) + Q(x,y)z(x,y-\sigma) \\ + R(x,y)z(x-\tau,y-\sigma) = 0, \quad (x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash J, \end{aligned} \tag{1}$$

$$z(x_n^+,y) - z(x_n^-,y) = L_n z(x_n^-,y), \quad (x_n,y) \in J, \tag{2}$$

where $z(x_n^+,y) = \lim\limits_{\substack{(q,s)\to(x_n,y) \\ q>x_n}} z(q,s)$, $z(x_n^-,y) = \lim\limits_{\substack{(q,s)\to(x_n,y) \\ q<x_n}} z(q,s)$.

In what follows, we shall assume that the following conditions are satisfied:

(i) $p_1 \geq 0$, $p_2, p_3 \geq p_4 > 0$ are real constants, $P$, $Q$, $R \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+\backslash\{0\})$,

(ii) $\{L_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\prod\limits_{n=1}^{\infty}(1+L_n) = L < \infty$,

(iii)

$$\begin{aligned} U(x,y) &= \min\{P(s,t) : x \leq s \leq x+a,\ y \leq t \leq y+b\}, \\ V(x,y) &= \min\{Q(s,t) : x \leq s \leq x+a,\ y \leq t \leq y+b\}, \\ W(x,y) &= \min\{R(s,t) : x \leq s \leq x+a,\ y \leq t \leq y+b\}, \\ F(x,y) &= \min\{U(x,y),\ V(x,y),\ W(x,y)\}, \end{aligned}$$

and $0 < \limsup\limits_{x,y\to\infty} F(x,y) < \infty$.

**Definition 1.** A function $z : [-\tau,\infty) \times [-\sigma,\infty) \to \mathbb{R}$ is called a solution of (1)–(2) if

(a) for $(x,y) \in (\mathbb{R}^+ \times \mathbb{R}^+)\backslash J$, $z$ is continuous and satisfies (1),

(b) for $(x,y) \in J$, $z(x^+,y)$ and $z(x^-,y)$ exist, $z(x^-,y) = z(x,y)$, and satisfy (2).

**Definition 2.** A solution $z(x,y)$ of (1)–(2) said to be eventually positive if $z(x,y) > 0$ for all large $x$ and $y$, eventually negative if $z(x,y) < 0$ for all large $x$ and $y$. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

## 2. MAIN RESULTS

For $(x,y) \in R^+ \times R^+$, define a set $E$ by

$$E = \left\{ \lambda > 0 : p_4 \prod_{x_0<x_m<x+a} (1+L_m) - \lambda F(x,y) > 0, \text{ eventually} \right\}. \tag{3}$$

Here, the symbol $\prod\limits_{x_0<x_m<u} a_m$ denotes the product of the members of the sequence $\{a_m\}$ over $m$ such that $x_m \in (x_0,u)\cap J_{imp}$. If $(x_0,u)\cap J_{imp}=\emptyset$, or $x_0 \ge u$, then we assume that $\prod\limits_{x_0<x_m<u} a_m = 1$. Let $\tau = ka+\theta$, $\sigma = lb+\eta$, where $k,\ l$ are nonnegative integers, $a>0$, $b>0$ are real numbers, and $\theta\in[0,a)$, $\eta\in[0,b)$. The proof of the following Lemmas 1 and 2 is similar to that of Lemma 1 in [3], and hence we omit the details.

**Lemma 1.** Assume that $z(x,y)$ be an eventually positive solution of (1)–(2). Define

$$w(x,y)=\int\limits_x^{x+a}\int\limits_y^{y+b}\left(\prod_{x_0<x_m<u}(1+L_m)^{-1}\right)z\,(u,v)\,dv\,du, \tag{4}$$

then $w(x,y)>0$, $\frac{\partial w}{\partial x}\le 0$, $\frac{\partial w}{\partial y}\le 0$ for all large $x$ and $y$.

**Lemma 2.** Assume that $z(x,y)$ be an eventually positive solution of (1)–(2). Then $w(x,y)$ defined in (4) is an eventually positive solution of the following difference inequality

$$\begin{aligned} p_1w(x+a,y+b)&+p_2w(x+a,y)+p_3w(x,y+b)\\ &-p_4\left(\prod_{x_0<x_m<x+a}(1+L_m)\right)w(x,y)\\ &+U(x,y)w(x-\tau,y)+V(x,y)w(x,y-\sigma)\\ &+W(x,y)w(x-\tau,y-\sigma)\le 0. \end{aligned}\tag{5}$$

Moreover,

$$\begin{aligned} p_1w(x+a,y+b)&+p_2w(x+a,y)+p_3w(x,y+b)\\ &-p_4\left(\prod_{x_0<x_m<x+a}(1+L_m)\right)w(x,y)\\ &+U(x,y)w(x-ka,y)+V(x,y)w(x,y-lb)\\ &+W(x,y)w(x-ka,y-lb)\le 0. \end{aligned}\tag{6}$$

**Lemma 3.** Let $a=b$. Assume that Eq. (1)–(2) have an eventually positive solution. Then for $Y\ge y_0$, inequalities

$$\begin{aligned} &\left(p_1+\frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x+a}(1+L_m)^{-1}\right)u(x)\\ &\quad-\left(p_4\prod_{x_0<x_m<x}(1+L_m)-\inf_{y\ge Y}U(x-a,y-a)\right)u(x-a)\le 0, \end{aligned}\tag{7}$$

and

$$\left(p_1+\frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x+a}(1+L_m)^{-1}\right)u(x)$$

$$- \left( p_4 \prod_{x_0 < x_m < x} (1 + L_m) - \inf_{y \ge Y} V(x-a, y-a) \right) u(x-a) \le 0 \tag{8}$$

have eventually positive solutions.

**Proof.** Let $z(x,y)$ be an eventually positive solution of (1)–(2). Then by Lemma 2, for $a = b$,

$$\begin{aligned} p_1 w(x+a, y+a) + p_2 w(x+a, y) + p_3 w(x, y+a) \\ - p_4 \left( \prod_{x_0 < x_m < x+a} (1 + L_m) \right) w(x,y) \\ + U(x,y) w(x-\tau, y) + V(x,y) w(x, y-\sigma) \\ + W(x,y) w(x-\tau, y-\sigma) \le 0 \end{aligned} \tag{9}$$

has an eventually positive solution. This inequality implies that

$$p_2 w(x+a, y) < p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) w(x,y)$$

and

$$p_3 w(x, y+a) < p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) w(x,y),$$

i.e.,

$$p_2 w(x,y) < p_4 \prod_{x_0 < x_m < x} (1 + L_m) w(x-a, y)$$

and

$$p_3 w(x,y) < p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) w(x, y-a).$$

Using above inequalities in (9) we obtain

$$\begin{aligned} \left( p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0 < x_m < x+a} (1 + L_m)^{-1} \right) w(x,y) \\ - p_4 \prod_{x_0 < x_m < x} (1 + L_m) w(x-a, y-a) \\ + U(x-a, y-a) w(x-\tau-a, y-a) \le 0. \end{aligned} \tag{10}$$

Let

$$u(x) = \int_{x-a}^{x} w(x,y) dy.$$

Integrating (10) with respect to $y$ from $x-a$ to $x$ we get

$$\begin{aligned} \left( p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0 < x_m < x+a} (1 + L_m)^{-1} \right) u(x) \\ - p_4 \prod_{x_0 < x_m < x} (1 + L_m) u(x-a) + \inf_{y \ge Y} U(x-a, y-a) u(x-a) \le 0, \end{aligned}$$

eventually. So, $u(x)$ is a positive solution of (7). Similarly it can be shown that $u(x)$ is a positive solution of (8). □

**Lemma 4.** Let $a = b$. Assume that Eq. (1)–(2) have an eventually positive solution. Then for $Y \geq y_0$, inequalities

$$\left(p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0<x_m<x+a} (1+L_m)^{-1}\right) u(x) - p_4 \prod_{x_0<x_m<x} (1+L_m) u(x-a) + \inf_{y\geq Y} W(x-a, y-a) u(x-\sigma-a) \leq 0, \quad \text{if } \tau \geq \sigma > 0, \tag{11}$$

and

$$\left(p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0<x_m<x+a} (1+L_m)^{-1}\right) u(x) - p_4 \prod_{x_0<x_m<x} (1+L_m) u(x-a) + \inf_{y\geq Y} W(x-a, y-a) u(x-\tau-a) \leq 0, \quad \text{if } \sigma \geq \tau > 0, \tag{12}$$

have eventually positive solutions.

We shall now prove the following two results which modify Theorems 1 and 3 in [3] respectively.

**Theorem 1.** Assume that there exist $X \geq x_0$, $Y \geq y_0$ such that if $k > l > 0$,

$$\sup_{\lambda\in E,\ x\geq X,\ y\geq Y} \Bigg\{ \lambda \prod_{i=1}^{l} \left(p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0<x_m<x-(i-2)a} (1+L_m)^{-1}\right)^{-1} \times \prod_{i=1}^{l} \left(p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \lambda F(x-ia, y-ib)\right) \times \prod_{j=1}^{k-l} \left(p_4 \prod_{x_0<x_m<x-(l+j-1)a} (1+L_m) - \lambda F(x-(l+j)a, y-lb)\right) \Bigg\} < p_2^{k-l}, \tag{13}$$

and if $l > k > 0$,

$$\sup_{\lambda\in E,\ x\geq X,\ y\geq Y} \Bigg\{ \lambda \prod_{i=1}^{k} \left(p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0<x_m<x-(i-2)a} (1+L_m)^{-1}\right)^{-1} \times \prod_{i=1}^{k} \left(p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \lambda F(x-ia, y-ib)\right) \times \prod_{j=1}^{l-k} \left(p_4 \prod_{x_0<x_m<x-(k-1)a} (1+L_m) - \lambda F(x-ka, y-(k+j)b)\right) \Bigg\} < p_3^{l-k}. \tag{14}$$

Then every solution of (1)–(2) is oscillatory.

**Proof.** Suppose, to the contrary, that there is a nonoscillatory solution of (1)–(2). Without lose of generality we may assume that $z(x, y)$ is an eventually positive solution of (1)–(2). Let $w(x, y)$ be defined as in Lemma 1. We now define a subset $S(\lambda)$ of the positive numbers as follows:

$$\begin{aligned} S(\lambda) = \{\lambda > 0 : p_1 w\,(x+a, y+b) + p_2 w\,(x+a, y) + p_3 w\,(x, y+b) \\ - \left( p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) - \lambda F(x, y) \right) w(x, y) \le 0, \quad \text{eventually} \right\}. \end{aligned}$$

Since $\frac{\partial w}{\partial x} \le 0$ and $\frac{\partial w}{\partial y} \le 0$, from (6) we obtain

$$\begin{aligned} p_1 w\,(x+a, y+b) + p_2 w\,(x+a, y) + p_3 w\,(x, y+b) \\ - \left( p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) - F(x, y) \right) w(x, y) \le 0 \end{aligned}$$

which implies that $1 \in S(\lambda)$. Hence $S(\lambda)$ is nonempty. For $\lambda \in S(\lambda)$, we eventually have

$$p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) - \lambda F(x, y) > 0$$

and therefore $S(\lambda) \subset E$. Since in view of conditions (i)–(iii), $E$ is bounded, we find that $S(\lambda)$ is bounded. Now from (6), we have

$$p_1 w\,(x+a, y+b) + p_2 w\,(x+a, y) + p_3 w\,(x, y+b) < p_4 \left( \prod_{x_0 < x_m < x+a} (1 + L_m) \right) w(x, y),$$

and hence,

$$\begin{aligned} w(x+a, y+b) &\le \frac{p_4}{p_2} \left( \prod_{x_0 < x_m < x+a} (1 + L_m) \right) w(x, y+b), \\ w(x+a, y+b) &\le \frac{p_4}{p_3} \left( \prod_{x_0 < x_m < x+2a} (1 + L_m) \right) w(x+a, y). \end{aligned} \tag{15}$$

Let $\mu \in S(\lambda)$. Then using (15), we obtain

$$\begin{aligned} &\left( p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0 < x_m < x+2a} (1 + L_m)^{-1} \right) w(x+a, y+b) \\ &\le p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b) \\ &\le \left( p_4 \prod_{x_0 < x_m < x+a} (1 + L_m) - \mu F(x, y) \right) w(x, y). \end{aligned} \tag{16}$$

*Case 1:* If $k > l > 0$, then from (16), we have

$$w(x, y) \le \prod_{i=1}^{l} \left( p_1 + \frac{2p_2p_3}{p_4} \prod_{x_0 < x_m < x-(i-2)a} (1 + L_m)^{-1} \right)^{-1}$$

$$\times \prod_{i=1}^{l}\left(p_4 \prod_{x_0<x_m<x-(i-1)a}(1+L_m) - \mu F(x-ia, y-ib)\right)$$
$$\times\, w(x-la, y-lb) \tag{17}$$

and

$$w(x-la, y-lb) \le \frac{1}{p_2}\left(p_4 \prod_{x_0<x_m<x-la}(1+L_m) - \mu F(x-la-a, y-lb)\right)$$
$$\times\, w(x-la-a, y-lb)$$
$$\le \cdots \le p_2^{l-k}\prod_{j=1}^{k-l}\left(p_4 \prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)\right.$$
$$\left. - \mu F(x-(l+j)a, y-lb)\right) w(x-ka, y-lb). \tag{18}$$

Hence, substituting (18) into (17), we get

$$w(x,y) \le p_2^{l-k}\prod_{i=1}^{l}\left(p_1 + \frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1}$$
$$\times \prod_{i=1}^{l}\left(p_4 \prod_{x_0<x_m<x-(i-1)a}(1+L_m) - \mu F(x-ia, y-ib)\right)$$
$$\times \prod_{j=1}^{k-l}\left(p_4 \prod_{x_0<x_m<x-(l+j-1)a}(1+L_m) - \mu F(x-(l+j)a, y-lb)\right)$$
$$\times\, w(x-ka, y-lb). \tag{19}$$

On the other hand, from (15) we have

$$w(x,y) \le \left(\frac{p_4}{p_2}\right)^k \prod_{i=1}^{k}\left(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)\right) w(x-ka, y), \tag{20}$$

and

$$w(x,y) \le \left(\frac{p_4}{p_3}\prod_{x_0<x_m<x+a}(1+L_m)\right)^l w(x, y-lb). \tag{21}$$

Using (19), (20) and (21), from (6) we obtain

$$p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b)$$
$$- \left\{ p_4 \prod_{x_0<x_m<x+a}(1+L_m) - F(x,y)\left[\left(\frac{p_2}{p_4}\right)^k \left(\prod_{i=1}^{k}\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\right)\right.\right.$$
$$+ \left(\frac{p_3}{p_4}\right)^l \left(\prod_{x_0<x_m<x+a}(1+L_m)^{-l}\right)$$

$$+ p_2^{k-l}\left[\sup_{x\geq X,\ y\geq Y}\prod_{i=1}^{l}\left(p_1+\frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1}\right.$$

$$\times\prod_{i=1}^{l}\left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu F(x-ia,y-ib)\right)$$

$$\left.\left.\left.\times\prod_{j=1}^{k-l}\left(p_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu F(x-(l+j)a,y-lb)\right)\right]^{-1}\right]\right\}$$

$$\times\ w(x,y)\leq 0,$$

which implies that

$$\left(\frac{p_2}{p_4}\right)^k\left(\prod_{i=1}^{k}\prod_{x_0<x_m<x-(i-1)a}(1+L_m)\right)^{-1}+\left(\frac{p_3}{p_4}\right)^l\left(\prod_{x_0<x_m<x+a}(1+L_m)^l\right)^{-1}$$

$$+ p_2^{k-l}\left[\sup_{x\geq X,\ y\geq Y}\prod_{i=1}^{l}\left(p_1+\frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1}\right.$$

$$\times\prod_{i=1}^{l}\left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu F(x-ia,y-ib)\right)$$

$$\left.\times\prod_{j=1}^{k-l}\left(p_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu F(x-(l+j)a,y-lb)\right)\right]^{-1}\in S(\lambda).$$

If $\lambda_1\in S(\lambda)$, then $\lambda_2\leq\lambda_1$ implies $\lambda_2\in S(\lambda)$. So,

$$p_2^{k-l}\left[\sup_{x\geq X,\ y\geq Y}\prod_{i=1}^{l}\left(p_1+\frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1}\right.$$

$$\times\prod_{i=1}^{l}\left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu F(x-ia,y-ib)\right)$$

$$\left.\times\prod_{j=1}^{k-l}\left(p_4\prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu F(x-(l+j)a,y-lb)\right)\right]^{-1}\in S(\lambda). \quad (22)$$

On the other hand (13) implies that there exists $\alpha_1\in(0,1)$ such that when $\lambda=\mu$,

$$\frac{\mu}{\alpha_1}\leq p_2^{k-l}\left\{\sup_{x\geq X,\ y\geq Y}\prod_{i=1}^{l}\left(p_1+\frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1}\right.$$

$$\times\prod_{i=1}^{l}\left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu F(x-ia,y-ib)\right)$$

$$\times \prod_{j=1}^{k-l}\left(p_4 \prod_{x_0<x_m<x-(l+j-1)a}(1+L_m)-\mu F(x-(l+j)a, y-lb)\right)\Bigg\}^{-1}. \tag{23}$$

Hence, it follows from (22)–(23) that $\frac{\mu}{\alpha_1} \in S(\lambda)$. Repeating the above arguments with $\mu$ replaced by $\frac{\mu}{\alpha_1}$, we get $\frac{\mu}{\alpha_1\alpha_2} \in S(\lambda)$, where $\alpha_2 \in (0,1)$. Continuing in this way, we obtain $\frac{\mu}{\prod_{i=1}^{\infty}\alpha_i} \in S(\lambda)$, where $\alpha_i \in (0,1)$. This contradicts the boundedness of $S(\lambda)$.

*Case 2.* If $l > k > 0$, then from (16) we have

$$\begin{aligned} w(x,y) \le p_3^{k-l} &\prod_{i=1}^{k}\left(p_1 + \frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1} \\ &\times \prod_{i=1}^{k}\left(p_4 \prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu F(x-ia, y-ib)\right) \\ &\times \prod_{j=1}^{l-k}\left(p_4 \prod_{x_0<x_m<x-(k-1)a}(1+L_m)-\mu F(x-ka, y-(k+j)b)\right) \\ &\times\ w(x-ka, y-lb). \end{aligned} \tag{24}$$

Using (20), (21) and (24), from (6), we get

$$\begin{aligned} &p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b) \\ &\quad - \Bigg\{p_4 \prod_{x_0<x_m<x+a}(1+L_m) - F(x,y)\Bigg[\left(\frac{p_2}{p_4}\right)^k \prod_{i=1}^{k}\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1} \\ &\quad + \left(\frac{p_3}{p_4}\right)^l \prod_{x_0<x_m<x+a}(1+L_m)^{-l} \\ &\quad + p_3^{l-k}\Bigg[\sup_{x\ge X,\ y\ge Y}\prod_{i=1}^{k}\left(p_1 + \frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1} \\ &\quad \times \prod_{i=1}^{k}\left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m)-\mu F(x-ia, y-ib)\right) \\ &\quad \times \prod_{j=1}^{l-k}\left(p_4\prod_{x_0<x_m<x-(k-1)a}(1+L_m)-\mu F(x-ka, y-(k+j)b)\right)\Bigg]^{-1}\Bigg]\Bigg\} \\ &\quad \times w(x,y) \le 0. \end{aligned}$$

The rest of the proof is similar to that of Case 1, and hence omitted. Analogously, if $z(x,y)$ is an eventually negative solution of (1)–(2), then also we get a contradiction. □

**Theorem 2.** Assume that there exist $X \geq x_0$, $Y \geq y_0$ such that if $k = l > 0$,

$$\sup_{\lambda\in E,\ x\geq X,\ y\geq Y} \lambda \prod_{i=1}^{k}\left(p_1 + \frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1} \times \prod_{i=1}^{k}\left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m) - \lambda F(x-ia, y-ib)\right) < 1. \tag{25}$$

Then every solution of (1)–(2) is oscillatory.

**Proof.** Suppose, to the contrary, that there is a nonoscillatory solution of (1)–(2). Without lose of generality we may assume that $z(x, y)$ is an eventually positive solution of (1)–(2). Let $\mu \in S(\lambda)$. Then from (15) and (16) we obtain

$$\left(\frac{p_2}{p_4}\right)^k \prod_{i=1}^{k}\left(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\right) w(x,y) \leq w(x-ka, y),$$

$$\left(\frac{p_3}{p_4}\right)^k \left(\prod_{x_0<x_m<x+a}(1+L_m)^{-1}\right)^k w(x,y) \leq w(x, y-kb),$$

and

$$\left[\prod_{i=1}^{k}\left(p_1 + \frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1} \times \left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m) - \mu F(x-ia, y-ib)\right)\right]^{-1} w(x,y) \leq w(x-ka, y-kb).$$

Substituting above inequalities in (6), we get

$$\begin{aligned} &p_1 w(x+a, y+b) + p_2 w(x+a, y) + p_3 w(x, y+b) \\ &\quad - \Bigg\{p_4 \prod_{x_0<x_m<x+a}(1+L_m) - F(x,y)\Bigg[\left(\frac{p_2}{p_4}\right)^k \prod_{i=1}^{k}\left(\prod_{x_0<x_m<x-(i-1)a}(1+L_m)^{-1}\right) \\ &\quad + \left(\frac{p_3}{p_4}\right)^k\left(\prod_{x_0<x_m<x+a}(1+L_m)^{-1}\right)^k \\ &\quad + \Bigg[\sup_{x\geq X,\ y\geq Y}\prod_{i=1}^{k}\left(p_1 + \frac{2p_2p_3}{p_4}\prod_{x_0<x_m<x-(i-2)a}(1+L_m)^{-1}\right)^{-1} \\ &\quad \times \left(p_4\prod_{x_0<x_m<x-(i-1)a}(1+L_m) - \mu F(x-ia, y-ib)\right)\Bigg]^{-1}\Bigg]\Bigg\} \\ &\quad \times\ w(x,y) \leq 0. \end{aligned}$$

The rest of the proof is similar to that of Theorem 1, and hence omitted. □

**Corollary 1.** Assume that if $k > l > 0$,

$$\liminf_{x,y\to\infty} F(x,y) = q > \left(p_1 + \frac{2p_2p_3}{p_4L}\right)^{-l} \frac{k^k}{(k+1)^{k+1}} (p_4L)^{k+1} p_2^{l-k}, \tag{26}$$

if $l > k > 0$,

$$\liminf_{x,y\to\infty} F(x,y) = q > \left(p_1 + \frac{2p_2p_3}{p_4L}\right)^{-k} \frac{l^l}{(l+1)^{l+1}} (p_4L)^{l+1} p_3^{k-l}, \tag{27}$$

and if $k = l > 0$,

$$\liminf_{x,y\to\infty} F(x,y) = q > \left(p_1 + \frac{2p_2p_3}{p_4L}\right)^{-k} \frac{k^k}{(k+1)^{k+1}} (p_4L)^{k+1}. \tag{28}$$

Then every solution of (1)–(2) is oscillatory.

**Proof.** We note that

$$\max_{0<\lambda<p_4L/q} \lambda(p_4L - \lambda q)^k = \frac{(p_4L)^{k+1}}{q} \frac{k^k}{(k+1)^{k+1}}.$$

Hence, (26), (27) and (28) imply (13), (14) and (25) respectively. Thus, by Theorems 1 and 2, every solution of (1)–(2) oscillates.

The following theorems extend the results established in [7].

**Theorem 3.** Let $0 < \limsup_{x,y\to\infty} U(x,y) < \infty$. Assume that there exist $X \geq x_0$, $Y \geq y_0$ such that

$$\sup_{\lambda\in E_U, x\geq X, y\geq Y} \lambda \prod_{i=1}^{k} \left(p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \lambda U(x-ia,y)\right) < p_2^k, \tag{29}$$

where

$$E_U = \left\{\lambda > 0 : p_4 \prod_{x_0<x_m<x+a} (1+L_m) - \lambda U(x,y) > 0,\ \text{eventually}\right\}.$$

Then every solution of (1)–(2) is oscillatory.

**Proof.** Suppose, to the contrary, $z(x,y)$ is an eventually positive solution of (1)–(2). Let $w(x,y)$ be as in Lemma 1. Then from (6), we have

$$\begin{aligned} &p_1w(x+a,y+b) + p_2w(x+a,y) + p_3(x,y+b) \\ &\qquad - p_4 \prod_{x_0<x_m<x+a} (1+L_m)w(x,y) + U(x,y)w(x-ka,y) \leq 0. \end{aligned} \tag{30}$$

Set

$$\begin{aligned} S_U(\lambda) = \Bigg\{\lambda > 0 : &\ p_1w(x+a,y+b) + p_2w(x+a,y) + p_3(x,y+b) \\ &- \left(p_4 \prod_{x_0<x_m<x+a} (1+L_m) - \lambda U(x,y)\right) w(x,y) \leq 0,\ \text{eventually}\Bigg\}. \end{aligned}$$

It can be seen that $S_U(\lambda) \subset E_U$ is not empty and $E_U$ is bounded. Let $\mu \in S_U(\lambda)$. Then we have

$$p_2 w(x+a,y) < \left( p_4 \prod_{x_0<x_m<x+a} (1+L_m) - \mu U(x,y) \right) w(x,y)$$

and

$$p_2^k w(x,y) < \prod_{i=1}^{k} \left( p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu U(x-ia,y) \right) w(x-ka,y).$$

From (30) and the above inequality, we have

$$\begin{aligned} &p_1 w(x+a,y+b) + p_2 w(x+a,y) + p_3 w(x,y+b) - \Bigg\{ p_4 \prod_{x_0<x_m<x+a} (1+L_m) \\ &\quad -U(x,y) p_2^k \left[ \sup_{\lambda \in E_U,\ x \geq X,\ y \geq Y} \prod_{i=1}^{k} \left( p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu U(x-ia,y) \right) \right]^{-1} \Bigg\} \\ &\quad \times\ w(x,y) \leq 0 \end{aligned}$$

which implies that

$$p_2^k \left[ \sup_{\lambda \in E_U,\ x \geq X,\ y \geq Y} \prod_{i=1}^{k} \left( p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu U(x-ia,y) \right) \right]^{-1} \in S_U(\lambda). \tag{31}$$

On the other hand, from (29) there exists $\alpha_1 \in (0,1)$ such that

$$\sup_{\lambda \in E_U,\ x \geq X,\ y \geq Y} \lambda \prod_{i=1}^{k} \left( p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \lambda U(x-ia,y) \right) \leq p_2^k \alpha_1.$$

Hence, when $\lambda = \mu$, we have

$$\frac{\mu}{\alpha_1} \leq p_2^k \left[ \sup_{\lambda \in E_U,\ x \geq X,\ y \geq Y} \prod_{i=1}^{k} \left( p_4 \prod_{x_0<x_m<x-(i-1)a} (1+L_m) - \mu U(x-ia,y) \right) \right]^{-1}.$$

Considering above inequality with (31), we get $\frac{\mu}{\alpha_1} \in S_U(\lambda)$. Repeating the above arguments with $\mu$ replaced by $\frac{\mu}{\alpha_1}$, it follows that there exists $\alpha_2 \in (0,1)$ such that $\frac{\mu}{\alpha_1 \alpha_2} \in S_U(\lambda)$. Continuing in this way, we obtain $\frac{\mu}{\prod_{i=1}^{\infty} \alpha_i} \in S_U(\lambda)$, $\alpha_i \in (0,1)$. This contradicts the boundedness of $S_U(\lambda)$. Similarly, if $z(x,y)$ is an eventually negative solution of (1)–(2), then get a contradiction. □

**Corollary 2.** If

$$\liminf_{x,y \to \infty} U(x,y) = q > (p_4 L)^{k+1} \frac{k^k}{(k+1)^{k+1}} p_2^{-k},$$

then every solution of (1)–(2) is oscillatory.

**Theorem 4.** Let $0 < \limsup_{x,y\to\infty} V(x,y) < \infty$. Assume that there exist $X \geq x_0$, $Y \geq y_0$ such that

$$\sup_{\lambda\in E_V,\ x\geq X,\ y\geq Y} \lambda \prod_{i=1}^{l} \left( p_4 \prod_{x_0<x_m<x+a} (1+L_m) - \lambda V(x, y-ib) \right) < p_3^l, \tag{32}$$

where

$$E_V = \left\{ \lambda > 0 : p_4 \prod_{x_0<x_m<x+a} (1+L_m) - \lambda V(x,y) > 0,\ \text{eventually} \right\}.$$

Then every solution of (1)–(2) is oscillatory.

The proof of Theorem 4 is similar to that of Theorem 3, and hence omitted.

**Corollary 3.** If

$$\liminf_{x,y\to\infty} V(x,y) = q > (p_4 L)^{l+1} \frac{l^l}{(l+1)^{l+1}} p_3^{-l},$$

then every solution of (1)–(2) is oscillatory.

**Theorem 5.** Let $a = b$. If there exists $Y \geq y_0$ such that one of the following hypoheses hold:

$$\limsup_{x\to\infty} \left\{ \inf_{y\geq Y} U(x-a, y-a) \right\} > p_4 L, \tag{33}$$

$$\limsup_{x\to\infty} \left\{ \inf_{y\geq Y} V(x-a, y-a) \right\} > p_4 L, \tag{34}$$

$$\limsup_{x\to\infty} \left\{ \inf_{y\geq Y} W(x-a, y-a) \right\} > p_4 L, \tag{35}$$

then every solution of (1)–(2) is oscillatory.

**Proof.** Let $z(x,y)$ be a positive solution of (1)–(2). If $a = b$, then from (7) we get

$$\inf_{y\geq Y} U(x-a, y-a) < p_4 \prod_{x_0<x_m<x} (1+L_m),\ \text{eventually}.$$

This inequality implies that

$$\limsup_{x\to\infty} \left\{ \inf_{y\geq Y} U(x-a, y-a) \right\} \leq p_4 L,$$

which contradicts (33). If (34) holds, the proof is similar and we omit it. For the last case, we note that $u'(x) \leq 0$, since $\frac{\partial w}{\partial x} \leq 0$ and $\frac{\partial w}{\partial y} \leq 0$. So, $u(x-\sigma-a) \geq u(x-a)$ and $u(x-\tau-a) \geq u(x-a)$. The rest of the proof is similar to the first case. If $z(x,y)$ is an eventually negative solution of (1)–(2), then get a similar contradiction. □

**Example 1.** Consider the impulsive partial difference equation with continuous variables

$$\begin{aligned} z(x+1, y+2) + ez(x+1, y) + z(x, y+2) - z(x,y) + z(x-3, y) \\ + (e^5 + e^2 + 1) z(x, y-2) \\ + (e^3 - e^2) z(x-3, y-2) = 0, \quad (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus J, \end{aligned} \tag{36}$$

$$z(x_n^+, y) - z(x_n^-, y) = \frac{1}{2^n} z(x_n^-, y), \quad (x_n, y) \in J, \tag{37}$$

where $J = \{(x, y) : x \in J_{imp}, \, y \in \mathbb{R}^+\}$, $J_{imp} = \{3n\}_{n=1}^{\infty}$. It can be seen that Eq. (36)–(37) satisfy all conditions of Corollary 3. So, every solution of (36)–(37) is oscillatory. Indeed, $z(x, y) = \left( \prod_{0<x_m<x} (1 + \frac{1}{2^m}) \right) e^{x+y} \cos \pi x$ is such a solution.

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