

## PREEXPONENTIAL AND PRETRIGONOMETRIC FUNCTIONS

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**Abstract:** This article develops pretrigonometric functions and prehyperbolic functions. Some properties of such functions are also developed. The theory developed will enrich the topic of special functions which are largely employed in applications.

**Keywords:** Preexponential functions, pretrigonometric functions, prehyperbolic functions, extended pretrigonometric and prehyperbolic functions.

### 1. INTRODUCTION.

The exponential, trigonometric and hyperbolic functions occupy significant space in mathematical literature. Generally these type of functions are classified under the title of “elementary functions”. It is known that such functions possess interesting properties such as continuity, differentiability, integrability, etc. Among these the trigonometric and hyperbolic functions possess periodic properties. Trigonometric and hyperbolic functions are in fact a part of the study of exponential functions. In fact, the entire study of such functions is the valuable contribution of the Euler’s number “e”. The various properties of the elementary functions are taught as a part of introductory course in mathematics since these play an important role in sequential course of mathematics.

One of the question that is at the centre of the present article is ‘can we determine as set of functions ( to be called as prefunctions ) possessing a sequence  $\{f_n(t), t \in \mathbb{R}\}$  which approaches as  $n \rightarrow 0$  to one of the elementary functions’. The answer to this question is positive. The newly discussed set of prefunctions indeed possess some of the properties of elementary functions, while some properties such as periodicity are lost. In turn, the study here generalises the scope of elementary functions.

The presently known elementary functions have natural generalisation which is classified under the title “extended elementary functions”. The study here also aims at obtaining the prefunction sets for the extended elementary functions and establishing some properties of such functions. The authors hope that the newly stated prefunctions – theory may equally get wide and meaningful applications in future.

## 2. EULER GAMMA FUNCTION.

The *gamma function* can be thought of as the natural way to generalize the concept of the factorial to non-integer arguments. Leonhard Euler came up with a formula for such a generalization in 1729. At around the same time, James Stirling independently arrived at a different formula, but was unable to show that it always converged. In 1900, Charles Hermite showed that the formula given by Stirling does work, and that it defines the same function as Euler's.

Euler's original formula for the gamma function is

$$\Gamma(z+1) = \lim_{n \rightarrow \infty} \frac{n^{z+1} n!}{\prod_{k=0}^n (z+1+k)} .$$

However, it is now more commonly defined by

$$\Gamma(\alpha+1) = \int_0^{\infty} e^{-t} t^{\alpha} dt \quad (1)$$

which converges for all  $\alpha > -1$ . Using integration by parts, we get the important property of the Gamma function namely,

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha) . \quad (2)$$

We let  $\alpha = 0$  in (1) which results  $\Gamma(1) = 1$ . For  $\alpha$ , a positive integer,  $\Gamma(\alpha+1) = \alpha!$ . It is interesting to note that  $\Gamma(\alpha)$  is defined by (2) for all real values except  $\alpha = 0, -1, -2, \dots$ . Here (1) does not give  $\Gamma(\alpha+1)$  for  $\alpha \leq -1$  because the behaviour of  $t^{\alpha}$  at  $t = 0$  makes the integral divergent. For details see [2, p 45-46]. The graph of  $\Gamma(\alpha)$  (Figure 1) has the appearance as given below and is given to indicate the nature of Gamma function which we use in subsequent development of article.

## 3. THE PREEXPONENTIAL FUNCTION OF A REAL VARIABLE.

Exponential function play a wide role in almost all branches of mathematics. A question may be raised : are there a sets of functions a sequence from which tends to  $\exp(z)$ ? We call such a set a prefunction set of the exponential function.

Define most general form of exponential function in the form of series which we term as preexponential function. For any real number  $t$  and for any  $\alpha \geq 0$ ,  $\alpha$  being a parameter, define

$$\text{pexp}(t, \alpha) = 1 + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{t^{3+\alpha}}{\Gamma(4+\alpha)} + \dots = 1 + \sum_{n=1}^{\infty} \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)} . \quad (3)$$

Here  $\text{pexp}(t, \alpha)$  for each  $\alpha \geq 0$  stands for prefunction of  $\exp(t)$ . Series (3) is absolutely convergent for all  $t \in \mathbb{R}$ . For each  $t \in \mathbb{R}$  and  $\alpha \geq 0$  graphs of such prefunctions can be drawn. In particular when  $\alpha = 0$ , (3) reduces to

$$\text{pexp}(t, 0) = \exp(t).$$

In general, it is easy to observe that

$$\text{pexp}(t, n) = \text{pexp}(t, n-1) - \frac{t^n}{n!} = \text{pexp}(t, n-2) - \frac{t^{n-1}}{(n-1)!} = \dots, n = 1, 2, 3, \dots$$

Special cases of this general representation are now below :

$$\text{pexp}(t, 1) = \exp(t) - t.$$

$$\text{pexp}(t, 2) = \exp(t) - t - \frac{t^2}{2!}.$$

$$\text{pexp}(t, 3) = \exp(t) - t - \frac{t^2}{2!} - \frac{t^3}{3!} = \exp(t) - S_3 \text{ where } S_3 \text{ is the partial sum of } e^t - 1.$$

$$\text{In general, for } n \in \mathbb{N}, \text{pexp}(t, n) = \exp(t) - S_n \text{ where } S_n = \sum_{r=1}^n \frac{t^r}{r!}. \quad (4)$$

Also, note that  $\lim_{n \rightarrow \infty} \text{pexp}(t, n) = 1$ , for all  $t \in \mathbb{R}$ .

It is interesting to compare the values of preexponential function at  $t = 0, 1, -1$  for different integral values of  $n$ . We have Table 1 given below.

Here the numbers in the second row keep on decreasing as  $n$  tends to  $\infty$  and tend to 1 while those in the third row keep on oscillating. The graphical representation of the functions  $\text{pexp}(t, 1)$  (Figure 2) and  $\text{pexp}(t, 2)$  (Figure 3) shown provide us the behavior of such functions.

Note that curve (Figure 2) is near parabolic in nature. The curve lies in the first and second quadrant. In the next graph (Figure 3) one can see the characteristic of Junction diode and Zener diode (reverse bias). Up to certain value the behavior is nonlinear and after it is like a linear function.

Now, for general values of  $-t, 0 \leq t < \infty$  we have from (3),

$$\begin{aligned} \text{pexp}(-t, \alpha) &= 1 - (-1)^\alpha \left\{ \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{t^{3+\alpha}}{\Gamma(4+\alpha)} - \dots \right\} \\ &= 1 + (-1)^\alpha \sum_{n=1}^{\infty} (-1)^n \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)}, \quad \alpha = 0, 1, 2, 3, \dots \end{aligned} \quad (5)$$

Clearly from (5)

$$\text{pexp}(-t, 0) = 1 - \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} - \frac{t^3}{\Gamma(4)} \dots = \exp(-t),$$

We write as in (4) above,

$$\text{pexp}(-t, n) = \exp(-t) - S_n \quad \text{where } S_n = \sum_{r=1}^n (-1)^r \frac{t^r}{r!}.$$

Now we define a number  $e_\alpha$  by

$$e_\alpha = \text{pexp}(1, \alpha) = 1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1+\alpha)}; \quad \alpha \geq 0.$$

and note that  $\lim_{\alpha \rightarrow 0} e_\alpha = e$ . Further,

$$e_\alpha^{-1} = \text{pexp}(-1, \alpha) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^\alpha}{\Gamma(n+1+\alpha)}.$$

When  $\alpha = 0$ ,  $e_\alpha + e_\alpha^{-1} = \frac{1}{2} \{e + e^{-1}\}$ , and  $e_\alpha - e_\alpha^{-1} = \frac{1}{2} \{e - e^{-1}\}$ .

Further, observe that  $e_\alpha > 1$  and

$$\frac{1}{\Gamma(2+\alpha)} < \frac{1}{\Gamma(2)} = 1, \quad \frac{1}{\Gamma(3+\alpha)} < \frac{1}{\Gamma(3)} = \frac{1}{2},$$

$$\frac{1}{\Gamma(4+\alpha)} < \frac{1}{\Gamma(4)} = \frac{1}{6} < \frac{1}{2^2}, \dots$$

$$e_\alpha < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 1 + \frac{1}{1 - \frac{1}{2}} < 3, \quad \text{hence } 1 < e_\alpha < 3.$$

The problem of discovering additional properties of  $e_\alpha$  is open.

#### 4. PRETRIGONOMETRIC FUNCTIONS

The prefunctions of  $\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}$  and  $\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}$

are defined below :

$$X_1(t, \alpha) = p \cos(t, \alpha) = 1 - \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{t^{4+\alpha}}{\Gamma(5+\alpha)} - \frac{t^{6+\alpha}}{\Gamma(7+\alpha)} + \dots$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)}, \quad t \in \mathbb{R}; \quad (6)$$

and

$$\begin{aligned}
X_2(t, \alpha) = p \sin(t, \alpha) &= \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{t^{3+\alpha}}{\Gamma(4+\alpha)} + \frac{t^{5+\alpha}}{\Gamma(6+\alpha)} - \dots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1+\alpha}}{\Gamma(2n+2+\alpha)}, \quad t \in \mathbb{R}.
\end{aligned} \tag{7}$$

Some particular cases for integral values of  $\alpha$  of these functions are listed below :

$$\begin{aligned}
p \cos(t, 1) &= \sin t - t + 1, & p \sin(t, 1) &= 1 - \cos t, \\
p \cos(t, 2) &= -\cos t - \frac{t^2}{2} + 2, & p \sin(t, 2) &= t - \sin t, \\
p \cos(t, 3) &= -\sin t + t - \frac{t^3}{6} + 1, & p \sin(t, 3) &= \cos t + \frac{t^2}{2} - 1,
\end{aligned}$$

From these relations it is easy to obtain many trigonometrical relations. For example

$$p \cos(t_1 + t_2, 1) = \sin(t_1 + t_2) - (t_1 + t_2) + 1.$$

Here the right hand side is completely known quantity. It is observed that the following formulae of psine and pcosine for  $\alpha = 2n$  and  $\alpha = 2n - 1$  takes the form as follows :

$$\begin{aligned}
p \cos(t, 2n) &= \{1 + (-1)^{n+1}\} + (-1)^n \cos t + (-1)^{n+1} \sum_{r=1}^n (-1)^r \frac{t^{2r}}{(2r)!}, \quad n=1, 2, 3, \dots; \\
p \cos(t, 2n-1) &= 1 + (-1)^{n+1} \sin t + (-1)^{n+1} \sum_{r=1}^n (-1)^r \frac{t^{2r-1}}{(2r-1)!}, \quad n=1, 2, 3, \dots
\end{aligned}$$

and

$$\begin{aligned}
p \sin(t, 2n) &= (-1)^n \sin t + (-1)^n \sum_{r=1}^n (-1)^r \frac{t^{2r-1}}{(2r-1)!}, \quad n=1, 2, 3, \dots, \\
p \sin(t, 2n+1) &= (-1)^n + (-1)^{n+1} \cos t + (-1)^n \sum_{r=1}^n (-1)^r \frac{t^{2r}}{(2r)!}, \quad n=1, 2, 3, \dots
\end{aligned}$$

Note that  $p \sin(t, 1) = 1 - \cos t$  as mentioned above.

Similarly for  $n \in \mathbb{N}$  we state,

$$\begin{aligned}
p \cos(-t, 2n) &= p \cos(t, 2n), \\
p \cos(-t, 2n-1) &= 1 + (-1)^n \sin t + (-1)^{n+1} \sum_{r=1}^n (-1)^{r+1} \frac{t^{2r-1}}{(2r-1)!}, \quad n=1, 2, 3, \dots
\end{aligned}$$

and

$$\begin{aligned}
p \sin(-t, 2n) &= (-1)^{n+1} \sin t + (-1)^n \sum_{r=1}^n (-1)^{r+1} \frac{t^{2r-1}}{(2r-1)!}, \quad n=1, 2, 3, \dots \\
p \sin(-t, 2n-1) &= p \sin(t, 2n-1).
\end{aligned}$$

## 5. EULER'S RESULT

We now generalize the Euler's formula below. For  $t$ , a real number (or Complex number) it is known that

$$\exp(it) = \cos t + i \sin t. \quad (8)$$

In view of (8), We obtain from (3),

$$\begin{aligned} \text{pexp}(it, \alpha) &= 1 - (i)^\alpha \left\{ \frac{(t)^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{(t)^{4+\alpha}}{\Gamma(5+\alpha)} + \frac{(t)^{6+\alpha}}{\Gamma(7+\alpha)} + \dots \right\} \\ &\quad + i(i)^\alpha \left\{ \frac{(t)^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{(t)^{3+\alpha}}{\Gamma(4+\alpha)} + \frac{(t)^{5+\alpha}}{\Gamma(6+\alpha)} - \frac{(t)^{7+\alpha}}{\Gamma(8+\alpha)} + \dots \right\} \\ &= 1 - (i)^\alpha \{ 1 - \text{pcos}(t, \alpha) \} + i(i)^\alpha \text{psin}(t, \alpha), \quad \alpha \geq 0. \end{aligned} \quad (9)$$

Thus we get the general form of Euler's Formulae for the preexponential function, for  $\alpha \geq 0$ .

Clearly

$$e^{it} = \text{pexp}(it, 0) = \text{pcos}(t, 0) + i \text{psin}(t, 0) = \cos t + i \sin t \text{ which is (8).}$$

Similarly, we get

$$\text{pexp}(-it, \alpha) = 1 - (-i)^\alpha \{ 1 - \text{pcos}(t, \alpha) \} - i(-i)^\alpha \text{psin}(t, \alpha). \quad (10)$$

When  $\alpha = 0$ , we have  $e^{-it} = \text{pexp}(-it, 0) = \text{pcos}(t, 0) - i \text{psin}(t, 0) = \cos t - i \sin t$ .

It is known that  $[\exp(it)]^2 + [\exp(-it)]^2 = 2 \cos(2t)$ . We have similar result for pretrigonometric functions.

$$\begin{aligned} &[\text{pexp}(it, \alpha) - 1]^2 + [\text{pexp}(-it, \alpha) - 1]^2 \\ &= 2(-1)^\alpha + 2(-1)^\alpha \{ \text{pcos}^2(t, \alpha) - \text{psin}^2(t, \alpha) \} - 4(-1)^\alpha \text{pcos}(t, \alpha). \end{aligned}$$

When  $\alpha = 0$ , as a particular case, we get,

$$[\exp(it) - 1]^2 + [\exp(-it) - 1]^2 = 2 + 2 \cos 2t - 4 \cos t \text{ which is true.}$$

Next, Euler's exponential values of  $\cos t$  and  $\sin t$  are given by

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}. \quad (11)$$

We have similar generalized results for pretrigonometric functions  $p\cos(t, \alpha)$  and  $p\sin(t, \alpha)$ . Observe that,

$$\begin{aligned} & \frac{p\exp(it, \alpha) + p\exp(-it, \alpha)}{2} \\ &= 1 - \left[ \frac{(-i)^\alpha + (i)^\alpha}{2} \right] + \left[ \frac{(i)^\alpha + (-i)^\alpha}{2} \right] p\cos(t, \alpha) + i \left[ \frac{(i)^\alpha - (-i)^\alpha}{2} \right] p\sin(t, \alpha), \text{ and} \end{aligned} \quad (12)$$

$$\begin{aligned} & \frac{p\exp(it, \alpha) - p\exp(-it, \alpha)}{2} \\ &= \left[ \frac{(-i)^\alpha - (i)^\alpha}{2} \right] + \left[ \frac{(i)^\alpha - (-i)^\alpha}{2} \right] p\cos(t, \alpha) + i \left[ \frac{(i)^\alpha + (-i)^\alpha}{2} \right] p\sin(t, \alpha). \end{aligned} \quad (13)$$

The relations (12) and (13), lead us to,

$$p\cos(t, \alpha) = \frac{(-i)^\alpha p\exp(it, \alpha) + (i)^\alpha p\exp(-it, \alpha) - (i)^\alpha + (-i)^\alpha}{2} + 1;$$

and

$$p\sin(t, \alpha) = \frac{(-i)^\alpha p\exp(it, \alpha) - (i)^\alpha p\exp(-it, \alpha) - (-i)^\alpha + (i)^\alpha}{2i}.$$

Clearly for  $\alpha = 0$ , these formulae reduce to (11). Now as in classical trigonometry, we define formulae for ptangent, psecant and pcotangent whenever they exist. We have,

$$\begin{aligned} p\tan(t, \alpha) &= i \frac{(-i)^\alpha + (i)^\alpha - (-i)^\alpha p\exp(it, \alpha) + (i)^\alpha p\exp(-it, \alpha)}{(-i)^\alpha p\exp(it, \alpha) + (i)^\alpha p\exp(-it, \alpha) - (i)^\alpha - (-i)^\alpha + 2}; \\ p\sec(t, \alpha) &= \frac{2}{(-i)^\alpha p\exp(it, \alpha) + (i)^\alpha p\exp(-it, \alpha) - (i)^\alpha - (-i)^\alpha + 2}; \\ p\operatorname{cosec}(t, \alpha) &= \frac{2i}{(-i)^\alpha p\exp(it, \alpha) - (i)^\alpha p\exp(-it, \alpha) - (i)^\alpha - (-i)^\alpha}. \end{aligned}$$

For  $\alpha = 0$ , these results reduce the Euler's formulae for tangent, secant and cosecant functions, namely,

$$\tan t = \frac{e^{it} - e^{-it}}{i(e^{it} + e^{-it})} \quad \sec t = \frac{2}{e^{it} + e^{-it}}, \quad \operatorname{cosec} t = \frac{2i}{e^{it} - e^{-it}}.$$

## 6. PREHYPERBOLIC FUNCTIONS

The prefunctions of  $\cosh t = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}$  and  $\sinh t = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}$  are defined as,

$$\begin{aligned}
Y_1(t, \alpha) = p \cosh(t, \alpha) &= 1 + \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{t^{4+\alpha}}{\Gamma(5+\alpha)} + \frac{t^{6+\alpha}}{\Gamma(7+\alpha)} + \dots \\
&= 1 + \sum_{n=1}^{\infty} \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)}, \quad \alpha \in \mathbb{N}, t \in \mathbb{R}.
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
Y_2(t, \alpha) = p \sinh(t, \alpha) &= \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{t^{3+\alpha}}{\Gamma(4+\alpha)} + \frac{t^{5+\alpha}}{\Gamma(6+\alpha)} - \dots \\
&= \sum_{n=0}^{\infty} \frac{t^{2n+1+\alpha}}{\Gamma(2n+2+\alpha)}, \quad \alpha \in \mathbb{N}, t \in \mathbb{R}.
\end{aligned} \tag{15}$$

Note that  $p \cosh(0, \alpha) = 1$  for all  $\alpha \in \mathbb{R}^+$  and  $p \sinh(0, \alpha) = 0$  for all  $\alpha \in \mathbb{R}^+$ . Further  $p \cosh(t, 0) = \cosh t$ ,  $p \sinh(t, 0) = \sinh t$ . We have defined the new number  $e_\alpha$  and  $e_\alpha^{-1}$  in section 3. The following property of these numbers is worthy of note.

$$e_\alpha + e_\alpha^{-1} = 2 + \{1 - (-1)^\alpha\} p \sinh(1, \alpha) + \{1 + (-1)^\alpha\} \{p \cosh(1, \alpha) - 1\}.$$

$$e_\alpha - e_\alpha^{-1} = \{1 + (-1)^\alpha\} p \sinh(1, \alpha) + \{1 - (-1)^\alpha\} \{p \cosh(1, \alpha) - 1\}.$$

Next, for  $\alpha = n$ , a positive integer we get the following phyperbolic sine and cosine functions.

$$\begin{aligned}
p \cosh(t, 1) &= \sinh t - t + 1, & p \sinh(t, 1) &= \cosh t - 1, \\
p \cosh(t, 2) &= \cosh t - \frac{t^2}{2}, & p \sinh(t, 2) &= \sinh t - t,
\end{aligned}$$

In general, the following relations are true :

$$p \cosh(t, 2n) = \cosh t - \sum_{r=1}^n \frac{t^{2r}}{(2r)!}, \tag{16}$$

$$p \cosh(t, 2n-1) = 1 + \sinh t - \sum_{r=1}^n \frac{t^{2r-1}}{(2r-1)!}, \quad n=1, 2, 3, \dots,$$

and

$$p \sinh(t, 2n) = \sinh t - \sum_{r=1}^n \frac{t^{2r-1}}{(2r-1)!}, \tag{17}$$

$$p \sinh(t, 2n+1) = \cosh t - 1 - \sum_{r=1}^n \frac{t^{2r}}{(2r)!}, \quad n=1, 2, 3, \dots$$

Note that  $p \sinh(t, 1) = \cosh t - 1$  as mentioned above.



For all real ( or complex ) values of  $t$  the quantities

$$\frac{e^t + e^{-t}}{2}, \quad \frac{e^t - e^{-t}}{2i} \quad (18)$$

are called the hyperbolic cosine and sine functions respectively and written as  $\cosh t$  and  $\sinh t$ . In view of these functions we now define  $\text{pcosh } t$  and  $\text{psinh } t$  functions using the series for  $\text{pexp } t$ . Thus using (3) and (5) we get ,

$$\begin{aligned} & \frac{\text{pexp}(t, \alpha) + \text{pexp}(-t, \alpha)}{2} \\ &= 1 + \left[ \frac{1 + (-1)^\alpha}{2} \right] \{ \text{pcosh}(t, \alpha) - 1 \} + \left[ \frac{1 - (-1)^\alpha}{2} \right] \text{psinh}(t, \alpha) \end{aligned} \quad (19)$$

and

$$\begin{aligned} & \frac{\text{pexp}(t, \alpha) - \text{pexp}(-t, \alpha)}{2} \\ &= 1 + \left[ \frac{1 - (-1)^\alpha}{2} \right] \{ \text{pcosh}(t, \alpha) - 1 \} + \left[ \frac{1 + (-1)^\alpha}{2} \right] \text{psinh}(t, \alpha). \end{aligned} \quad (20)$$

The relations (19) and (20) lead us to

$$\begin{aligned} \text{pcosh}(t, \alpha) &= \frac{(-1)^\alpha \text{pexp}(t, \alpha) + \text{pexp}(-t, \alpha)}{2} - \frac{1 - (-1)^\alpha}{2}, \text{ and} \\ \text{psinh}(t, \alpha) &= \frac{(-1)^\alpha \text{pexp}(t, \alpha) - \text{pexp}(-t, \alpha)}{2} - \frac{(-1)^\alpha - 1}{2}. \end{aligned}$$

Clearly for  $\alpha = 0$ , these formulae reduce to those given in (18). We also define the following hyperbolic functions whenever they exist ,

$$\begin{aligned} \text{ptanh}(t, \alpha) &= \frac{(-1)^\alpha \text{pexp}(t, \alpha) - \text{pexp}(-t, \alpha) - (-1)^\alpha + 1}{(-1)^\alpha \text{pexp}(t, \alpha) + \text{pexp}(-t, \alpha) - 1 + (-1)^\alpha}, \\ \text{psech}(t, \alpha) &= \frac{2}{(-1)^\alpha \text{pexp}(t, \alpha) + \text{pexp}(-t, \alpha) - 1 + (-1)^\alpha}, \\ \text{pcosech}(t, \alpha) &= \frac{2}{(-1)^\alpha \text{pexp}(t, \alpha) - \text{pexp}(-t, \alpha) - (-1)^\alpha + 1}. \end{aligned}$$

For  $\alpha = 0$ , above relations reduce to the Euler formulae for hyperbolic functions :

$$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad \text{sech } t = \frac{2}{e^t + e^{-t}}, \quad \text{cosech } t = \frac{2}{e^t - e^{-t}}.$$

## 7. PREFUNCTIONS USING DIFFERENTIAL EQUATION AND LAPLACE TRANSFORM.

In view of the relations for  $\text{pcos}(t, n)$ ,  $\text{psin}(t, n)$ ,  $\text{pcosh}(t, n)$ ,  $\text{psinh}(t, n)$  for an integer  $n > 0$ , which have been stated earlier, it is clearly seen that the Laplace Transform of all such pretrigonometric functions can be obtained and as

applications can be employed in solving differential equation involving preexponential, pretrigonometric, prehyperbolic functions. We do not present a table of such functions but provide one illustrative example.

The function  $X_1(t, \alpha)$  defined in (3) satisfies the second order nonhomogeneous differential equation of the form

$$X_1''(t, \alpha) + X_1(t, \alpha) = 1 - \frac{t^\alpha}{\Gamma(1+\alpha)} \quad (21)$$

with the initial conditions

$$X_1(0, \alpha) = 1, \quad X_1'(0, \alpha) = 0. \quad (22)$$

We solve this differential equation by using two methods (1) variation of parameters and (2) using Laplace Transforms for positive integer values for  $\alpha$  and show that  $p\cos(t, \alpha)$  and  $p\sin(t, \alpha)$  are two solutions of the IVP (21) and (22).

By employing the variation of parameters [2, p.51] formula we obtain

$$X_1(t, \alpha) = C_1 \sin t + C_2 \cos t + \left( \sin t - \frac{1}{\Gamma(1+\alpha)} C_\alpha \right) \sin t + \left( \frac{1}{\Gamma(1+\alpha)} S_\alpha + \cos t \right) \cos t$$

where  $C_\alpha = \int t^\alpha \cos t \, dt$  and  $S_\alpha = \int t^\alpha \sin t \, dt$  are known through the recurring relations

$$C B_{\alpha B} = (t \sin t + \alpha \cos t) t P^{\alpha-1P} - \alpha(\alpha-1) C B_{\alpha-2B},$$

$$S B_{\alpha B} = (\alpha \sin t - t \cos t) t P^{\alpha-1P} - \alpha(\alpha-1) S B_{\alpha-2}.$$

and the constants  $C_1$  and  $C_2$  can be found by using the initial conditions (22).

To verify the validity of the above discussion, let us consider (i)  $\alpha=1$  and (ii)  $\alpha=2$ . By evaluating  $C_1$ ,  $C_2$ ,  $C_\alpha$  and  $S_\alpha$ , we show that

$$p\cos(t, 1) = \sin t + 1 - t, \quad (23)$$

$$p\cos(t, 2) = -\cos t + 2 - \frac{t^2}{2}. \quad (24)$$

For  $\alpha=1$ , IVP (21) and (22) takes the form

$$X_1''(t, 1) + X_1(t, 1) = 1 - t, X_1(0, 1) = 1, \quad X_1'(0, 1) = 0;$$

for which the solution is given by

$$X_1(t, 1) = \sin t + 1 - t = p\cos(t, 1); \quad (25)$$

and for  $\alpha = 2$ , the IVP

$$X_1''(t, 2) + X_1(t, 2) = 1 - \frac{t^2}{2}, \quad X_1(0, 2) = 1, \quad X_1'(0, 2) = 0.$$

has the solution

$$X_1(t, 2) = -\cos t + 2 - \frac{t^2}{2} = p\cos(t, 2). \quad (26)$$

These are the relations established earlier using (6). Also using the Laplace transform method, we take the Laplace transform of both sides of the equation (21) to get

$$\begin{aligned} L\{X_1''(t, \alpha)\} + L\{X_1(t, \alpha)\} &= L\left\{1 - \frac{t^\alpha}{\Gamma(1+\alpha)}\right\} \\ X_1(t, \alpha) &= L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{1}{p^{\alpha+1}(p^2+1)}\right\} \end{aligned}$$

For  $\alpha = 1$  and  $\alpha = 2$  we obtain (25) and (26).

The case for  $X_2(t, \alpha)$  defined in (7) can be treated similarly. The details are omitted here. At this stage we are in a position to provide the addition formulae for pretrigonometric functions  $p\cos(t, \alpha)$  and  $p\sin(t, \alpha)$ .

## 8. GENERALIZED PRETRIGONOMETRIC AND PREHYPERBOLIC FUNCTIONS.

We have,

$$\begin{aligned} p\exp(-t, \alpha) &= 1 - (-1)^\alpha \left\{ \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} - \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{t^{3+\alpha}}{\Gamma(4+\alpha)} - \dots \right\} \\ &= 1 + (-1)^\alpha \sum_{n=0}^{\infty} (-1)^n \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)}, \quad \alpha = 0, 1, 2, 3, \dots \end{aligned}$$

Trisecting the series for  $p\exp(-t, \alpha)$  we form three infinite series (absolutely convergent for  $t \in \mathbb{R}$  and  $\alpha \geq 0$ ) given as follows:

$$\begin{aligned} M_{3,0}(t, \alpha) &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{3n+\alpha}}{\Gamma(3n+1+\alpha)} \\ M_{3,1}(t, \alpha) &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+1+\alpha}}{\Gamma(3n+2+\alpha)} \\ M_{3,2}(t, \alpha) &= \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2+\alpha}}{\Gamma(3n+3+\alpha)} \end{aligned} \quad (27)$$

With initial data  $M_{3,0}(0, \alpha) = 1$ ,  $M_{3,1}(0, \alpha) = 0$ ,  $M_{3,2}(0, \alpha) = 0$ . Clearly

$$M_{3,0}(t, \alpha) = -M_{3,2}(t, \alpha)$$

$$M_{3,1}(t, \alpha) = (-1)^\alpha \frac{t^\alpha}{\Gamma(1+\alpha)} + M_{3,0}(t, \alpha) - 1$$

$$M_{3,2}(t, \alpha) = M_{3,1}(t, \alpha)$$

In the form of system, we have

$$\begin{aligned} \begin{bmatrix} M_{3,0}(t, \alpha) \\ M_{3,1}(t, \alpha) \\ M_{3,2}(t, \alpha) \end{bmatrix}' &= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_{3,0}(t, \alpha) \\ M_{3,1}(t, \alpha) \\ M_{3,2}(t, \alpha) \end{bmatrix} + \begin{bmatrix} 0 \\ (-1)^\alpha \frac{t^\alpha}{\Gamma(t+\alpha)} - 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} M_{3,0}(t, \alpha) \\ M_{3,1}(t, \alpha) \\ M_{3,2}(t, \alpha) \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \quad (28)$$

The infinite series represented by  $M_{3,0}(t, \alpha)$ ,  $M_{3,1}(t, \alpha)$ ,  $M_{3,2}(t, \alpha)$  is obviously the solution of the system of non-homogeneous equations given by (28). The relations in (27) define the extended pretrigonometric functions for  $n = 3$ .

In particular when  $\alpha = 1$ , we have

$$M_{3,0}(t, 1) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{3n+1}}{\Gamma(3n+2)} = M_{3,1}(t, 0) - t + 1$$

$$M_{3,1}(t, 1) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{\Gamma(3n+3)} = -M_{3,2}(t, 0)$$

$$M_{3,2}(t, 1) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+3}}{\Gamma(3n+4)} = M_{3,0}(t, 0) - 1$$

## 9. GRAPHICAL REPRESENTATION OF PREFUNCTIONS.

We now address the graphical representation of extended trigonometric functions  $M_{3,0}(t)$ ,  $M_{3,1}(t)$ ,  $M_{3,2}(t)$  are which are obtained as linear independent solutions of  $x''' + x = 0$ . These functions have many properties similar to the classical trigonometric functions. But these functions lack periodic properties although we can see from above graphical representation that they are oscillatory with interlacing zeros. These functions can be expressed in terms of classical trigonometric functions although such relations are complicated. The parametric representations of these functions given by

$$x = r M_{3,0}(t), \quad y = r M_{3,1}(t), \quad z = r M_{3,2}(t)$$

will generate a surface  $x^3 - y^3 + z^3 + 3xyz = r^3$  (Figure 4).

We have two graphs of Pretrigonometric functions namely  $p\cos(t, \alpha)$  and  $p\sin(t, \alpha)$  are drawn for  $0 \leq \alpha \leq 1$ . (Figure 5 & Figure 6) From the graphical representation we conclude that these functions are also oscillatory but they loose the periodicity property. Also these functions assume the values greater than 1 and less than  $-1$  at various points.

Lastly, we have the functions  $M_{30}(t, \alpha)$ ,  $M_{31}(t, \alpha)$ ,  $M_{32}(t, \alpha)$  which are known as the extended pretrigonometric functions. (Figures 7, 8, 9) gives the graphical representation of these functions for  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq 10$ . These functions are also oscillatory and they loose the periodicity property. Defined with the help of Gamma function instead of Factorial function, (such series may be termed as Khandeparkar-Deo-Dhaigude / KDD series), the nature of the function remains same but values of the functions at various points increase with large difference compared to extended trigonometric functions  $M_{30}(t)$ ,  $M_{31}(t)$ ,  $M_{32}(t)$ .

**Remark 1.**

$$M_{3,0}(t_1 + t_2, 1) = M_{3,1}(t_1 + t_2, 0) - (t_1 + t_2) + 1 \quad (29)$$

The right hand side of (29) is completely known and hence  $M_{3,0}(t_1 + t_2, 1)$  is completely known. Similarly addition formulae for  $M_{3,1}(t, 1)$ ,  $M_{3,2}(t, 1)$  can be computed.

**Remark 2.**

Trisecting the series (3) for  $p\exp(t, \alpha)$  we similarly form three infinite series (absolutely convergent for  $t \in \mathbb{R}$  and  $\alpha \geq 0$ )  $N_{3,0}(t, \alpha)$ ,  $N_{3,1}(t, \alpha)$ ,  $N_{3,2}(t, \alpha)$ . These series define extended hyperbolic functions for  $n = 3$ . The details are omitted here.

Generalized extended trigonometric functions and hyperbolic functions can be obtained from  $p\exp(-t, \alpha)$  and  $p\exp(t, \alpha)$  respectively by  $n$ -secting these infinite series and treating these on similar lines as has been done for  $n = 2$  and  $n = 3$  above. The details are omitted.

We expect that some applications of pretrigonometric and prehyperbolic relations will enlarge the scope of the newly defined functions.

**Remark 3. :** The following is one of the illustration for generalization of special functions. Hope this will work in new direction of development of special functions.

Extended Generating Function for **Pre Laguerre Polynomial :**

$$\begin{aligned}
& \frac{1}{(1-t)} \exp\left(\frac{-tx}{1-t}, \alpha\right) \\
= & \frac{1}{1-t} \left\{ 1 + (-1)^\alpha \sum_{r=0}^{\infty} (-1)^r \frac{t^{r+\alpha} x^{r+\alpha}}{(1-t)^{r+\alpha} \Gamma(r+1+\alpha)} \right\} \\
= & \frac{1}{1-t} + \sum_{r=0}^{\infty} \frac{(-1)^{r+\alpha}}{\Gamma(r+\alpha+1)} \frac{t^{r+\alpha} x^{r+\alpha}}{(1-t)^{r+\alpha+1}} \\
= & \frac{1}{1-t} + \sum_{r=0}^{\infty} \frac{(-1)^{r+\alpha}}{\Gamma(r+\alpha+1)} t^{r+\alpha} x^{r+\alpha} (1-t)^{-(r+\alpha+1)} \\
= & \frac{1}{1-t} + \sum_{r=0}^{\infty} \frac{(-1)^{r+\alpha}}{\Gamma(r+\alpha+1)} t^{r+\alpha} x^{r+\alpha} \sum_{s=0}^{\infty} \frac{(r+\alpha+s)!}{(r+\alpha)! s!} t^s \\
= & \frac{1}{1-t} + \sum_{\substack{r=0 \\ s=0}}^{\infty} (-1)^{r+\alpha} \frac{(r+\alpha+s)!}{\Gamma(r+\alpha+1) (r+\alpha)! s!} x^{r+\alpha} \sum_{s=0}^{\infty} t^{r+r+\alpha}
\end{aligned}$$

Coefficient of  $t^n$ , when  $r+s=n$ , for a fixed value of  $r$  is given by

$$(-1)^{r+\alpha} \frac{(n+\alpha)!}{\Gamma(r+\alpha+1) (r+\alpha)! (n-r)!} x^{r+\alpha}$$

The total coefficient of  $t^n$  is obtained by summing over all allowed values of  $r$ .  
 $S = n-r, s \geq 0 \Rightarrow (n-r) \geq 0$  or  $r \leq n$ . Coefficient of  $t^n$  is

$$\sum_{r=0}^n (-1)^{r+\alpha} \frac{(n+\alpha)!}{\Gamma(r+\alpha+1) (r+\alpha)! (n-r)!} x^{r+\alpha} = L_n(x, \alpha)$$

$L_n(x, \alpha)$  is the **PreLaguerre Polynomial**. when  $\alpha = 0$ ,

$$\sum_{r=0}^n (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r = L_n(x, 0) = L_n(x)$$

$$\frac{1}{(1-t)} \left\{ \exp\left(\frac{-tx}{1-t}, \alpha\right) - 1 \right\} = \sum_{n=0}^{\infty} t^{n+\alpha} L_n(x, \alpha)$$

### References :

[ 1 ] I. S. Sokolnikoff, R. M. Redheffer, *Mathematics of Physics and Engineering*, International Student Edition, (1966).

[ 2 ] Deo S. G., V. Lakshmikantham, V. Raghavendra, *Ordinary Differential Equations, 2<sup>nd</sup> Edition*, Tata McGraw Hill Publishing Company Limited (1997).

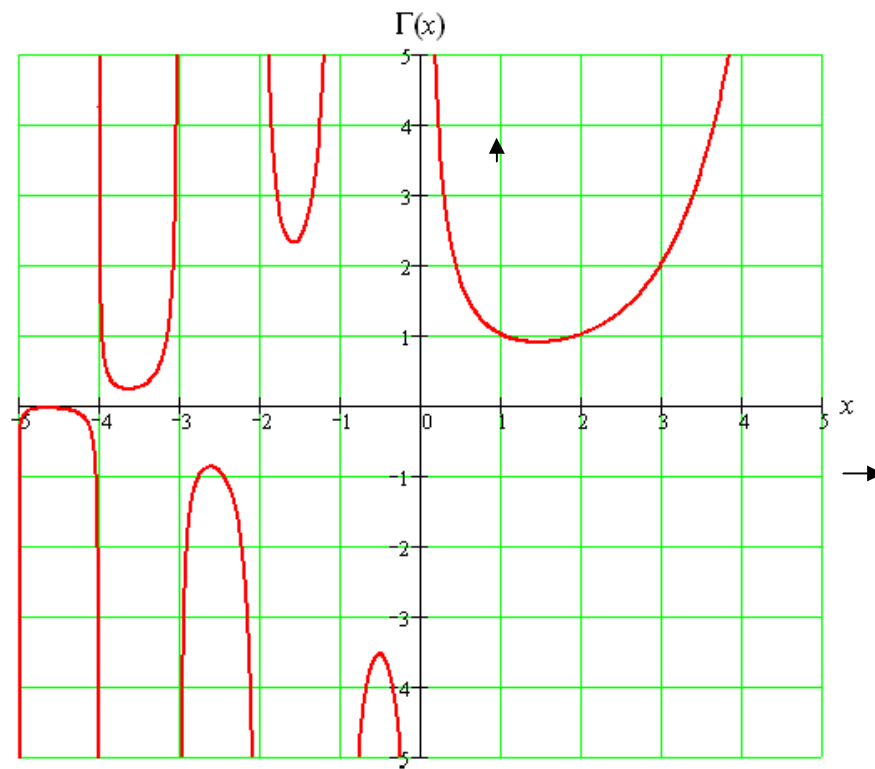


Figure 1 : Graph of  $\Gamma(\alpha)$

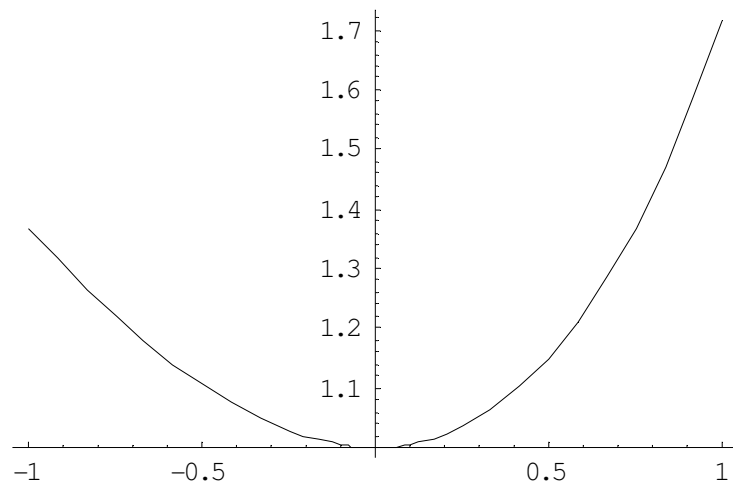
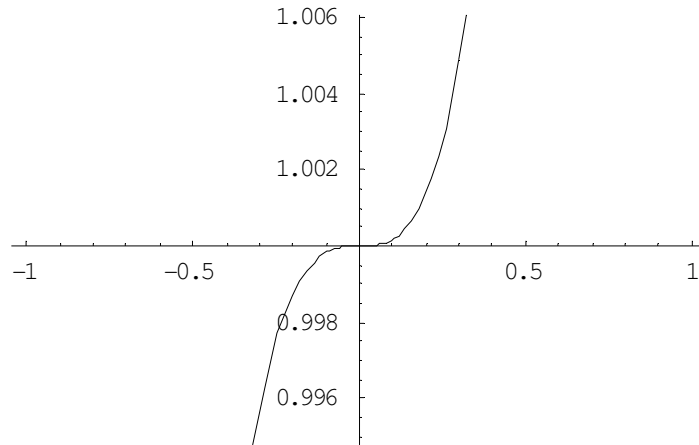


Figure 2.: Graph of  $\text{pexp}(t, 1)$

Figure 3 : Graph of  $\text{pexp}(t, 2)$ 

t	$\text{pexp}(t, 0)$	$\text{pexp}(t, 1)$	$\text{pexp}(t, 2)$	$\text{pexp}(t, 3)$	$\text{pexp}(t, 4)$
0	1.00000	1.00000	1.00000	1.00000	1.00000
1	2.71828	1.71828	1.21828	1.05162	1.00995
-1	0.36788	1.36788	0.86788	1.03455	0.99288

Table 1 : preexponential function at  $t = 0, 1, -1$ .

$x = r M_{30}(t)$ ,  $y = r M_{31}(t)$ ,  $z = r M_{32}(t)$  will generate a surface  
 $x^3 - y^3 + z^3 + 3xyz = r^3$

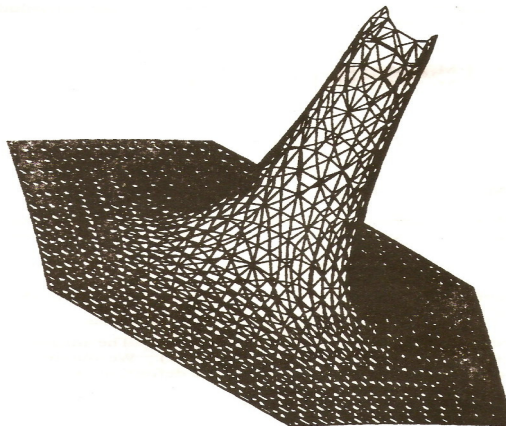


Figure 4 . .



Graph of  $p\cos(t, \alpha) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)}$ ;  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq 2\pi$ .

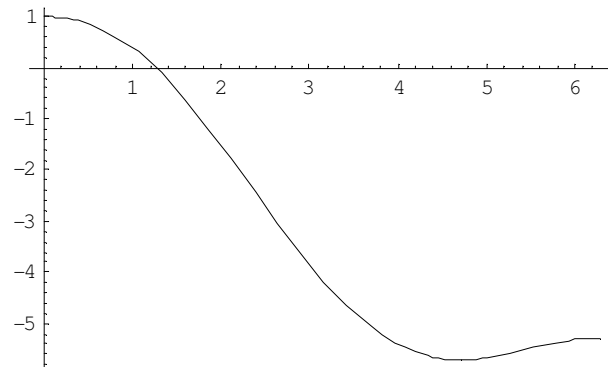


Figure 5.

Graph of  $p\sin(t, \alpha) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1+\alpha}}{\Gamma(2n+2+\alpha)}$ ;  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq 2\pi$ .

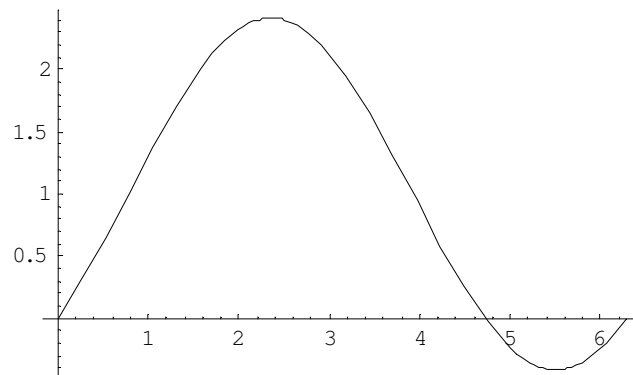


Figure 6.

Graph of  $M_{30}(t, \alpha)$ ;  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq 10$ .

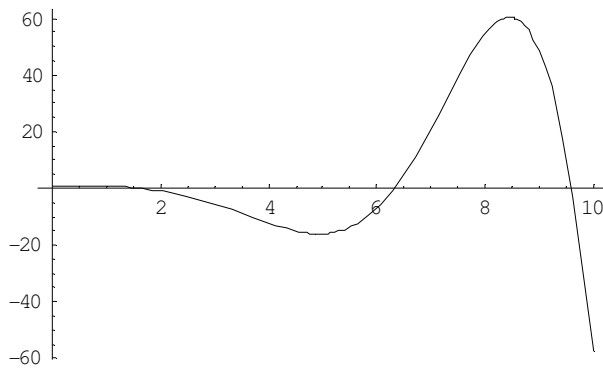


Figure 7.

Graph of  $M_{31}(t, \alpha)$ ;  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq 10$ .

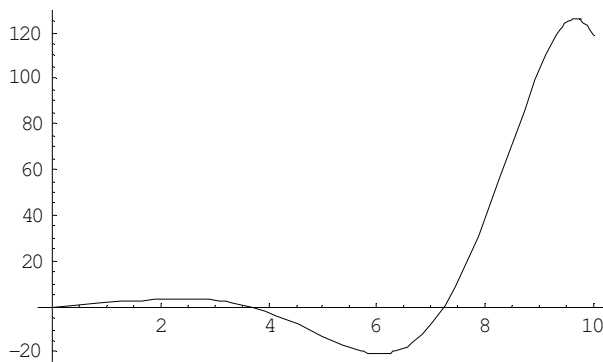


Figure 8.

Graph of  $M_{32}(t, \alpha)$ ;  $0 \leq \alpha \leq 1$ ,  $0 \leq t \leq 10$ .

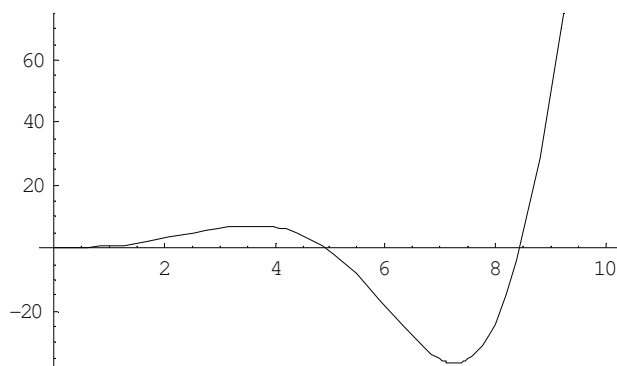


Figure 9.