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POSITIVE SOLUTIONS FOR A SECOND ORDER TWO POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. We employ fixed point index properties and quadrature technics to obtain existence and multiplicity results for positive solutions to a second order two point boundary value problem.

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1. INTRODUCTION

In this paper we present some results of existence and multiplicity of positive solutions for the boundary value problem

$$\begin{cases} -u''(x) = f(x, u(x)) \ x \in (0,1) \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0, \end{cases} \tag{1.1}$$

where $a,\ b,\ c$ and d are nonegative real numbers such that $ac+ad+cb > 0$ and $f\ : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $f(t,x) > 0$ for all $(t,x) \in [0,1] \times (0,+\infty)$.

By a positive solution to problem (1.1) we understand a function $u \in C^2([0,1])$ satisfying all equations in (1.1).

We can find in many papers conditions which guarantee existence or multiplicity of positive solutions for problem (1.1), see [2], [3], [4], [5], [7], [8] and [9] and refernces therein. Often in these hypotheses is involved the position of the ratio $f(t,x)/x$ about λ_1 at 0 or ∞ (see [3], [4], [5] and[9]); Here λ_1 is the first eigenvalue of

$$\begin{cases} -u''(x) = \lambda u(x) \ x \in (0,1) \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0. \end{cases}$$

In the same spirit, we will prove in this paper that if there exists two bounded intervals I and J satisfying some conditions at there endpoints (see Theorem 3.4 in section 3) and such that the ratio $f(t,x)/x$ is great than λ_1 in I and is less than λ_1 in J, then the boundary value problem (1.1) admits a positive solution. This result is obtained by combining fixed

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point index properties with quadratures technics. Throughout, for $u \in C([0,1])$ $\|u\| = \sup\{|u(x)|, x \in [0,1]\}$.

The paper is organized as follows. In the following section we recall breefly some basic facts related to the fixed point index theory. In the third section we state main results. In the fourth section we prove some technical lemmas. Proofs of main results are posteponed to the fifth section and we end the paper by two examples of Dirichlet boundary value problems having multiple positive solutions.

2. FIXED POINT INDEX THEORY

Let E be a real Banach space and K a closed subset of E.

K is called a cone if

- K is convex
- $tx \in K$ for all $t \geq 0$ and $x \in K$,
- if $x \in K$ and $(-x) \in K$ then $x = 0$.

K is called a retract of E if there exists a continuous mapping $r : E \to K$ such that $r(x) = x$ for all $x \in K$. A such mapping is called a retraction.

From a theorem proved by Dugundji, every nonempty closed convex set of E is a retract of E. In particular every cone of E is a retract of E.

Let K be a retract of E and U an open bounded subset of K such that $U \subset B(0,R)$. For any completly continuous mapping $f : \overline{U} \to K$ with $f(x) \neq x$ for all $x \in \partial U$, the integer given by

$$i(f,U,K) = \deg\left(I - f \circ r, B(0,R) \cap r^{-1}(U), 0\right)$$

where $\deg$ is the Leray-Schauder degree, is well defined and is called fixed point index.

Properties of fixed point index:

1. **Normality :** $i(f,U,K) = \mathbf{1}$ if $f(x) = x_0 \in \overline{U}$ for all $x \in \overline{U}$
2. **Homotopy invariance :** Let $H : [0,1] \times \overline{U} \to K$ be a completly continuous mapping such that $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial U$. The integer $i(H(t,\cdot), U, K)$ is independent of t.
3. **Additivity** :
$$i(f,U,K) = i(f,U_1,K) + i(f,U_2,K)$$
whenever U_1 and U_2 are two disjoint open subsets of U such that f has no fixed point in $\overline{U} \setminus (U_1 \cup U_2)$.
4. **Permanence** : If K' is a retract of K with $f\left(\overline{U}\right) \subset K'$ then
$$i(f,U,K) = i(f,U \cap K', K').$$
5. **Solution property** : If $i(f,U,K) \neq 0$ then f admits a fixed point in U.

Now we assume that K is a cone and for all $R>0$, we denote by $K_R = B(0,R) \cap K$. We need in this work the following lemmas which give a computation of $i(f,U,K)$.

Lemma 2.1. *If $f(x) \neq \lambda x$ for all $x \in \partial K_R = \partial B(0,R) \cap K$ and $\lambda \geq 1$ then*

$$i(f, K_R, K) = 1.$$

Lemma 2.2. *If*

- $f(x) \neq \lambda x$ *for all* $x \in \partial K_R = \partial B(0,R) \cap K$ *and* $\lambda \in]0,1]$ *and*
- $\inf \{\|f(x)\| : \ x \in \partial K_R\} > 0$

then

$$i(f, K_R, K) = 0.$$

For more details and proofs we refer the reader to [6].

3. MAIN RESULTS

The statement of main results need the following notations. Let D be the subset of $\mathbb{R}^4$ defined by

$$D = \left\{(a,b,c,d) \in \mathbb{R}^4 : a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0 \text{ and } ac + ad + cb > 0.\right\}$$

For all $x \in \mathbb{R}$

$$sgn(x) = \begin{cases} \dfrac{|x|}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

For all $(a,b,c,d) \in D$ we denote by

$$2^* = 2(a,b,c,d) = 2^{sgn(ac)}.$$

Throughout this paper, for $(a,b,c,d) \in D$, $\lambda_1 = \lambda_1(a,b,c,d)$ is the first eigenvalue of the boundary value problem

$$\begin{cases} -u''(x) = \lambda u(x) \ x \in (0,1), \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0. \end{cases}$$

For $(a,b,c,d) \in D$ and $\beta \in (0, \lambda_1)$, $\eta^* = \eta(a,b,c,d,\beta)$ is the positive real number defined by

$$\eta^* = \begin{cases} \inf\left(\sqrt{\dfrac{b^2\lambda_1}{a^2 + b^2\lambda_1}}, \sqrt{\dfrac{d^2\lambda_1}{c^2 + d^2\lambda_1}}\right) \text{ if } b \neq 0 \text{ and } d \neq 0 \\ \sqrt{\dfrac{b^2\lambda_1}{a^2 + b^2\lambda_1}} \text{ if } b \neq 0 \text{ and } d = 0 \\ \sqrt{\dfrac{d^2\lambda_1}{c^2 + d^2\lambda_1}} \text{ if } b = 0 \text{ and } d \neq 0 \\ \sin\left(\dfrac{\pi - \sqrt{\beta}}{2}\right) \text{ if } b = 0 \text{ and } d = 0. \end{cases}$$

For $(a,b,c,d) \in D$ let $G^* : (0,+\infty) \to (0,+\infty)$ be the function defined by

$$G^*(x) = \sqrt{x} + \arcsin\left(\sqrt{\frac{b^2 x}{a^2 + b^2 x}}\right) + \arcsin\left(\sqrt{\frac{d^2 x}{c^2 + d^2 x}}\right) - \pi.$$

Proposition 3.1. *λ_1 is the unique solution of the equation $G^*(x) = 0$. Moreover $\lambda_1(a,b,c,d)$ is nondecreasing with respect to the variables b and d and is nonincreasing with respect to the variables a and c.*

Proposition 3.2. *Problem (1.1) admits no positive solution whenever one of the following situations*

$$\frac{f(t,x)}{x} > \lambda_1 \text{ for all } (t,x) \in [0,1] \times (0,+\infty),$$

and

$$\frac{f(t,x)}{x} < \lambda_1 \text{ for all } (t,x) \in [0,1] \times (0,+\infty)$$

holds true.

Remark 3.3. We deduce immediatly from Proposition 3.2 that a necessary condition for existence of a positive solution to Problem (1.1) is that the ratio $f(t,x)/x$ must change its position relatively to the eigenvalue λ_1.

The main result of this paper is the following theorem.

Theorem 3.4. *Suppose that there exist six real numbers $p,\ q,\ r,\ s,\ \alpha$ and β such that*

$$0 \le p < q \le r < s$$

and

$$0 < \beta < \lambda_1 < \alpha.$$

If one of the following situations (3.1) and (3.2)

$$\begin{gathered} f(t,x) \ge \alpha x\ \forall (t,x) \in [0,1] \times [p,q], \\ f(t,x) \le \beta x\ \forall (t,x) \in [0,1] \times [r,s], \\ \frac{2^*}{\sqrt{\alpha}}\left(\frac{p}{\sqrt{q^2 - p^2}} + \frac{\pi}{2} - \arcsin\left(\frac{p}{q}\right)\right) < 1 \text{ and} \\ \frac{r}{s} < \eta^*. \end{gathered} \tag{3.1}$$

$$\begin{gathered} f(t,x) \le \beta x\ \forall (t,x) \in [0,1] \times [p,q], \\ f(t,x) \ge \alpha x\ \forall (t,x) \in [0,1] \times [r,s], \\ \frac{2^*}{\sqrt{\alpha}}\left(\frac{r}{\sqrt{s^2 - r^2}} + \frac{\pi}{2} - \arcsin\left(\frac{r}{s}\right)\right) < 1 \text{ and} \\ \frac{p}{q} < \eta^*. \end{gathered} \tag{3.2}$$

holds true, then Problem (1.1) admits a positive solution u with $q < \|u\| < s$.

Remark 3.5. 1. The condition $\beta < \lambda_1 < \alpha$ in Theorem 3.4 is suggested by Remark 3.3.

2. Note that a necessary condition for existence of an interval $[p,q]$ such that the condition

$$\frac{2^*}{\sqrt{\alpha}}\left(\frac{p}{\sqrt{q^2-p^2}}+\frac{\pi}{2}-\arcsin\left(\frac{p}{q}\right)\right)<1$$

is satisfied is

$$\alpha>\left(\frac{2^*}{2}\pi\right)^2=\begin{cases}\lambda_1(a,0,c,0) & \text{if } ac\neq 0\\ \lambda_1(0,b,c,0) & \text{if } bc\neq 0\\ \lambda_1(a,0,0,d) & \text{if } ad\neq 0.\end{cases}$$

Since $\lambda_1(a,b,c,d)$ nondecreasing with respect of the variables b and d and is nondecreasing with respect of the variables a and c, it is easy to see that $\alpha>\lambda_1$ implies $\alpha>\left(\frac{2^*}{2}\pi\right)^2$.

Remark 3.6. It is proved in [8] (see Theorem 3) that if there exist two continuous functions $g_i:\mathbb{R}^+\to\mathbb{R}^+$ for $i=1,2$ such that

$$g_1(x)\leq f(t,x)\leq g_2(x) \text{ for all } (t,x)\in[0,1]\times(0,+\infty) \tag{3.3}$$

and there exist $w,\overline{w}\in\mathbb{R}^+$ with

$$\int_0^w (g_2(s)-\lambda_1 s)\,ds<0<\int_0^{\overline{w}}(g_1(s)-\lambda_1 s)\,ds \tag{3.4}$$

then Problem (1.1) admits a positive solution.

It is easy to see that Hypotheses (3.3) and (3.4) implies that there exist two bounded intervals I and J such that

$$\begin{aligned} f(t,x) &< \lambda_1 x \text{ for all } (t,x)\in[0,1]\times I \text{ and}\\ \lambda_1 x &< f(t,x) \text{ for all } (t,x)\in[0,1]\times J. \end{aligned}$$

Theorem 3.4 recover the following result proved in [4].

Corollary 3.7. *Suppose that one of the following hypotheses*

$$\liminf_{x\to 0}\left(\min_{t\in[0,1]}\frac{f(t,x)}{x}\right)>\lambda_1 \quad \textit{and} \quad \limsup_{x\to+\infty}\left(\max_{t\in[0,1]}\frac{f(t,x)}{x}\right)<\lambda_1 \tag{3.5}$$

and

$$\limsup_{x\to 0}\left(\max_{t\in[0,1]}\frac{f(t,x)}{x}\right)<\lambda_1 \quad \textit{and} \quad \liminf_{x\to+\infty}\left(\min_{t\in[0,1]}\frac{f(t,x)}{x}\right)>\lambda_1 \tag{3.6}$$

holds true, then Problem (1.1) *admits a positive solution.*

Proof. If Hypothesis (3.5) holds true (the other case is proved similarly) then for $\varepsilon>0$ small enough there exists $0<q<r$ such that

$$\begin{aligned} f(t,x) &\geq (\lambda_1+\varepsilon)x \quad \text{for } x\in[0,q] \text{ and}\\ f(t,x) &\leq (\lambda_1-\varepsilon)x \quad \text{for } x\in[r,+\infty). \end{aligned}$$

Thus, taking $p=0$ and $s=\delta r$ with δ large enough Hypothesis (3.1) is satisfied and Theorem 3.4 ensures existence of positive solution to Problem (1.1). □

We derive from Theorem 3.4 the following multiplicity result;

Corollary 3.8. *Suppose that there exists six finite sequences,* $(p_i)_{i=1}^{i=n+1}$, $(q_i)_{i=1}^{i=n+1}$, $(r_i)_{i=1}^{i=n}$, $(s_i)_{i=1}^{i=n}$, $(\alpha_i)_{i=1}^{i=n+1}$ *and* $(\beta_i)_{i=1}^{i=n}$ *(*$n\geq 1$*) such that*

$$0\leq p_1<q_1\leq r_1<s_1\leq p_2<q_2\leq r_2<s_2\leq\cdots\leq r_n<s_n\leq p_{n+1}<q_{n+1}.$$

If one of the following situations (3.7) *and* (3.8)

$$\begin{gathered}
f(t,x)\geq\alpha_j x\quad\forall\,(t,x)\in[0,1]\times[p_j,q_j]\,,\\
f(t,x)\leq\beta_i x\quad\forall\,(t,x)\in[0,1]\times[r_i,s_i]\,,\\
0<\beta_i<\lambda_1<\alpha_j,\\
\frac{2^*}{\sqrt{\alpha_j}}\left(\frac{p_j}{\sqrt{q_j^2-p_j^2}}+\frac{\pi}{2}-\arcsin\left(\frac{p_j}{q_j}\right)\right)<1 \textit{ and}\\
\frac{r_i}{s_i}<\eta_i^*=\eta\,(a,b,c,d,\beta_i)\,.
\end{gathered}\tag{3.7}$$

$$\begin{gathered}
f(t,x)\leq\alpha_j x\quad\forall\,(t,x)\in[0,1]\times[p_j,q_j]\,,\\
f(t,x)\geq\beta_i x\quad\forall\,(t,x)\in[0,1]\times[r_i,s_i]\,,\\
0<\alpha_j<\lambda_1<\beta_i\\
\frac{2^*}{\sqrt{\beta_i}}\left(\frac{r_i}{\sqrt{s_i^2-r_i^2}}+\frac{\pi}{2}-\arcsin\left(\frac{r_i}{s_i}\right)\right)<1 \textit{ and}\\
\frac{p_j}{q_j}<\eta_j^*=\eta\,(a,b,c,d,\alpha_i)\,.
\end{gathered}\tag{3.8}$$

holds true for all $1\leq i\leq n$ *and* $1\leq j\leq n+1$, *then Problem* (1.1) *admits* $2n$ *positive solutions* $(u_i)_{i=1}^{i=n}$ *and* $(v_i)_{i=1}^{i=n}$ *with* $q_i<\|u_i\|<s_i$ *and* $s_i<\|v_i\|<q_{i+1}$ *for all* $1\leq i\leq n$.

Proof. We give the proof in the case where Hypothesis (3.7) is satisfied, the other case is checked similarly.

Taking $p=p_i,\ q=q_i, r=r_i$ and $s=s_i$ then Hypothesis (3.1) of Theorem 3.4 is satisfied for all $i=1,\dots,n$. Thus, Porblem (1.1) admits n solutions $(u_i)_{i=1}^{i=n}$ with $q_i<\|u_i\|<s_i$ for all $i=1,2,\dots,n$.

Also, taking $p=r_i, q=s_i, r=p_{i+1}$ and $s=q_{i+1}$ then Hypothesis (3.2) of Theorem 3.4 is satisfied for all $1\leq i\leq n$. Thus, Porblem (1.1) admits n solutions $(v_i)_{i=1}^{i=n}$ with $s_i<\|v_i\|<q_{i+1}$ for all $1\leq i\leq n$. □

4. TECHNICAL LEMMAS

Let u be a positive solution to the boundary value problem

$$\begin{cases} -u''(x) = g(x, u(x)) \ x \in (0,1) \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0, \end{cases} \tag{4.1}$$

where $(a, b, c, d) \in D$ and $g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $g(t, x) > 0$ for all $(t, x) \in [0, 1] \times (0, +\infty)$.

It is clear that u is concave and admits a unique critical point in $[0,1]$ denoted throughout by θ, at which u reaches its maximum value. Moreover, if $a \neq 0$, u is increasing on $[0, \theta]$ and if $c \neq 0$, u is decreasing on $[\theta, 1]$.

Multiplying the differential equation in (4.1) by u' and integrating between θ and $s \in [0, 1]$ we get

$$(u'(s))^2 = 2 \int_s^\theta g(\tau, u(\tau))u'(\tau)d\tau \tag{4.2}$$

Thus, if $a \neq 0$ then for all $t \in [0, \theta]$

$$\theta - t = \int_{u(t)}^{u(\theta)} \frac{du(s)}{u'(s)} = \int_{u(t)}^{u(\theta)} \frac{du(s)}{\sqrt{2\lambda \int_s^\theta f(\tau, u(\tau))u'(\tau)d\tau}} \tag{4.3}$$

and if $c \neq 0$ then for all $t \in [\theta, 1]$

$$t - \theta = \int_{u(t)}^{u(\theta)} -\frac{du(s)}{u'(s)} = \int_{u(t)}^{u(\theta)} \frac{du(s)}{\sqrt{2 \int_s^\theta f(\tau, u(\tau))u'(\tau)d\tau}}. \tag{4.4}$$

Lemma 4.1. **a)** *Suppose that $a \neq 0$ and there exists some $t_1 \in [0, \theta]$ and $A > 0$ such that*

$$g(\tau, u(\tau)) \geq Au(\tau) \quad \textit{for all } \tau \in [t_1, \theta]. \tag{4.5}$$

Then for all $t \in [0, \theta]$

$$\theta - t \leq \frac{1}{\sqrt{A}} \begin{cases} \frac{\pi}{2} - \arcsin\left(\frac{u(t_1)}{u(\theta)}\right) & \textit{if } t \in [t_1, \theta] \\ \frac{u(t_1)}{\sqrt{u^2(\theta) - u^2(t_1)}} + \frac{\pi}{2} - \arcsin\left(\frac{u(t_1)}{u(\theta)}\right) & \textit{if } t \in [0, t_1] \end{cases} \tag{4.6}$$

Moreover if $t_1 = 0$ then

$$\theta \leq \frac{1}{\sqrt{A}} \left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{b^2 A}{a^2 + b^2 A}}\right)\right). \tag{4.7}$$

b) *Suppose that $c \neq 0$ and there exists some $t_2 \in [\theta, 1]$ and $A > 0$ such that*

$$g(\tau, u(\tau)) \geq Au(\tau) \quad \textit{for all } \tau \in [\theta, t_2]. \tag{4.8}$$

Then for all $t \in [\theta, 1]$

$$t-\theta \leq \frac{1}{\sqrt{A}} \begin{cases} \dfrac{\pi}{2} - \arcsin\left(\dfrac{u(t_2)}{u(\theta)}\right) & \text{if } t \in [\theta, t_2] \\ \dfrac{u(t_2)}{\sqrt{u^2(\theta)-u^2(t_2)}} + \dfrac{\pi}{2} - \arcsin\left(\dfrac{u(t_2)}{u(\theta)}\right) & \text{if } t \in [t_2, 1] \end{cases} \tag{4.9}$$

Moreover if $t_2 = 0$ *then*

$$1-\theta \leq \frac{1}{\sqrt{A}}\left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{d^2A}{c^2+d^2A}}\right)\right) \tag{4.10}$$

Proof. **a)** For $t \in [0, \theta]$ we distinguish two cases.

• If $t \in [t_1, \theta]$ then we derive from (4.2) and (4.6) that

$$\begin{aligned} \theta - t &\leq \int_{u(t)}^{u(\theta)} \frac{du(s)}{\sqrt{A\left(u^2(\theta)-u^2(s)\right)}} \\ &= \frac{1}{\sqrt{A}}\left(\frac{\pi}{2} - \arcsin\left(\frac{u(t)}{u(\theta)}\right)\right) \\ &\leq \frac{1}{\sqrt{A}}\left(\frac{\pi}{2} - \arcsin\left(\frac{u(t_1)}{u(\theta)}\right)\right) \end{aligned} \tag{4.11}$$

• If $t \in [0, t_1]$, since the function $s \to \int_s^\theta g(\tau, u(\tau))u'(\tau)d\tau$ is decreasing on $[0, \theta]$, we derive from (4.2) and (4.6) that

$$\begin{aligned} \theta - t &\leq \int_{u(t_1)}^{u(\theta)} \frac{du(s)}{\sqrt{A\left(u^2(\theta)-u^2(s)\right)}} + \int_{u(t)}^{u(t_1)} \frac{du(s)}{\sqrt{A\left(u^2(\theta)-u^2(t_1)\right)}} \\ &= \frac{1}{\sqrt{A}}\left(\frac{u(t_1)}{\sqrt{u^2(\theta)-u^2(t_1)}} + \frac{\pi}{2} - \arcsin\left(\frac{u(t_1)}{u(\theta)}\right)\right). \end{aligned} \tag{4.12}$$

Now if $t_1 = 0$ it follows from (4.11)

$$\theta \leq \frac{1}{\sqrt{A}}\left(\frac{\pi}{2} - \arcsin\left(\frac{u(0)}{u(\theta)}\right)\right). \tag{4.13}$$

Moreover, we obtain from (4.2)

$$a^2u^2(0) = b^2\left(u'(0)\right)^2 = b^2\int_0^\theta g(\tau, u(\tau))u'(\tau)d\tau \geq b^2A\left(u^2(\theta) - u^2(0)\right)$$

whence

$$\frac{u(0)}{u(\theta)} \geq \sqrt{\frac{b^2A}{a^2+b^2A}}. \tag{4.14}$$

Inserting (4.14) in (4.13) we obtain

$$\theta \leq \frac{1}{\sqrt{A}}\left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{b^2A}{a^2+b^2A}}\right)\right).$$

b) is checked similarly since the function $s \to \int_s^\theta g(\tau, u(\tau))u'(\tau)d\tau$ is increasing on $[\theta, 1]$. □

Lemma 4.2. ***a)*** *Suppose that* $a \neq 0$ *and there exists some* $t_3 \in [0, \theta]$ *and* $B > 0$ *such that*

$$g(\tau, u(\tau)) \leq Bu(\tau) \quad \textit{for all } \tau \in [t_3, \theta]\,. \tag{4.15}$$

Then for all $t \in [0, \theta]$

$$\theta - t \geq \frac{1}{\sqrt{B}} \begin{cases} \frac{\pi}{2} - \arcsin\left(\frac{u(t)}{u(\theta)}\right) & \textit{if } t \in [t_3, \theta] \\ \frac{\pi}{2} - \arcsin\left(\frac{u(t_3)}{u(\theta)}\right) & \textit{if } t \in [0, t_3]\,. \end{cases} \tag{4.16}$$

Moreover if $t_3 = 0$ *then*

$$\theta \geq \frac{1}{\sqrt{B}}\left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{b^2 A}{a^2 + b^2 A}}\right)\right) \tag{4.17}$$

b) *Suppose that* $c \neq 0$ *and there exists some* $t_4 \in [\theta, 1]$ *and* $B > 0$ *such that*

$$g(\tau, u(\tau)) \leq Bu(\tau) \quad \textit{for all } \tau \in [\theta, t_4]\,. \tag{4.18}$$

Then for all $t \in [\theta, 1]$

$$t - \theta \geq \frac{1}{\sqrt{B}} \begin{cases} \frac{\pi}{2} - \arcsin\left(\frac{u(t)}{u(\theta)}\right) & \textit{if } t \in [\theta, t_4] \\ \frac{\pi}{2} - \arcsin\left(\frac{u(t_4)}{u(\theta)}\right) & \textit{if } t \in [t_4, 1] \end{cases} \tag{4.19}$$

Moreover if $t_4 = 1$ *then*

$$1 - \theta \geq \frac{1}{\sqrt{B}}\left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{d^2 A}{c^2 + d^2 A}}\right)\right) \tag{4.20}$$

Proof. We present the proof of the case a) the other case is checked similarly. It follows from (4.3) for $t \in [t_3, \theta]$

$$\begin{aligned} \theta - t &\geq \int_{u(t)}^{u(\theta)} \frac{du(s)}{\sqrt{2\int_s^\theta f(\tau, u(\tau))u'(\tau)d\tau}} \\ &\geq \int_{u(t)}^{u(\theta)} \frac{du(s)}{\sqrt{B\left(u^2(\theta) - u^2(s)\right)}} \\ &= \frac{1}{\sqrt{B}}\left(\frac{\pi}{2} - \arcsin\left(\frac{u(t)}{u(\theta)}\right)\right), \end{aligned}$$

and for $t \in [0, t_3]$

$$\begin{aligned} \theta - t &\geq \int_{u(t_3)}^{u(\theta)} \frac{du(s)}{\sqrt{2\int_s^\theta g(\tau, u(\tau))u'(\tau)d\tau}} \\ &\geq \int_{u(t_3)}^{u(\theta)} \frac{du(s)}{\sqrt{B\left(u^2(\theta) - u^2(s)\right)}} \\ &= \frac{1}{\sqrt{B}}\left(\frac{\pi}{2} - \arcsin\left(\frac{u(t_3)}{u(\theta)}\right)\right). \end{aligned}$$

In particular if $t_3 = 0$ then

$$\theta \geq \frac{1}{\sqrt{B}}\left(\frac{\pi}{2} - \arcsin\left(\frac{u(0)}{u(\theta)}\right)\right).$$

As in the proof of Lemma 4.1, we check from (4.2)

$$\frac{u(0)}{u(\theta)} \leq \sqrt{\frac{b^2 B}{a^2 + b^2 B}}$$

and then

$$\theta \geq \frac{1}{\sqrt{B}}\left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{b^2 B}{a^2 + b^2 B}}\right)\right).$$

□

5. PROOFS

5.1. **Proof of Proposition 3.1.** Applying Lemmas 4.1 and 4.2 for $g = f$ and $A = B = \lambda_1$ we deduce from (4.7), (4.10), (4.17) and (4.20)

$$1 = \frac{1}{\sqrt{\lambda_1}}\left(\pi - \arcsin\left(\sqrt{\frac{b^2 A}{a^2 + b^2 A}}\right) - \arcsin\left(\sqrt{\frac{d^2 A}{c^2 + d^2 A}}\right)\right)$$

that is

$$G^*(\lambda_1) = 0.$$

At the end, using Implicit Function Theorem yield the monotonicity properties of $\lambda_1(\cdot,\cdot,\cdot,\cdot)$.

5.2. **Proof of Proposition 3.2.** Let ϕ be the positive eigenfunction associated to the eigenvalue λ_1. If u is a positve solution to Problem (1.1) then multiplying the differential equation in (1.1) by ϕ and integrating over $[0,1]$ we get the relation

$$\int_0^1 (f(\tau, u(\tau)) - \lambda_1 u(\tau))\,\phi(\tau)\,d\tau = 0.$$

which is impossible in both the cases

$$\frac{f(t,x)}{x} > \lambda_1 \text{ for all } (t,x) \in [0,1] \times (0,+\infty),$$

and

$$\frac{f(t,x)}{x} < \lambda_1 \text{ for all } (t,x) \in [0,1] \times (0,+\infty).$$

5.3. **Proof of Theorem 3.4.** Let E be the Banach space of all continuous function defined on $[0,1]$ equiped with its norm

$$\|u\| = \sup\{|u((x))|\,,\ x \in [0,1]\}\,.$$

We denote $C^2_\# = \{u \in C^2([0,1]) : au(0) - bu'(0) = 0 \text{ and } cu(1) + du'(1) = 0\}$. Let $L : C^2_\# \to E$ be the operator defined by $Lu = -u''$ with the inverse $L^{-1} : E \to C^2_\#$ is given by $L^{-1}u(x) = \int_0^1 g(x,t)\,u(t)\,dt$ where g is the Green function associated to the operator $u \to -u''$ with the boundary conditions $au(0) - bu'(0) = 0$, $cu(1) + du'(1) = 0$.

It is well known that u is a solution to Problem (1.1) if and only if $u = Tu$ where

$$Tu(x) = \int_0^1 g(x,t)\,f(t,u(t))dt.$$

T is completely continuous since $T = j \circ L^{-1} \circ N : E \to E$ where $j : C^2_\# \to E$ is the compact embeding of $C^2_\#$ in E and $N : E \to E$ is the Nymitski operator defined by $Nu(x) = f(x,u(x))$.

Let K be the cone defined by

$$K = \{u \in E : u(x) \geq p(x)\,\|u\| \text{ for all } x \in [0,1]\},$$

where for $x \in [0,1]$

$$p(x) = \min(x, 1-x).$$

Let us show that $T(K) \subset K$. If $u \in K$ and $v = Tu$ then the positivity on $[0,1]$ of $f(t,u(t))$ implies that v is concave and if v reaches its maximum at t_0 then we have in all the situations

- $t_0 \in (0,1)$, $v(x) = v((\frac{x}{t_0})t_0 + (1-(\frac{x}{t_0}))0) \geq (\frac{x}{t_0})v(t_0) + (1-(\frac{x}{t_0}))v(0) \geq p(x)\,\|v\|$ if $x \in [0,t_0]$ and $v(x) = v((\frac{1-x}{1-t_0})t_0 + (\frac{x-t_0}{1-t_0})) \geq (\frac{1-x}{1-t_0})v(t_0) + (\frac{x-t_0}{1-t_0})v(1) \geq p(x)\,\|v\|$ if $x \in [t_0,1]$,
- $t_0 = 1$, $v(x) = v(x + (1-x)0) \geq xv(1) + (1-x)v(0) \geq p(x)\,\|v\|$ and
- $t_0 = 0$, $v(x) = v(x + (1-x)0) \geq xv(1) + (1-x)v(0) \geq p(x)\,\|v\|$.

We suppose in the following $ac \neq 0$ and Hypothesis (3.1) is satisfied (the other cases can be proved in similar way).

Let us compute $i(T, K_s, K)$ where $K_s = K \cap B(0,s)$. In view of Lemma 2.1, let $u \in E$ be such that $Tu = \mu u$ with $\mu \geq 1$ and $\|u\| = s$. Then u satisfies

$$\begin{cases} -u''(x) = \mu^{-1} f(x,u(x))\ x \in (0,1) \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0. \end{cases}$$

Applying Lemma 4.2 for $g = \mu^{-1} f,\ t_3 < t_4$ are such that $u(t_3) = u(t_4) = r$ and $B = \mu^{-1}\beta$ we obtain from (4.16), (4.17), (4.19) and (4.20)

$$1 \geq \frac{\sqrt{\mu}}{\sqrt{\beta}}\left(\pi - \arcsin(\rho_1) - \arcsin(\rho_2)\right) \tag{5.1}$$

where

$$\rho_1 = \max\left(\sqrt{\frac{\beta\mu^{-1}b^2}{a^2+\beta\mu^{-1}b^2}}, \frac{r}{s}\right) \text{ and } \rho_2 = \max\left(\sqrt{\frac{\beta\mu^{-1}d^2}{c^2+\beta\mu^{-1}d^2}}, \frac{r}{s}\right).$$

Since $\frac{r}{s} < \eta^*$ and $\beta\mu^{-1} < \lambda_1$, the inequality (5.1) becomes

$$1 > \frac{1}{\sqrt{\lambda_1}}\left(\pi - \arcsin\left(\sqrt{\frac{b^2\lambda_1}{a^2+b^2\lambda_1}}\right) - \arcsin\left(\sqrt{\frac{d^2\lambda_1}{c^2+d^2\lambda_1}}\right)\right)$$

that is

$$G^*(\lambda_1) > 0$$

which is impossible.

So, hypothesis of Lemma 2.1 is satisfied and

$$i(T, K_s, K) = 1.$$

Now, let us compute $i(T, K_q, K)$. In view of Lemma 2.2, let $u \in E$ be such that $Tu = \mu u$ with $\mu \leq 1$ and $\|u\| = q$. Then u satisfies

$$\begin{cases} -u''(x) = \mu^{-1}f(x, u(x))\ x \in (0,1) \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0. \end{cases}$$

Applying Lemma 4.1 for $g = \mu^{-1}f$, $t_1 < t_2$ are such that $u(t_1) = u(t_2) = p$ and $A = \alpha\mu^{-1}$ we obtain from (4.6) and (4.9)

$$1 \leq \frac{2\sqrt{\mu}}{\sqrt{\alpha}}\left(\frac{p}{\sqrt{q^2-p^2}} + \frac{\pi}{2} - \arcsin\left(\frac{p}{q}\right)\right),$$

this is impossible since $\mu \leq 1$ and

$$\frac{2}{\sqrt{\alpha}}\left(\frac{p}{\sqrt{q^2-p^2}} + \frac{\pi}{2} - \arcsin\left(\frac{p}{q}\right)\right) < 1.$$

It remains to prove that $\inf\{\|Tu\|,\ u \in K \cap \partial B(0,q)\} > 0$. Denote $m = \inf\{f(t,x)/x,$ $(t,x) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}q, q\right]\} > 0$ and let $u \in K \cap \partial B(0,q)$. We have for all $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$

$$\begin{aligned} Tu(x) &= \int_0^1 g(x,t)f(t,u(t))dt \geq M\int_0^1 g(t,t)f(t,u(t))dt \\ &\geq M\int_{\frac{1}{4}}^{\frac{3}{4}} g(t,t)f(t,u(t))dt \geq Mm\int_{\frac{1}{4}}^{\frac{3}{4}} g(t,t)u(t)dt \\ &\geq Mmq\int_{\frac{1}{4}}^{\frac{3}{4}} g(t,t)p(t)dt > 0, \end{aligned}$$

where

$$M = \min\left(\frac{c+4d}{4(c+d)}, \frac{a+4b}{4(a+b)}\right)$$

is computed as in formula (2.4) of [4].

Therefore $\inf\{\|Tu\|\,,\ u\in K\cap\partial B(0,q)\}\geq Mmq\int_{\frac{1}{4}}^{\frac{3}{4}}g(t,t)p(t)dt>0.$

Thus, hypotheses of Lemma 2.2 are satisfied and

$$i\left(T,K_q,K\right)=0.$$

At the end, we deduce from the additivity of the fixed point index and the solution property that

$$i\left(T,K_{qs},K\right)=i\left(T,K_s,K\right)-i\left(T,K_q,K\right)=1$$

and T admits a fixed point in $K_{qs}=\{x\in K,\ q<\|x\|<s\}$ which is a positive solution to Problem (1.1). This completes the proof of Theorem 3.4.

6. EXAMPLES

6.4. **Example 1.** Consider the boundary value problem

$$\begin{cases}-u''(x)=f(u(x)),\ x\in(0,1)\\ u(0)=u(1)=0,\end{cases}\tag{6.1}$$

where $f(u)=\pi^2\left(u^2-\dfrac{11}{4}u+2\right)$.

By simple computations we can see that

$$\begin{aligned}f(x)&\geq\frac{3}{2}\pi^2x\text{ for }x\in\left[0,\frac{1}{2}\right],\\ f(x)&\leq\frac{1}{4}\pi^2x\text{ for }x\in[1,2]\text{ and}\\ f(x)&\geq 9\pi^2x\text{ for }x\in[12,13].\end{aligned}$$

So, taking $p_1=0,\ q_1=\dfrac{1}{2},\ r_1=1,\ s_1=2,\ p_2=12$, $q_2=13,\ \alpha_1=\frac{3}{2}\pi^2,\ \beta_1=\frac{1}{4}\pi^2$ and $\alpha_2=9\pi^2$ we deduce from Corollary 3.8 that Problem (6.1) has two positive solutions u_1 and u_2 such that $0<\|u_1\|<2<\|u_2\|<13$.

6.5. **Example 2.** Consider the boundary value problem

$$\begin{cases}-u''(x)=f(u(x)),\quad x\in(0,1)\\ u(0)=u(1)=0,\end{cases}\tag{6.2}$$

where $f(u)=\pi^2u\left(1+\omega\sin\left(u\right)\right)$ and $\omega=0.99$.

Choosing $p_k=\sigma\pi+2\left(k-1\right)\pi$ and $q_k=\left(1-\sigma\right)\pi+2\left(k-1\right)\pi$ for all $k\in\mathbb{N}^*$ with $\sigma\in\left(0,\frac{1}{2}\right)$, we get

$$f(u)\geq\pi^2\left(1+\omega\sin\left(\sigma\pi\right)\right)u\quad\forall u\in\left[p_k,q_k\right],$$

that is $\alpha_k=\pi^2\left(1+\omega\sin\left(\sigma\pi\right)\right)$ for all $k\in\mathbb{N}^*$.

One can prove by simple computations that the function H defined on $\mathbb{R}^+$ by

$$H(x) = \frac{2}{\pi\sqrt{1+\omega\sin(\sigma\pi)}}\left(\frac{\sigma+2x}{\sqrt{(1-2\sigma)(1+4x)}} + \frac{\pi}{2} - \arcsin\left(\frac{\sigma+2x}{1-\sigma+2x}\right)\right)$$

is increasing and has

$$\lim_{x\to\infty} H(x) = \frac{2}{\pi\sqrt{1+\omega\sin(\sigma\pi)}}\frac{1}{\sqrt{(1-2\sigma)}}.$$

Thus, choosing σ small enough, we get

$$H(k-1) < 1.$$

Now, taking $r_k = \pi(1+\theta) + 2(k-1)\pi$ and $s_k = \pi(2-\theta) + 2(k-1)\pi$ for all $k \in \mathbb{N}^*$ with $\theta \in \left(0, \frac{1}{2}\right)$, we get

$$f(u) \leq \pi^2(1-\omega\sin(\theta\pi))\, u \;\forall u \in [r_k, s_k],$$

that is $\beta_k = \pi^2(1-\omega\sin(\theta\pi))$ for all $k \in \mathbb{N}^*$.

Numerical computations give for $\theta = \dfrac{11}{24}$,

$$\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} < \eta^* = \sin\left(\frac{\pi}{2}\left(1-\sqrt{1-\omega\sin(\theta\pi)}\right)\right) < \frac{r_4}{s_4}.$$

Thus, we deduce from Corollary 3.8 that Problem (6.2) admits six positive solutions, $(u_i)_{i=1}^{i=6}$ such that

$$q_1 < \|u_1\| < s_1 < \|u_2\| < q_2 < \|u_3\| < s_2 < \|u_4\| < q_3 < \|u_5\| < s_3 < \|u_6\| < q_4.$$

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