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MONOTONE METHOD FOR PERIODIC BOUNDARY VALUE PROBLEMS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, the monotone iterative technique is developed to study existence of solutions of PBVP for Fractional Differential Equation.

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1. INTRODUCTION

The concept of derivative of an arbitrary order or fractional derivative was more than 300 years old beginning with a query posed by L'Hospital to Leibnitz. In course of time the notions were well defined and the fractional calculus was fully developed. In the past few decades it was realized that these fractional derivatives appear to have natural framework to model physical phenomeno in a transient state. This gave new impetus to this field and recently there has been a rekindled interest to study the theory of fractional differential equations [1, 3, 4, 5, 6, 8, 9].

The monotone iterative technique [7] is an effective and flexible mechanism that offers theoretical, as well as constructive results in a closed set, namely, the sector. The first order periodic boundary value problem is a resonance problem, has useful applications and so is of considerable interest.

In this paper, the PBVP for Caputo fractional differential equation is considered and the monotone iterative technique is developed.

2. PRELIMINARIES

We begin with the definition of the Riemann Liouville fractional differential equation, the Caputo fractional differential equation and then proceed to give the relation between these derivatives.

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The Riemann-Liouville fractional differential equation is given by

$$D^q x = f(t,x), \quad x(t_0) = x^0 = x(t)(t-t_0)^{1-q}\mid_{t=t_0}, \quad t_0 \le t \le T, \tag{2.1}$$

and the corresponding Volterra fractional integral equation is given by

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} x(s) ds \tag{2.2}$$

where $x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$.

The Caputo fractional differential equation is given by

$$^cD^q x = f(t,x), \quad x(t_0) = x_0 \tag{2.3}$$

and the corresponding Volterra fractional integral equation is given by

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} x(s) ds \tag{2.4}$$

The relation between the Caputo fractional derivative and the Riemann fractional derivative is given by

$$^cD^q x(t) = D^q[x(t) - x(t_0)]$$

Using this relation we can show that the following results that are true for Riemann Liouville fractional derivative also hold for Caputo fractional derivative.

We need the following notation before proceeding further.

$$C_p[[t_0,T],R] = [u \in C((t_0,T],R] \quad \text{and} \quad (t-t_0)^p u(t) \in C[[t_0,T],R].$$

Now we state the following lemmas without proof.

Lemma 2.1. *Let $m \in C_p([t_0,T],R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0,T]$, we have*

$$m(t_1) = 0 \quad \textit{and} \quad m(t) \le 0 \quad \textit{for} \quad t_0 \le t \le t_1. \tag{2.5}$$

Then it follows that,

$$D^q m(t_1) \ge 0. \tag{2.6}$$

Lemma 2.2. *Let $\{x_\epsilon(t)\}$ be a family of continuous functions on $[t_0,T]$, for each $\epsilon > 0$, where $D^q x_\epsilon(t) = f(t, x_\epsilon(t))$, $x_\epsilon^0 = x_\epsilon(t)(t-t_0)^{1-q}\mid_{t=t_0}$, and $\mid f(t,x_\epsilon(t)) \mid \le M$ for $t_0 \le t \le T$. Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $[t_0,T]$.*

In order to develop the monotone iterative technique for PBVP of Caputo fractional differential equation we need the explicit solution of the nonhomogeneous linear fractional differential equation of Caputo's type given by

$$^cD^q x = \lambda x + f(t), \quad x(t_0) = x_0, \tag{2.7}$$

where $f \in C_q([t_0, T], \mathbb{R})$, Hölder continuous with exponent q. Following the method of successive approximations we get the unique solution of (2.4) as

$$x(t) = x_0 E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)ds, \quad t \in [t_0, T], \tag{2.8}$$

where $E_q(t) \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+1)}$ and $E_{q,q}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+q)}$ are Mittag-Leffler functions of one parameter and two parameters respectively.

3. MONOTONE METHOD FOR PERIODIC BOUNDARY VALUE PROBLEMS

Consider the IVP

$$^cD^q x = f(t,x), \quad x(0) = x_0 \tag{3.1}$$

where $f \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$, $^cD^q x$ is the Caputo fractional derivative of x of order q, $0 < q < 1$. The following Volterra integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds, \quad 0 \le t \le 2\pi \tag{3.2}$$

is equivalent to (3.1), where Γ is the Gamma function. We are interested in the following periodic boundary value problem (PBVP)

$$^cD^q x = f(t,x), \quad x(0) = x(2\pi). \tag{3.3}$$

Since (3.3) is a resonance problem, we have no way of obtaining right away an equivalent fractional Volterra integral equation. We need to employ the ideas of Lyapunov Schmidt method [2]. For this purpose, we develop needed results. Let us start with the following theorem involving basic inequalities.

Theorem 3.1. *Let $f \in C([0, 2\pi] \times \mathbb{R}, \mathbb{R})$, $v, w \in C([0, 2\pi], \mathbb{R})$, v, w be Hölder continuous with exponent $\lambda > q$, $0 < \lambda < 1$ and for $0 < t \le 2\pi$,*

$$\left.\begin{array}{ll} ^cD^q v(t) & \le f(t, v(t)), \quad v(0) \le v(2\pi) \\ ^cD^q w(t) & \ge f(t, w(t)), \quad w(0) \ge w(2\pi) \end{array}\right\}. \tag{3.4}$$

Suppose further $f(t,x)$ is strictly decreasing in x for each t. Then

$$v(t) \le w(t), \quad 0 \le t \le 2\pi. \tag{3.5}$$

Proof. If (3.5) is not true, then there exists an $\epsilon > 0$ and $t_0 \in [0, 2\pi]$ such that

$$v(t_0) = w(t_0) + \epsilon \text{ and } v(t) \le w(t) + \epsilon, \quad 0 \le t \le 2\pi. \tag{3.6}$$

Setting $m(t) = v(t) - w(t) - \epsilon$, we find that, if $t_0 \in (0, 2\pi]$,

$$m(t_0) = 0, \quad m(t) \le 0, \quad 0 \le t \le t_0 \le 2\pi.$$

If $t_0 = 0$, we get, because of (3.4), $v(2\pi) \geq v(0) = w(0) + \epsilon \geq w(2\pi) + \epsilon$, and hence, in all cases, we have

$$m(t_0) \geq 0 \quad \text{and} \quad m(t) \leq 0 \quad \text{for } 0 \leq t \leq t_0 \leq 2\pi \tag{3.7}$$

We therefore obtain, using (3.4), strictly decreasing nature of $f(t,x)$ in x and Lemma 2.1,

$$f(t_0, v(t_0)) \geq {}^cD^q v(t_0) \geq {}^cD^q w(t_0) \geq f(t_0, w(t_0)) > f(t_0, v(t_0)),$$

which is a contradiction. Hence (3.5) is valid and the proof is complete. □

Corollary 3.2. *Let $m : [0, 2\pi] \to \mathbb{R}$ be Hölder continuous and satisfy*

$${}^cD^q m(t) \leq -Mm(t), \; 0 \leq t \leq 2\pi, \quad m(0) \leq m(2\pi), \; M > 0.$$

Then $m(t) \leq 0$, $0 \leq t \leq 2\pi$.

We next consider the PBVP for linear nonhomogeneous fractional differential equation given by

$${}^cD^q u(t) = -Mu(t) + h(t), \quad u(0) = u(2\pi) \tag{3.8}$$

where $M > 0$ and $h \in C([0, 2\pi], \mathbb{R})$.

In order to prove the existence of a solution for the PBVP (3.8), we begin with the solution of the IVP

$${}^cD^q u(t) = -Mu(t) + h(t), \quad u(0) = u_0 \tag{3.9}$$

which, from (2.7) and (2.8), reduces as

$$u(t) = u_0 E_q(Mt^q) + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) h(s) ds.$$

Now setting $t = 2\pi$ and $u(2\pi) = u(0) = u_0$, we get

$$u_0[1 - E_q(Mt^q)] = \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}(M(2\pi - s)^q) h(s) ds$$

which gives,

$$u_0 = \frac{1}{[1 - E_q(Mt^q)]} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}(M(2\pi - s)^q) h(s) ds.$$

Thus the solution of the PBVP (3.9) is given by

$$\begin{aligned} u(t) \;\; = \;\; & \tfrac{E_q(Mt^q)}{1 - E_q(Mt^q)} \int_0^{2\pi} (2\pi - s)^{q-1} E_{q,q}(M(2\pi - s)^q) h(s) ds \\ & + \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) h(s) ds. \end{aligned} \tag{3.10}$$

We shall next develop the monotone iterative technique for the PBVP (3.3). The following result provides the existence of extremal solutions of (3.3) and used the method of upper and lower solutions.

Theorem 3.3. *Assume that*

(i) $v_0, w_0 : [0, 2\pi] \to \mathbb{R}$ *are Hölder continuous with exponent* $\lambda > q$ *such that*

$$\begin{aligned} &{}^cD^q v_0 \le (t, v_0), \qquad f\ v_0(0) \le_0 (2\pi), \\ v\ &{}^cD^q w_0 \ge f(t, w_0), \quad w_0(0) \ge w_0(2\pi), \\ &0 < t \le 2\pi \\ \textit{and} \qquad &v_0(t) \le w_0(t) \qquad \textit{for}\ \ 0 \le t \le 2\pi; \end{aligned}$$

(ii) $f(t, u) - f(t, v) \ge -M(u - v)$ *for* $v_0 \le v \le u \le w_0$ *and* $M > 0$.

Then there exist monotone sequences $\{v_n\}, \{w_n\}$ *such that* $v_n \to \rho$, $w_n \to r$ *as* $n \to \infty$ *uniformly on* $[0, 2\pi]$ *and* (ρ, r) *are extremal solutions of PBVP* (3.3).

Proof. For any $\eta \in [v_0, w_0]$ such that $\eta \in C([0, 2\pi], \mathbb{R})$ and $v_0 \le \eta \le w_0$, consider the fractional linear differential equation with periodic boundary conditions

$$ {}^cD^q u(t) = G(t, u(t)) = f(t, \eta(t)) - M[u(t) - \eta(t)], \quad u(0) = u(2\pi) \tag{3.11}$$

The linear PBVP (3.11) can be solved and a solution of (3.11) is given by relation (3.10). Of course, in the present context $h(t) = f(t, \eta) + M\eta$ and is to be replaced in (3.10) by this new value.

We shall show that the solution u(t) of (3.11) thus obtained is unique. If not, let $u_1(t), u_2(t)$ be two solutions of (3.11). Set $p(t) = u_1(t) - u_2(t)$. Then we find that ${}^cD^q p(t) = -Mp(t)$, $p(0) = p(2\pi)$. Now using Corollary 3.1 it follows that $u_1(t) = u_2(t)$. This proves uniqueness.

We now define a mapping A by $A\eta = u$, where u is the unique solution of PBVP (3.11) and show that

(a) $v_0 \le Av_0$, $w_0 \ge Aw_0$
(b) A is monotone nondecreasing on $[v_0, w_0]$.

In order to prove (a), we set $Av_0 = v_1$, where v_1 is the unique solution of the PBVP (3.11) and v_0 is the lower solution of (3.3). Let $p = v_0 - v_1$, then using (i) and (3.11) with $\eta = v_0$, we get

$$ {}^cD^q p = {}^cD^q v_0 -^c D^q v_1 \le -Mp, \quad p(0) \le p(2\pi), \quad 0 \le t \le 2\pi. $$

Hence, by Corollary 3.1, we have $p(t) \le 0$ for $0 \le t \le 2\pi$, which proves that $v_0 \le Av_0$. Similarly, we can show $w_0 \ge Aw_0$. To prove (b), let $\eta, \mu \in [v_0, w_0]$ such that $\eta \le \mu$. Suppose that $u_1 = A\eta$ and $u_2 = A\mu$.

Setting $p(t) = u_1(t) - u_2(t)$, we find that

$$\begin{aligned} {}^cD^q p &= {}^cD^q u_1 - {}^cD^q u_2 \\ &= f(t,\eta) - M(u_1 - \eta) - f(t,\mu) + M(u_2 - \mu) \\ &\leq M(\mu - \eta) - M(u_1 - \eta) + M(u_2 - \mu) \\ &= -Mp \text{ and } p(0) = p(2\pi). \end{aligned}$$

Consequently, Corollary 3.1 gives $u_1 \leq u_2$ on $[0, 2\pi]$ proving (b).

It is now easy to define sequences $\{v_n\}, \{w_n\}$ such that $v_n = Av_{n-1}$, $w_n = Aw_{n-1}$ and then to conclude

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0, \quad 0 \leq t \leq 2\pi. \tag{3.12}$$

Since the sequences $\{v_n\}, \{w_n\}$ are uniformly bounded by (3.12), we see that $\{{}^cD^q v_n\}$, $\{{}^cD^q w_n\}$ are also uniformly bounded on $[0, 2\pi]$ in view of the fact

$$\begin{aligned} {}^cD^q v_{n+1} &= f(t, v_n) - M(v_{n+1} - v_n), \qquad v_{n+1}(0) = v_{n+1}(2\pi) \\ {}^cD^q w_{n+1} &= f(t, w_n) - M(w_{n+1} - w_n), \quad w_{n+1}(0) = w_{n+1}(2\pi) \end{aligned} \tag{3.13}$$

Then utilizing Lemma 2.2, we can conclude the equicontinuity of the sequences $\{v_n\}, \{w_n\}$ and as a result, Ascoli-Arzela Theorem and (3.12) shows that $v_n \to \rho$, $w_n \to r$ as $n \to \infty$, uniformly on $[0, 2\pi]$. Clearly ρ, r are solutions of PBVP (3.3), since v_n, w_n satisfy the corresponding fractional Volterra integral equations of type (3.10).

To prove that ρ, r are extremal solutions of PBVP (3.3), let us suppose that, for some $k > 0$, $v_{k-1} \leq u \leq w_{k-1}$ on $[0, 2\pi]$, where u is any solution of PBVP (3.3) such that $v_0 \leq u \leq w_0$. Then setting $p = v_0 - v_1$, we get

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_k - {}^cD^q u \\ &= f(t, v_{k-1}) - M(v_k - v_{k-1}) - f(t, u) \\ &\leq M(u - v_{k-1}) - M(v_k - v_{k-1}) = -Mp \\ \text{and } p(0) &= p(2\pi). \end{aligned}$$

This implies by Corollary 3.1 that $p(t) \leq 0$ on $0 \leq t \leq 2\pi$, which yields $v_k \leq u$. Using similar arguments, we have $u \leq w_k$ on $[0, 2\pi]$. Since $v_0 \leq u \leq w_0$, it follows by induction that $v_n \leq u \leq w_n$, for all n, on $[0, 2\pi]$. Hence we have $\rho \leq u \leq r$ on $[0, 2\pi]$, proving ρ, r are minimal and maximal solutions of PBVP (3.3) respectively. □

Corollary 3.4. *If in addition to the assumptions of Theorem* 3.1*, we assume that, for* $0 < N < M$*, we have*

$$f(t, u) - f(t, v) \leq -N(u - v), \quad v_0 \leq v \leq u \leq w_0,$$

that $\rho = r = u$ *is the unique solution of the PBVP* (3.3).

Proof. The proof is immediate since $\rho \leq r$, setting $p = r - \rho$,

$$\begin{aligned} D^q p &= D^q r - D^q \rho = f(t,r) - f(t,\rho) \\ &\leq -Mp \\ p(0) &= p(2\pi), \end{aligned}$$

and Corollary 3.1, yields $\rho = r = u$, as claimed. □

REFERENCES

[1] Kilbas, A. A., Srivatsava, H. M. and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[2] Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol I and II, Academic Press, New York, 1969.

[3] Lakshmikantham, V., and Vasundhara Devi, J., Theory of fractional differential equations in a Banach space, *European Jour. Pure and Appl. Math.*, Vol 1, No. 1, (2008), 38–45.

[4] Lakshmikantham, V., and Vatsala, A. S., Theory of fractional differential inequalities and applications, *Commun. Appl. Anal.* 11 (2007), no. 3-4, 395–402.

[5] Lakshmikantham, V., and Vatsala, A. S., Basic theory of fractional differential equations, *Nonlinear Anal.* 69 (2008), no. 8, 2677–2682.

[6] Lakshmikantham, V., and Vatsala, A. S., General uniqueness and monotone iterative technique for fractional differential equations, *Appl. Math. Lett.* 21 (2008), no. 8, 828–834.

[7] Ladde, G. S., Lakshmikantham, V. and Vatsala, A. S., *Monotone Iterative Techniques for Nonlinear Differential Equations,* Pitman Advanced Publishing Program, London, 1985.

[8] McRae, F. A., Monotone iterative technique and existence results for fractional differential Equations, *Nonlinear Anal.* 71 (2009), no. 12, 6093–6096.

[9] Podlubny, I., *Fractional Differential Equations,* Academic Press, San Diego, 1999.