Communications in Applied Analysis **14** (2010), no. 2, 123–136

OSCILLATIONS IN LINEAR NEUTRAL DELAY IMPULSIVE DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT. This paper presents the conditions for the oscillation of the solutions of the neutral delay impulsive differential equation

$$\begin{cases} [x(t) + px(t-\tau)]' + qx(t-\sigma) = 0, & t \neq t_k \\ \Delta[x(t_k) + p\,x(t_k-\tau)] + q_0 x(t_k-\sigma) = 0, & \forall t = t_k \end{cases}$$

for constant coefficients and delays. The relevance of the resulting theorems is manifested in many extensions, particularly in the investigation involving neutral impulsive differential equations with variable coefficients and delays.

AMS (MOS) Subject Classification. 99Z00. 34K11, 34K40, 34K45

1. INTRODUCTION

It is observed that a great deal of interest is still being focused on the oscillations of ordinary and neutral delay differential equations in spite of the existence of extensive literature in these fields ([3], [6], [5], [9], [8], [7]). More recently, the investigation of the oscillatory properties of yet another interesting area of impulsive differential equations known as the neutral delay impulsive differential equations, has again captured the attention of many applied mathematicians as well as other scientists around the world ([10], [2]).

A neutral delay impulsive differential equation of the nth order is a differential system comprising an nth-order differential equation and its impulsive conditions in which the highest-order derivative of the unknown function appears in the differential equation both with and without delays. Thus, the linear neutral delay impulsive differential equation

$$\begin{cases} [x(t) + px(t-\tau)]' + qx(t-\sigma) = 0, & t \neq t_k \\ \Delta[x(t_k) + p\,x(t_k-\tau)] + q_0 x(t_k-\sigma) = 0, & \forall t = t_k, \end{cases} \tag{1.1}$$

Received June 6, 2009

where $t, t_k \in R\ \forall k \in N$, $p,\ q,\ q_0 \in R$ and $\tau, \sigma \in R_+$ is an example of a first-order neutral delay impulsive differential equation.

Usually, the solution $x(t)$ for $t \in [t_0, T)$ of the impulsive differential equation or its first derivative $x'(t)$ is a piece-wise continuous function with points of discontinuity $t_k \in [t_0, T), t_k \neq t$. Therefore, in order to simplify the statements of our assertions later, we introduce the set of functions PC and PC^r which are defined as follows.

Let $r \in N, D := [T, \infty) \subset R$ and let $S := \{t_k\}_{k \in E}$, where E represents a subscript set which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout our discussion, we will assume that the sequence $\{t_k\}_{k \in E}$ are moments of impulse effect and satisfy the properties:

C1.1 $\{t_k\}_{k \in E}$ is defined with $E := N$, then $0 < t_1 < t_2 < \cdots$ and

$$\lim_{k \to +\infty} t_k = +\infty$$

C1.2 If $\{t_k\}_{k \in E}$ is defined with $E := Z$, then $t_0 \leq 0 < t_1,\ t_k < t_{k+1}$ for all $k \in Z$, $k \neq 0$, and

$$\lim_{k \to \pm\infty} t_k = \pm\infty.$$

We denote by $PC(D, R)$ the set of all functions $\varphi : D \to R$, which are continuous for all $t \in D$, $t \notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$.

By $PC^r(D, R)$, we denote the set of functions $\varphi : D \to R$ having derivative $\dfrac{d^j \varphi}{dt^j} \in PC(D, R),\ 0 \leq j \leq r$ ([1], [4]).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols $PC(D, R; S)$ and $PC^r(D, R; S), r \in N$.

In the sequel, all functional inequalities that we write are assumed to hold finally, that is, for all sufficiently large t.

Let $\gamma = \max\{\tau, \sigma\}$ and let $t_1 \geq t_0$. By a solution of equation (1.1), we mean a function $x(t) \in PC[[t_1 - \gamma, \infty), R]$ such that $x(t) + px(t - \tau)$ is piece-wise continuously differentiable for $t \geq t_1$ and such that equation (1.1) is satisfied for all $t \geq t_1$.

Let $t_1 \geq t_0$ be a given initial point and let $\varphi \in PC[[t_1 - \gamma, t_1], R]$ be a given initial function. Then if $x(t - \tau) \in PC^1[[t_1 - \gamma, t_1], R],\ x(t - \sigma) \in PC[[t_1 - \gamma, t_1], R]$ and

$$\lim_{t \to \infty} (t - \tau) = \lim_{t \to \infty} (t - \sigma) = \infty, \forall t \geq t_1,\ \tau, \sigma \in R_+,$$

equation (1.1) has a unique solution on $[t_1, \infty)$ satisfying the initial condition ([1])

$$x(t) = \varphi(t)\ for\ t_1 - \gamma \leq t \leq t_1\ . \tag{1.2}$$

A solution x of the initial value problem (1.1) and (1.2) on $[t_1, \infty)$ is said to be

(i) finally positive, if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) > 0$ for all $t \geq T$;

(ii) finally negative, if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) < 0$ for all $t \geq T$;

(iii) regular, if it is defined in some half line $[T_x, \infty)$ for some $T_x \in R$ and

$$\sup\{|x(t)|\} : t \geq T\} > 0, \quad \forall T > T_x \text{ ([4]).}$$

Unlike the classical definition of oscillations, when we say that every solution of equation (1.1) oscillates, we mean that for every initial point $t_1 \geq t_0$ and for every initial function $\varphi \in PC[[t_1 - \gamma, t_1], R]$, the solution of the initial value problem (1.1) and (1.2) on $[t_1, \infty)$ is neither finally positive nor finally negative. If it is false that every solution of the initial value problem (1.1) and (1.2) oscillates, then there exists a $t_1 \geq t_0$ such that the solution of the initial value problem (1.1) and (1.2) is either finally positive or finally negative.

2. STATEMENT OF THE PROBLEM

We return to the linear neutral delay impulsive differential equation of the first order and now write it as follows:

$$\begin{cases} [x(t) + px(t-\tau)]' + qx(t-\sigma) = 0, & t \notin S \\ \Delta[x(t_k) + p\,x(t_k - \tau)] + q_0 x(t_k - \sigma) = 0, & \forall t_k \in S, \end{cases} \tag{2.1}$$

where $t, t_k \in R \; \forall k \in N$. Our aim in this paper is to fill some of the gaps identified in the work by Bainov and Simeonov ([1]). We introduce the following conditions:

C2.1 There exist nonnegative integers m_1 and m_2 such that

$$t_{k+m_1} = t_k + \tau, t_{k+m_2} = t_k + \sigma, \quad k \in N.$$

The following lemmas will be useful in putting the main results together.

Lemma 2.1. *Let condition* C2.1 *be satisfied and let*

$$\tau, \quad \sigma \in R_+, \quad qq_0 \geq 0. \tag{2.2}$$

Suppose further, that equation (2.1) *has a finally positive solution* $x(t)$. *Then,*

(a) *the functions*

$$z(t) = x(t) + px(t-\tau) \tag{2.3}$$

and

$$w(t) = z(t) + pz(t-\tau) \tag{2.4}$$

are solutions of equation (2.1) *and* $z \in PC^1$, $w \in PC^2$;

(b) *if* $q + q_0 > 0$, *then* $z(t)$ *is a non-increasing function and either*

$$\lim_{t\to+\infty} z(t) = -\infty \tag{2.5}$$

or

$$\lim_{t\to+\infty} z(t) = 0; \tag{2.6}$$

(c) *if* $q + q_0 < 0$*, then* $z(t)$ *is a non-decreasing function and either*

$$\lim_{t \to +\infty} z(t) = +\infty \tag{2.7}$$

or

$$\lim_{t \to +\infty} z(t) = 0. \tag{2.8}$$

Proof. (a) The proof follows from equations (2.3) and (2.4).

(b) Let $q + q_0 > 0$, $q \geq 0$, $q_0 \geq 0$, then

$$\begin{cases} z'(t) = -qx(t-\sigma), & t \notin S \\ \Delta z(t_k) = -q_0 x(t_k - \sigma), & \forall t_k \in S, \end{cases} \tag{2.9}$$

which implies z is a non-increasing function. It converges therefore, either to $-\infty$ or to a number L, where $-\infty < L < +\infty$, as $t \to \infty$. If z converges to $-\infty$, then the proof of (b) is complete.

Let us consider the case when $z(t) \to L$ as $t \to \infty$. We integrate equation (2.9) from t to $+\infty$ and obtain

$$L - z(t) = -\int_t^{+\infty} qx(s-\sigma)ds - \sum_{t \leq t_k} q_0 x(t_k - \sigma).$$

Since the integral is finite, it means

$$\int_{T_0}^{\infty} x(t)dt < +\infty \quad \text{(since } q > 0 \text{ and } q_0 > 0\text{)}$$

and

$$\sum_{T_0 \leq t_k} x(t_k) < +\infty \quad \text{(since } q_0 > 0\text{)}.$$

Then

$$-\infty < \int_{T_1}^{\infty} z(t)dt = \int_{T_1}^{\infty} x(t)dt + p\int_{T_1}^{\infty} x(t-\tau)dt < +\infty$$

and

$$-\infty < \sum_{T_1 \leq t_k} z(t_k) = \sum_{T_1 \leq t_k} x(t_k) + p\sum_{T_1 \leq t_k} x(t_k - \tau) < +\infty$$

for a suitable $T_1 \geq T_0$.

On the other hand, if $z(t) \to L > 0$ ($L < 0$), then

$$-\infty < \int_{T_2}^{\infty} z(t)dt = +\infty \left(\int_{T_2}^{\infty} z(s)ds = -\infty \right).$$

This contradicts our above assertions. Hence $z(t) \to 0$.

(c) The proof is analogous to that of Lemma 2.1(b). This completes the proof. □

Remark 2.2. If $q > 0$, then the function $z(t)$ is strictly decreasing and it is strictly increasing if $q < 0$.

Lemma 2.3. *Let condition* C2.1 *be satisfied and let*

$$\tau, \sigma \in R_+, \quad q > 0, \quad q_0 > 0. \tag{2.10}$$

Suppose further, that equation (2.1) *has a finally positive solution* $x(t)$ *and the functions* $z(t)$ *and* $w(t)$ *are defined by equations* (2.3) *and* (2.4) *respectively.*

(a) *The following assertions are equivalent.*

(i) $\lim_{t\to+\infty} z(t) = -\infty$*;*

(ii) $p < -1$*;*

(iii) $\lim_{t\to+\infty} x(t) = +\infty$*;*

(iv) $w(t) > 0$, $w'(t) > 0$, $w''(t) > 0$,

$$\lim_{t\to+\infty} w(t) = +\infty, \quad \Delta w(t_k) > 0, \quad \Delta w'(t_k) > 0. \tag{2.11}$$

(b) *The following assertions are equivalent.*

(j) $\lim_{t\to+\infty} z(t) = 0$*;*

(jj) $p > -1$*;*

(jjj) $\lim_{t\to+\infty} x(t) = 0$*;*

(jv) $w(t) > 0$, $w'(t) < 0$, $w''(t) > 0$,

$$\lim_{t\to+\infty} w(t) = 0, \quad \Delta w(t_k) < 0, \quad \Delta w'(t_k) > 0. \tag{2.12}$$

Proof. (a) (i) $\Rightarrow$ (ii) Let (i) hold, that is, condition (2.5) is fulfilled. By definition,

$$z(t) = x(t) + px(t-\tau).$$

Both $x(t)$ and $x(t-\tau)$ are positive functions, meaning that the above expression can be negative only if $p < 0$. Consequently, $z(t) \to -\infty$ only if $x(t)$ is unbounded.

We show that there exists $T_0 \in R$ such that

$$z(T_0^+) < \sup_{t\le T_0} x(t) \quad \forall T_0 \in R.$$

Let us assume conversely that such T_0 does not exist. Then

$$x(T_0^+) < 0 \; and \; x(T_0^+) \ge \sup_{t\le T_0} x(t).$$

Consequently, $\exists \varepsilon > 0$ such that $\forall s$, $T_0 < s < T_0 + \varepsilon$, $x(s) \ge \sup_{t\le T_0} x(t)$. Hence

$$\sup\left\{ s : x(s) < \sup_{t\le T_0} x(t) \right\} = T \in R$$

must exist, otherwise $x(s)$ is bounded contrary to our earlier assertion.

But then for T,

$$\sup_{t\le T} x(t) = x(T)$$

holds. With this $T_0 := T$, we obtain the inequality

$$0 > z(T_0^+) = x(T_0^+) + p\,x(T_0 - \tau^+) \ge x(T_0^+)(1+p).$$

This is only possible if $p < -1$, since $x(T_0^+) > 0$.

(ii) ⇒ (iii) Let $p < -1$. Also, let us assume that z is finally positive. Then z is decreasing and $z \to 0$ by Lemma 2.1. If

$$0 < z(t) = x(t) + p\,x(t-\tau)$$

then

$$x(t) > (-p)\,x(t-\tau). \tag{2.13}$$

On the other hand, by $z(t) \to 0$ and

$$\begin{cases} z'(t) = -qx(t-\sigma), & t \notin S \\ \Delta z(t_k) = -q_0 x(t_k - \sigma), & \forall t_k \in S, \end{cases}$$

$$0 - z(t) = -q \int_t^{+\infty} x(s-\sigma)ds - q_0 \sum_{t\le t_k} x(t_k - \sigma) > -\infty.$$

Hence,

$$\int_{t-\sigma}^{\infty} x(s)ds < \infty, \ \sum_{t\le t_k} x(t_k-\sigma) < \infty. \tag{2.14}$$

Inequality (2.13) gives a contradiction since

$$x(t_k + i\sigma) > (-p)^i x(t_k - \sigma), \quad 1 \le i < \infty \tag{2.15}$$

would have led to infinity in (2.14).

Indeed, $x(t_k + i\sigma) = x(t_k + im_2)$ by C2.1, $1 \le i < \infty$. Hence, by inequality (2.15),

$$x(t_k + i\sigma) = x(t_k + im_2) > (-p)^i x(t_k - \sigma).$$

In the statement (ii) ⇒ (iii), $p < -1 \Leftrightarrow -p > 1$. Thus,

$$x(t_k + i\sigma) = x(t_k + im_2) > (-p)^i x(t_k - \sigma) > 0$$

and x is finally positive by the condition of Lemma 2.1. Therefore, the set $\{x(t_k - \sigma) \mid t_k \ge t\}$ is an infinite subsequence with each value greater than a positive constant $x(t_k - \sigma)$. Thus, the sum adds up to $+\infty$. Hence z cannot be finally positive. Thus, by Lemma 2.1, $z \to -\infty$ if $t \to \infty$. Consequently, there exists T_0 such that $z(s) < 0$ if $s > T_0$.

Since

$$z(t) = x(t) + p\,x(t-\tau)$$

and $z(t) \to -\infty$,

$$0 > z(t) > p\,x(t-\tau)$$

which implies

$$0 < \frac{z(t)}{p} < x(t-\tau) \to +\infty.$$

<u>(iii) ⇒ (i)</u> Assume that $x(t) \to \infty$ for $t \to \infty$. We show that if $z(t) \to 0$, it implies that $x(t) \not\to \infty$. Really,

$$\begin{cases} z'(t) = -qx(t-\sigma), & t \notin S \\ \Delta z(t_k) = -q_0 x(t_k - \sigma), & \forall t_k \in S. \end{cases}$$

Hence,

$$0 - z(t) = -q \int_t^{+\infty} x(s-\sigma)ds - q_0 \sum_{t \le t_k} x(t_k - \sigma) < \infty.$$

Thus,

$$\int_t^{\infty} x(s-\sigma)ds < \infty \; and \; \sum_{t \le t_k} x(t_k - \sigma) < \infty$$

which contradicts the statement that $x(t) \to \infty$. Hence, $z(t) \to 0$ means $x(t) \not\to \infty$. Therefore, $x(t) \to \infty$ leads to $z(t) \to -\infty$ by Lemma 2.1. This completes the proof.

<u>(i) ⇒ (iv)</u> $q > 0$, $q_0 > 0$ and equation (2.5) leads to

$$\begin{cases} w'(t) = -qz(t-\sigma), & t \notin S \\ \Delta w(t_k) = -q_0 z(t_k - \sigma), & \forall t_k \in S. \end{cases} \tag{2.16}$$

The integration of (2.16) gives (iv).

<u>(iv) ⇒ (i)</u> If $w(t) \to +\infty$ for $t \to \infty$, then by equation (2.16), it implies that $z(s) < 0$ for some $s > T_0$. Hence by the proof of Lemma 2.1a(i), $z(t) \to -\infty$.

(b) Applying contraposition to the statements of Lemma 2.2(a), we obtain

$$\neg(j) \Rightarrow \neg(jj) \Rightarrow \neg(jjj) \Rightarrow \neg(jv) \Rightarrow \neg(j).$$

Thus,

$$\neg(j) \Rightarrow \neg(jj) \text{ means } z(t) \to 0 \Rightarrow p \ge -1;$$

$$\neg(j) \Rightarrow \neg(jjj) \text{ means } z(t) \to 0 \Rightarrow x(t) \not\to \infty;$$

$$\neg(j) \Rightarrow \neg(jv) \text{ means } z(t) \to 0 \Rightarrow w(t) \not\to \infty.$$

<u>(j) ⇒ (jj)</u> We know that $z(t) \to 0 \Rightarrow p \ge -1$. Let us assume that $p = -1$. If z, being a decreasing function, has negative values, then $z(t)$ finally tends to $-\infty$ by Lemma 2.1. Hence, $z(t) \to 0$ implies that z is finally positive. Thus,

$$0 < z(t) = x(t) - x(t-\tau), \quad \forall t > T_0.$$

Hence,

$$x(t-\tau) < x(t), \quad \forall t > T_0.$$

Iterating the above inequality, we obtain

$$x(t+i\tau) > x(t-\tau) > 0. \tag{2.17}$$

On the other hand,

$$\begin{cases} z'(t) = -qx(t-\sigma), & t \notin S \\ \Delta z(t_k) = -q_0 x(t_k-\sigma), & \forall t_k \in S, \end{cases}$$

where t_k belongs to the set of points of impulse effect. Hence

$$0 - z(t) = -q \int_t^{+\infty} x(s-\sigma)ds - q_0 \sum_{t \le t_k} x(t_k - \sigma).$$

From this $\sum_{t \le t_k} x(t_k-\sigma) < \infty$ which contradicts condition (2.17). Hence the assumption that, $z(t) \to 0$ when $p = -1$ leads to a contradiction. Therefore, only $p > -1$ is admissible.

<u>(jj) ⇒ (jjj)</u> Now we are familiar with the fact when $p > -1$, $x(t) \not\to \infty$. Let us check what happens when $p \le 0$. Since, whenever $x(t) \not\to \infty$ implies $z(t) \not\to -\infty$, it follows by Lemma 2.1, that $z(t) \to 0$. Therefore,

$$z(t) = x(t) + px(t-\tau) > x(t) > 0, \quad \forall t > T_0.$$

Hence, $x(t) \to 0$.

Let $-1 < p < 0$. Then, the fact that z is a strictly decreasing function and $t \in [T_0, T_0+\tau]$ implies

$$x(t) = (-p)x(t-\tau) + z(t) < (-p)x(t-\tau) + z(T_0-\tau).$$

We rewrite the above inequality in the form

$$x(t) < (-p)x(t-\tau) + z(T_0-\tau)$$

and replace the function $x(t-\tau)$ with its supremum

$$x(t-\tau) \le \sup_{s \in [T_0-\tau, T_0]} x(s).$$

Then,

$$x(t) < (-p) \sup_{s \in [T_0-\tau, T_0]} x(s) + z(T_0-\tau),$$

and hence

$$\sup_{s \in [T_0, T_0+\tau]} x(s) < (-p) \sup_{s \in [T_0-\tau, T_0]} x(s) + z(T_0-\tau). \tag{2.18}$$

Let

$$\theta_k := T_0 + k\tau, \quad M_k := \sup_{s \in [\theta_k-\tau, \theta_k]} x(s) \quad \forall\ -1 \le k < \infty.$$

Then we get

$$M_{k+1} < (-p)M_k + z(\theta_k - \tau). \tag{2.19}$$

We recall that $-1 < p < 0 \Leftrightarrow 0 < -p < 1$. Multiplying both sides of (2.19) by $-p$, we obtain

$$(-p)M_{k+1} < (-p)^2 M_k + (-p)z(\theta_k - \tau).$$

By the assumption that z is a strictly decreasing function,

$$z(\theta_{k+1} - \tau) \leq z(\theta_k - \tau), \quad 1 \leq k < \infty. \tag{2.20}$$

Hence,

$$M_{k+2} < (-p)M_{k+1} + z(\theta_{k+1} - \tau) < (-p)^2 M_k + (-p)z(\theta_k - \tau) + z(\theta_k - \tau)$$

or

$$M_{k+2} < (-p)^2 M_k + z(\theta_k - \tau)(1 + (-p)).$$

We repeat the process again and obtain the inequality

$$\begin{aligned} M_{k+3} &< (-p)M_{k+2} + z(\theta_{k+2} - \tau) < (-p)^3 M_k + z(\theta_k - \tau)(1 + (-p) + (-p)^2) \\ &\leq (-p)^3 M_k + z(\theta_k - \tau)\sum_{s=0}^{2}(-p)^s. \end{aligned}$$

Employing the principles of induction, we finally obtain, for $\ell > k$

$$M_\ell \leq (-p)^{\ell-k} M_k + z(\theta_k - \tau)\sum_{j=k}^{\ell}(-p)^j \leq (-p)^{\ell-k} M_k + z(\theta_k - \tau)\frac{1}{1+p}. \tag{2.21}$$

We return to inequality (2.19) and immediately observe that

$$\sup_{t>\ell} M_t \leq (-p)^{\ell-k} M_k + z(\theta_k - \tau)\frac{1}{1+p}.$$

On the other hand,

$$\inf_{\ell\geq k}\sup_{t>\ell} M_t \leq (-1)^{\ell-k} M_k + z(\theta_k - \tau)\frac{1}{1+p}, \quad \forall k \leq \ell < \infty. \tag{2.22}$$

We take the limit of both sides of (2.22) in ℓ and obtain

$$\limsup M_\ell \leq z(\theta_k - \tau)\frac{1}{1+p}.$$

Hence, as $\ell \to \infty$,

$$\limsup M_\ell \leq z(\theta_k - \tau)\frac{1}{1+p} \to 0, \tag{2.23}$$

therefore

$$M_\ell \to 0 \Rightarrow x(t) \to 0.$$

(jjj) $\Rightarrow$ (jv) From the fact that $w(t) \not\to \infty$ when $t \to \infty$, it follows, from (jjj), that

$$z(t) = x(t) + p\,x(t - \tau) \to 0$$

and hence z is finally positive. Therefore applying Lemma 2.1 to z and w, it follows that $w(t) \not\to \infty$ implies that $w(t) \to 0$ since $w(t) = z(t) + pz(t-\sigma)$ and z is finally positive.

(jv) ⇒ (j) Since $w(t) \not\to 0$ as $t \to \infty$, it implies that $z(t) \not\to -\infty$ hence, by Lemma 2.1, $z(t) \to 0$ as $t \to \infty$. This completes the proof of Lemma 2.2. □

From Lemmas 2.1 and 2.2, we deduce the following additional assertions.

Lemma 2.4. *Let us assume that condition* C2.1 *is fulfilled and that*

$$\tau, \sigma \in R, \quad qq_0 > 0.$$

Suppose also, that equation (2.1) *has a finally positive solution* $x(t)$ *and the functions* $z(t)$ *and* $w(t)$ *are defined by equations* (2.3) *and* (2.4) *respectively. Then,*

(a) *conditions* (2.11) *are satisfied provided* $x(t)$ *is an unbounded function;*
(b) *conditions* (2.12) *are satisfied provided* $x(t)$ *is a bounded function.*

Proof. This follows from Lemma 2.2:

(a) $(iii) \Leftrightarrow (iv)$; and
(b) $(jjj) \Leftrightarrow (jv)$.

□

In what follows, we try to deduce the oscillatory conditions taking advantage of the above lemmas.

3. MAIN RESULTS

Consider the impulsive delay differential equation

$$\begin{cases} y'(t) + \sum\limits_{i=1}^{n} q_i(t) y(t-\tau_i(t)) = 0, & t \notin S \\ \Delta y(t_k) + \sum\limits_{i=1}^{n} q_{ik} y(t_k - \tau_i(t_k)) = 0, & \forall t_k \in St. \end{cases} \tag{3.1}$$

and the impulsive delay inequalities

$$\begin{cases} x'(t) + \sum\limits_{i=1}^{n} p_i(t) x(t-\tau_i(t)) \le 0, & t \notin S \\ \Delta x(t_k) + \sum\limits_{i=1}^{n} p_{ik} x(t_k - \tau_i(t_k)) \le 0, & \forall t_k \in S \end{cases} \tag{3.2}$$

and

$$\begin{cases} z'(t) + \sum\limits_{i=1}^{n} r_i(t) z(t-\tau_i(t)) \ge 0, & t \notin S \\ \Delta z(t_k) + \sum\limits_{i=1}^{n} r_{ik} z(t_k - \tau_i(t_k)) \ge 0, & \forall t_k \in S. \end{cases} \tag{3.3}$$

We introduce the condition:

C3.1

$$\begin{cases} p_i,\ q_i,\ r_i \in PC(R_+, R_+),\ \tau_i \in C(R_+, R_+),\ i = 1, 2, \ldots, n \\ p_{ik},\ q_{ik},\ r_{ik} \geq 0, k \in N,\ \ i = 1, 2, \ldots, n. \end{cases}$$

Let $t_0 \geq 0$. The initial interval associated with the above equation and inequalities is the segment $[t_{-1}, t_0]$, where

$$t_{-1} = \min_{1 \leq i \leq n} \left\{ \inf_{t \geq t_0} \{t - \tau_i(t)\} \right\}. \tag{3.4}$$

Theorem 3.1. *Let condition* C3.1 *be fulfilled and*

$$\begin{cases} p_i(t) \geq q_i(t) \geq r_i(t); & \forall t \in R_+,\ i = 1, 2, \ldots, n \\ p_{ik} \geq q_{ik} \geq r_{ik}; & k \in N,\ i = 1, 2, \ldots, n. \end{cases} \tag{3.5}$$

Assume that $y(t)$, $x(t)$ and $z(t)$ are solutions of equation (3.1) *and inequalities* (3.2) *and* (3.3) *respectively and belong to the space $PC([t_{-1}, +\infty), R)$ and such that*

$$x(t) > 0,\ t \geq t_0, \tag{3.6}$$

$$z(t_0^+) \geq y(t_0^+) \geq x(t_0^+), \tag{3.7}$$

$$\frac{x(t)}{x(t_0)} \geq \frac{y(t)}{y(t_0)} \geq \frac{z(t)}{z(t_0)} \geq 0, \quad t_{-1} \leq t \leq t_0 \tag{3.8}$$

(see [1]*). Then,*

$$z(t) \geq y(t) \geq x(t), \quad \forall t \geq t_0. \tag{3.9}$$

Now consider the impulsive differential inequality (3.2) together with the impulsive differential equation

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0, & t \notin S \\ \Delta x(t_k) + \sum_{i=1}^{n} p_{ik} x(t_k - \tau_i(t_k)) = 0, & \forall t_k \in S. \end{cases} \tag{3.10}$$

Using Theorem 3.1, we obtain the following.

Corollary 3.2. *Let condition* C3.1 *be fulfilled. Then the following statements are equivalent:*

(a) *The inequality* (3.2) *has a finally positive solution.*
(b) *The equation* (3.10) *has a finally positive solution.*

Comparison results are also needed for differential equations and inequalities with advanced arguments. The following result is an analogue of Corollary 3.1.

Corollary 3.3. ([1]) *Suppose that*

$$p_i \in PC(R_+, R_+),\ \tau_i \in C(R_+, R_+),\ p_{ik} \geq 0,\ k \in N. \tag{3.11}$$

The following statements are equivalent:

(i) *The inequality*

$$\begin{cases} x'(t) - \sum_{i=1}^{n} p_i(t)x(t+\tau_i(t)) \geq 0, & t \notin S \\ \Delta x(t_k) - \sum_{i=1}^{n} p_{ik}x(t_k+\tau_i(t_k)) \geq 0, & \forall t_k \in S, \end{cases}$$

has a finally positive solution.

(ii) *The equation*

$$\begin{cases} x'(t) - \sum_{i=1}^{n} p_i(t)x(t+\tau_i(t)) = 0, & t \notin S \\ \Delta x(t_k) - \sum_{i=1}^{n} p_{ik}x(t_k+\tau_i(t_k)) = 0, & \forall t_k \in S, \end{cases}$$

has a finally positive solution.

Armed with the above tools, we can now go ahead to establish the expected oscillatory conditions.

Theorem 3.4. *Let conditions* C2.1 *and* (2.2) *be fulfilled. Then assuming that x is non-oscillatory,*

(a) $p < -1$ *if and only if* $\lim_{t\to+\infty} |x(t)| = +\infty$ *;*
(b) $p > -1$ *if and only if* $\lim_{t\to+\infty} |x(t)| = 0$.

This implies that it can be finally positive or finally negative. Hence, its proof follows immediately from Lemma 2.2. Precisely,

(a) $(ii) \Leftrightarrow (iii)$;

and

(b) $(jj) \Leftrightarrow (jjj)$.

Theorem 3.5. *Consider the neutral delay impulsive differential equation*

$$\begin{cases} [x(t) - x(t-\tau)]' + qx(t-\sigma) = 0, & t \notin S \\ \Delta[x(t_k) - x(t_k-\tau)] + q_0x(t_k-\sigma) = 0, & \forall t_k \in S \end{cases} \tag{3.12}$$

that is, equation (2.1) *when $p = -1$. Let condition* C2.1 *be fulfilled and suppose that*

$$\tau, \quad \sigma \in R_+, \quad qq_0 \geq 0, \quad q + q_0 > 0.$$

Then each regular solution of equation (3.12) *is oscillatory.*

Proof. If equation (3.12) has a finally positive solution $x(t)$, then in view of Lemma 2.1, we have for the solution $z(t) = x(t) - x(t-\tau)$ either

$$\lim_{t\to+\infty} z(t) = -\infty \text{ or } \lim_{t\to+\infty} z(t) = -\infty.$$

Then by Lemma 2.2, either $p > -1$ or $p < -1$, that is, the equality $p = -1$ is impossible. This completes the proof of the theorem. □

The following theorem gives sufficient conditions for the oscillations of the solutions of equation (2.1) in the case $p \neq -1$.

Theorem 3.6. *Let condition* C2.1 *be fulfilled and suppose*

$$\tau, \quad \sigma \in R_+, \quad p \neq -1, \quad q > 0, \quad q_0 > 0, \quad (1+p)(\sigma - \tau) > 0.$$

Assume further, that each regular solution of the following equation

$$\begin{cases} w'(t) + \dfrac{q}{1+q} w(t - (\sigma - \tau)) = 0, & t \notin S \\ \Delta w(t_k) + \dfrac{q_0}{1+p} w(t_k - (\sigma - \tau)) = 0, & \forall t_k \in S \end{cases} \tag{3.13}$$

is oscillatory. Then each regular solution of equation (2.1) *is also oscillatory.*

Proof. Let us assume that equation (2.1) has a finally positive solution $x(t)$. Then the function $w(t)$ defined by equation (2.4) is finally positive and moreover, satisfies the following equation

$$\begin{cases} w'(t) + p\, w'(t-\tau) + q w(t-\sigma) = 0, & t \notin S \\ \Delta w(t_k) + p \Delta w(t_k - \tau) + q_0 w(t_k - \sigma) = 0, & \forall t_k \in S. \end{cases} \tag{3.14}$$

Since $w'(t)$ is an increasing function and $z(t)$ is decreasing, we have

$$\begin{cases} w'(t) \geq w'(t-\tau), & t \notin S \\ \Delta w(t_k) \geq \Delta w(t_k - \tau), & \forall t_k \in S. \end{cases}$$

Then it follows from equation (3.14) that

$$\begin{cases} (1+p) w'(t-\tau) + q w'(t-\sigma) \leq 0, & t \notin S \\ (1+p)\Delta w(t_k - \tau) + q_0 \Delta w(t_k - \sigma) \leq 0, & \forall t_k \in S, \end{cases}$$

where we deduce that

$$\begin{cases} w'(t) + \dfrac{q}{1+q} w(t - (\sigma - \tau)) \leq 0, & t \notin S \\ \Delta w(t_k) + \dfrac{q_0}{1+p} w(t_k - (\sigma - \tau)) \leq 0, & \forall t_k \in S \end{cases} \tag{3.15}$$

if $1 + p > 0$, and

$$\begin{cases} w'(t) - \left[\dfrac{q}{-(1+q)}\right] w(t + (\tau - \sigma)) \geq 0, & t \notin S \\ \Delta w(t_k) - \left[\dfrac{q_0}{-(1+p)}\right] w(t_k + (\tau - \sigma)) \geq 0, & \forall t_k \in S \end{cases} \tag{3.16}$$

if $1 + p < 0$.

Therefore, the differential inequalities (3.15) and (3.16) as well as Corollaries 3.1 and 3.2 imply that equation (3.13) has a finally positive solution, which leads to a contradiction. This proves Theorem 3.4. □

Remark 3.7. Let condition C2.1 be fulfilled. Then, in view of the conditions of Theorems 3.3 and 3.4, each regular solution of the equation

$$\begin{cases} [x(t) + px(t-\tau)]' + qx(t-\sigma) = 0, & t \notin S \\ \Delta[x(t_k) + px(t_k-\tau)] + q_0 x(t_k-\sigma) = 0, & \forall t_k \in S \end{cases} \tag{3.17}$$

is oscillatory provided $q \neq 0$.

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